A CHARACTERIZATION OF THE SUM AND INTEGRAL SUM LABELLINGS OF SOME CLASSES OF GRAPHS

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Abstract

A finite simple graph G is called a *sum graph* (respectively, *integral sum graph*) if there is a bijection f from the vertices of G to a set of positive integers (respectively, integers) S such that uv is an edge of G if and only if $f(u) + f(v) \in S$. For a connected graph G, the *sum number* (respectively, *integral sum number*) of G, $\sigma(G)$ (respectively, $\zeta(G)$), is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (respectively, integral sum graph). The *spum* (respectively, the *integral spum*) of a graph G is the minimum difference between the largest and smallest integer in any set S that corresponds to a sum graph (respectively, integral sum graph) containing G. The *integral radius* of a graph G is the minimum $r = r(G)$ for which there exists an integral labelling lying in the interval $[-r, r]$. We characterize sum and integral sum labellings of complete graphs, symmetric complete bipartite graphs and star graphs, and deduce the *spum*, *integral spum*, and *integral radius* for these classes of graphs.

1. Introduction

The notion of sum graph was introduced by Harary [5]. A graph $G(V, E)$ is called a *sum graph* if there is a bijection f from $V(G)$ to a set of positive integers S such that $uv \in E(G)$ for $u \neq v$ if and only if $f(u) + f(v) \in S$. We call S a set of *labels* for the sum graph G, and denote this set by $\mathcal{L}(G)$. Conversely, any set of positive integers S induces a sum graph G_S with vertex set S and edges $s_i s_j$ whenever $s_i + s_j \in S$. Thus every sum graph can be realized as one induced by a (finite) set of positive

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integers. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain at least one isolated vertex. For a connected graph G, the *sum number* of G, denoted by $\sigma(G)$, is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. The sum number of various classes of graphs are known, including $\mathcal{K}_n$, $\mathcal{K}_{m,n}$, $\mathcal{C}_n$ and trees; see [3, Table 20, p. 238].

Harary [6] also generalized the notion of sum graphs by allowing the set S to contain any set of integers in the definition of sum graphs. The corresponding graph is called an *integral sum graph*, and the *integral sum number* of a connected graph G, denoted by $\zeta(G)$, is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Unlike sum graphs, integral sum graphs need not have isolated vertices. In fact, a conjecture of Harary [6] states that all trees have integral sum number 0. The integral sum number of a few classes of graphs are known, including $\mathcal{K}_n$ and $\mathcal{C}_n$; see [3, p. 232–233].

Goodell et al. [4] investigated the difference between the largest and smallest labels in a sum graph G, and called the minimum possible such difference the *spum* of G. They proved the spum of $\mathcal{K}_n$ is $4n-6$, and the spum of $\mathcal{C}_n$ is at most $4n-10$, but their work seems to be unpublished [3, p. 230]. Singla et al. [10] extended this definition to *integral spum*, which they defined as the minimum difference between largest and smallest labels in integral sum graphs. They determined the *spum* and *integral spum* for several classes of graphs, including $\mathcal{K}_n$. $\mathcal{K}_{1,n}$, $\mathcal{K}_{n,n}$, and found bounds in the case of $\mathcal{P}_n$ and $\mathcal{C}_n$. They also obtained a sharp lower bound for the *spum* and for the *integral spum* of graphs of order n in terms of maximum and minimum vertex degree Δ and δ, respectively.

Melnikov and Pyatkin [8] defined the *integral radius* of an integral sum graph G to be the least positive integer $r = r(G)$ for which there exists an integral sum labelling $\mathcal{L}(G) \subseteq [-r, r]$. Among other results, they conjectured that the *integral radius* of the family of graphs of fixed order n is bounded above by Cn for some constant C.

In this work, we characterize sum labellings and integral sum labellings of $\mathcal{K}_n$, $\mathcal{K}_{1,n}$, and $\mathcal{K}_{n,n}$ in Theorems 1, 3, and 5, respectively. We use these characterizations to determine the *spum*, *integral spum*, and *integral radius* of these classes of graphs, as well as characterize labellings that give these numbers in Theorems 2, 4, and 6. A summary of our results is given in Table 1.

Throughout this paper, X and Y are sets of integers, $X \backslash Y := \{x : x \in X, x \notin Y\}$. By $\mathrm{AP}(a, d, k)$ we mean the k-term arithmetic progression with first term a and common difference d. By $\mathcal{L}(G)$ we mean a labelling on the vertices of the graph G, so that the *spum G (integral spum G)* equals $\max \mathcal{L}(G) - \min \mathcal{L}(G)$, and by *integral radius G* we mean the smallest positive integer $r = r(G)$ for which $\mathcal{L}(G) \subseteq [-r, r]$.

G	Characterization of integral sum labellings of G	Spum G	Integral Spum G	$r(G)$
$\mathcal{K}_n$	Theorem 1	$4n - 6$, $n \geq 5$ (Theorem 2 (a))	$4n - 6$, $n \geq 5$ (Theorem 2 (b))	$4n - 6$, $n \geq 5$ (Theorem 2 (c))
$\mathcal{K}_{1,n}$	Theorem 3	$2n - 1$, $n \geq 2$ (Theorem 4 (a))	$2n - 2$, $n \geq 2$ (Theorem 4 (b))	$\lfloor \frac{3n-2}{2} \rfloor$, $n \geq 2$ (Theorem 4 (c))
$\mathcal{K}_{n,n}$	Theorem 5	$7n - 7$, $n \geq 2$ (Theorem 6 (a))	$7n - 7$, $n \geq 2$ (Theorem 6 (b))	$6n - 5$, $n \geq 2$ (Theorem 6 (c))

Table 1: Summary of results on characterization of sum labellings and integral sum labellings of $\mathcal{K}_n$, $\mathcal{K}_{1,n}$, $\mathcal{K}_{n,n}$

2. Complete Graphs

In this section, we study sum labellings and integral sum labellings of complete graphs. Bergstrand et al. [1] showed that the sum number $\sigma(\mathcal{K}_n) = 2n - 3$ for $n \geq 4$. It is known and easy to see that $\sigma(\mathcal{K}_2) = 1$ and $\sigma(\mathcal{K}_3) = 2$; see [3, Table 20, p. 238]. Chen [2], Sharary [9], and Xu [13] proved a conjecture of Harary that the integral sum number $\zeta(\mathcal{K}_n) = 2n - 3$ for $n \geq 4$. It is known and easy to see that $\zeta(\mathcal{K}_2) = 0$ (see [12]). We note that the labelling $\{-1, 0, 1\}$ shows that $\zeta(\mathcal{K}_3) = 0$. For $n \geq 5$, we characterize sum labellings and integral sum labellings of $\mathcal{K}_n$ (Theorem 1). We show that with three exceptions for ordered pairs (a, d), every integral sum labelling of a complete graph is a union of an n-term arithmetic progression and a $(2n - 3)$-term arithmetic progression.

The study of *spum* was initiated by Goodell et al. [4] with the calculation of *spum* of complete graphs and of cycles. Although they determined the *spum* of complete graphs and found an upper bound for cycles, their paper was unpublished. Singla et al. [10] extended this definition to *integral spum*, which they defined as the minimum difference between largest and smallest labels in integral sum graphs. They showed

$$spum\ \mathcal{K}_n = 4n - 6, n \geq 2 \qquad integral\ spum\ \mathcal{K}_n = \begin{cases} 4n - 6, & n \geq 4, \\ n - 1, & n = 2, 3, \end{cases}$$

by providing a suitable sum labelling and proving that there is no integral sum labelling with a smaller difference than the one that yields a difference $4n - 6$ between minimum and maximum labels.

In this section, we deduce the results on *spum* and *integral spum* of $\mathcal{K}_n$, $n \geq 5$, from the characterization in Theorem 1. We also use this characterization to determine the integral radius $r(\mathcal{K}_n)$, and characterize all integral labellings that yield the *spum*, *integral spum*, and *integral radius* of $\mathcal{K}_n$ (Theorem 2).

Theorem 1. *For $n \geq 5$, every integral sum labelling of $\mathcal{K}_n$ is of the form*

$$AP(a, d, n) \bigcup AP(2a + d, d, 2n - 3)$$

except when

(i) $a = \lambda d$, $\lambda \in [-3n+5, 2n-4] \cap \mathbb{Z}$,

(ii) $2a = \lambda d$, $\lambda \in [-4n+8, 2n-6] \cap \mathbb{Z}$, *or*

(iii) $3a = \lambda d$, $\lambda \in [-4n+7, n-4] \cap \mathbb{Z}$.

Proof. Let $L = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_{2n-3}\}$ be an integral sum labelling of $\mathcal{K}_n$, where the graph induced by the labels $A = \{a_1, \ldots, a_n\}$ is a clique and the graph induced by the labels $B = \{b_1, \ldots, b_{2n-3}\}$ is an independent set. Without loss of generality, we may assume $a_1 < a_2 < a_3 < \ldots < a_n$ and $b_1 < b_2 < b_3 < \ldots < b_{2n-3}$. Since $0 \notin L$, we may further assume that $a_{n-2} > 0$, by replacing L by $-L$ if necessary. We denote the vertices induced by $A \cup B$ by $v_1, \ldots, v_n, w_1, \ldots, w_{2n-3}$, with $\ell(v_i) = a_i$ and $\ell(w_i) = b_i$.

CLAIM 1. We claim that $a_n + a_i \in B$ for each $i \in \{1, \ldots, n-1\}$. Since $a_n + a_i \in L$, this is equivalent to showing that $a_n + a_i \notin A$. Observe that the claim holds for $i \in \{n-1, n-2\}$ since $a_n + a_{n-1} > a_n + a_{n-2} > a_n$. Assume, for some $i \in \{1, \ldots, n-3\}$, that $a_n + a_i = a_j \in A$. Clearly, $j \neq i, n$.

Suppose $j = n-1$. Since $a_n + a_{n-2} > a_n$, we must have $a_n + a_{n-2} = \ell(w_k)$ for some k. Now $a_{n-2} + a_j \in L$ and $a_{n-2} + a_j = a_i + (a_n + a_{n-2})$ show $d(w_k) > 0$, a contradiction.

Suppose $j < n-1$. Since $a_n + a_{n-1} > a_n$, we must have $a_n + a_{n-1} = \ell(w_k)$ for some k. Again $a_{n-1} + a_j \in L$ and $a_{n-1} + a_j = a_i + (a_n + a_{n-1})$ show $d(w_k) > 0$, a contradiction.

Thus we have our claim that $a_n + a_i \in B$ for each $i \in \{1, \ldots, n-1\}$.

CLAIM 2. We claim that $a_1 + a_i \in B$ for each $i \in \{2, \ldots, n\}$. The proof of this claim follows on lines similar to that in Claim 1. Since $a_1 + a_i \in L$, this is equivalent to showing that $a_1 + a_i \notin A$. Observe that the claim holds for $i = n$, by Claim 1. Assume, for some $i \in \{2, \ldots, n-1\}$, that $a_1 + a_i = a_j \in A$. Clearly, $j \neq 1, i$.

Suppose $j = n$ and $i = n-1$. Since $a_{n-1} + a_{n-2} > a_{n-1} + a_1 = a_n$ and $a_{n-1} + a_{n-2} \in L$, we must have $a_{n-1} + a_{n-2} = \ell(w_k)$ for some k. But now $a_n + a_{n-2} = a_1 + (a_{n-1} + a_{n-2})$ shows $d(w_k) > 0$, a contradiction.

Suppose $j = n$ and $i < n-1$. Since $a_{n-1} + a_i > a_i + a_1 = a_n$ and $a_{n-1} + a_i \in L$, we must have $a_{n-1} + a_i = \ell(w_k)$ for some k. But again $a_n + a_{n-1} = a_1 + (a_{n-1} + a_i)$ shows $d(w_k) > 0$, a contradiction.

Suppose $j \leq n-1$. From Claim 1, $a_n + a_j \in B$; so $a_n + a_1 = \ell(w_k)$ for some k. Hence $a_n + a_j = a_i + (a_n + a_1)$ shows $d(w_k) > 0$, a contradiction.

Thus we have our claim that $a_1 + a_i \in B$ for each $i \in \{2, \ldots, n\}$.

CLAIM 3. We claim that A is sum free. Let, if possible, $a_i + a_j = a_k$, with $i, j \in \{1, \ldots, n\}$, $i \neq j$. By Claims 1 and 2, we may assume that $i, j \notin \{1, n\}$.

If $k = n$, then $a_i + (a_j + a_1) = a_n + a_1$. But then $a_j + a_1 \in A$, which contradicts Claim 2.

If $k \leq n-1$, then $a_i + (a_j + a_n) = a_n + a_k$. But then $a_j + a_n \in A$, which contradicts Claim 1.

CLAIM 4. We claim that the terms in A and the terms in B are in AP. Consider the $(2n-3)$-term sequence:

$$a_1 + a_2 < a_1 + a_3 < \ldots < a_1 + a_n < a_2 + a_n < \ldots < a_{n-1} + a_n.$$

Each of these terms is in B, by Claims 1 and 2, and so these terms must be the sequence $b_1, \ldots, b_{2n-3}$.

Now consider the $(2n-3)$-term sequence:

$$a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < a_2 + a_4 < \ldots$$
$$< a_2 + a_{n-1} < a_3 + a_{n-1} < \ldots$$
$$< a_{n-2} + a_{n-1} < a_{n-2} + a_n < a_{n-1} + a_n.$$

Each of these terms is in B, by Claim 3, and so these terms must be the sequence $b_1, \ldots, b_{2n-3}$.

Comparing terms in the two sequences yields $a_1 + a_k = a_2 + a_{k-1}$ for $k \in \{2, \ldots, n\}$. Thus the terms in A are in AP. The first sequence yields

$$b_k = \begin{cases} a_1 + a_{k+1} & \text{if } 1 \le k \le n-1; \\ a_n + a_{k-n+2} & \text{if } n \le k \le 2n-3. \end{cases}$$

Hence, with $a_1 = a$ and $a_2 - a_1 = d$, we get $b_k = 2a + kd$ for $k \in \{1, \ldots, 2n-3\}$. This proves that the terms in B are in AP.

Thus $L = \mathrm{AP}(a, d, n) \cup \mathrm{AP}(2a+d, d, 2n-3)$, for some choice of $a \in \mathbb{Z}$ and $d \in \mathbb{N}$.

Since the terms in L are distinct, we must have $a + id \ne 2a + jd$ with $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, 2n-3\}$. Thus, we must exclude $a = (i - j)d$, with i, j from the sets above. With $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, 2n+3\}$ we get $i - j \in \{-(2n-3), \ldots, n-2\}$.

The set A is sum free by Claim 3. Hence $a_i + a_j \in B$ for $a_i, a_j \in A$, $a_i \ne a_j$. This translates to the requirement $(a + id) + (a + jd) = 2a + (i + j)d \in \{2a + kd : 1 \le k \le 2n-3\}$, and hence imposes no restrictions on a, d since $1 \le i + j \le 2n-3$.

We must also ensure that $a_i + b_j \notin L$ and that $b_i + b_j \notin L$ for $i \ne j$.

The first of these yields $(a + id) + (2a + jd) = 3a + (i + j)d \ne a + kd$ with $i, k \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, 2n-3\}$ and $(a+id)+(2a+jd) = 3a + (i+j)d \ne 2a + kd$ with $j, k \in \{1, \ldots, 2n-3\}$ and $i \in \{0, \ldots, n-1\}$. Thus, we must exclude $2a = (k - i - j)d$ with i, j, k from the sets given in the first subcase, and also exclude $a = (k - i - j)d$ with i, j, k from the sets given in the second subcase. In the first subcase, $k - i - j \in \{-(3n-4), \ldots, n-2\}$; in the second subcase, $k - i - j \in \{-(3n-5), \ldots, 2n-4\}$.

The second of these yields $(2a + id) + (2a + jd) = 4a + (i + j)d \ne a + kd$ with $i, j \in \{1, \ldots, 2n-3\}$, $i \ne j$, and $k \in \{0, \ldots, n-1\}$ and $(2a + id) + (2a + jd) = 4a + (i+j)d \ne 2a + kd$ with $i, j, k \in \{1, \ldots, 2n-3\}$, $i \ne j$. Thus, we must exclude $3a = (k - i - j)d$ with i, j, k from the sets given in the first subcase, and also exclude $2a = (k - i - j)d$ with i, j, k from the sets given in the second subcase.

In the first subcase, $k - i - j \in \{-(4n - 7), \ldots, n - 4\}$; in the second subcase, $k - i - j \in \{-(4n - 8), \ldots, 2n - 6\}$.

This completes the proof. $\qquad\square$

Theorem 1 provides us a means to find *spum*, *integral spum*, and *integral radius* of $\mathcal{K}_n$ for $n \geq 5$. We also characterize the extremal labellings that yield these three parameters in the following theorem.

Theorem 2. *Let $n \geq 5$.*

(a) *spum $\mathcal{K}_n = 4n - 6$. Moreover, the only labelling that achieves the spum is*

$$AP(2n - 3, 1, n) \bigcup AP(4n - 5, 1, 2n - 3).$$

(b) *integral spum $\mathcal{K}_n = 4n - 6$. Moreover, the only labelling (up to sign) that achieves the integral spum is*

$$AP(2n - 3, 1, n) \bigcup AP(4n - 5, 1, 2n - 3).$$

(c) *integral radius $\mathcal{K}_n = 4n - 6$. Moreover, the only labelling (up to sign) that achieves the integral radius is*

$$AP(-2n + 1, 4, n) \bigcup AP(-4n + 6, 4, 2n - 3).$$

Proof. This follows from Theorem 1. Observe that $\mathrm{AP}(2n - 3, 1, n) \cup \mathrm{AP}(4n - 5, 1, 2n - 3)$ is a permissible labelling of $\mathcal{K}_n$, so that both *spum $\mathcal{K}_n$* and *integral spum $\mathcal{K}_n$* are bounded above by $(4n - 5) + (2n - 4) - (2n - 3) = 4n - 6$.

To show this bound cannot be improved, we need to make separate arguments for sum labellings and integral sum labellings.

(a) If L is a sum labelling, then $\max L - \min L = \big(2a + (2n-3)d\big) - a = a + (2n-3)d$ by Theorem 1, with some pairs (a, d) excluded. If $d > 1$, $\max L - \min L > 4n - 6$. Hence we may assume $d = 1$, so that $L = \mathrm{AP}(a, 1, n) \cup \mathrm{AP}(2a+1, 1, 2n-3)$, $a \notin [1, 2n - 4] \cap \mathbb{N}$. But then $a \geq 2n - 3$ and $\max L - \min L \geq 4n - 6$, with equality if and only if $a = 2n - 3$. This completes the claim for *spum* as also the claim for the extremal labelling for the *spum*.

(b) If L is an integral sum labelling, then $\max L = \max\{a + (n-1)d, 2a + (2n-3)d\}$ and $\min L = \min\{a, 2a + d\}$. If $d > 1$, then $\max L - \min L \geq \big(2a + (2n - 3)d\big) - (2a + d) = (2n - 2)d \geq 4n - 4$. Hence, we may assume $d = 1$, so that $L = \mathrm{AP}(a, 1, n) \cup \mathrm{AP}(2a + 1, 1, 2n - 3)$, $a \notin [-3n + 5, 2n - 4] \cap \mathbb{Z}$. If $a \leq -3n + 4$, then $\max L - \min L \geq (a + n - 1) - (2a + 1) = -a + n - 2 \geq 4n - 6$, with equality if and only if $a = -3n + 4$. Note that the extremal labelling is covered by the statement of the theorem by replacing L by $-L$. If $a \geq 2n - 3$, then $\max L - \min L \geq (2a + 2n - 3) - a = a + 2n - 3 \geq 4n - 6$, with equality if and only if $a = 2n - 3$. This completes the claim for *integral spum* as also the claim for the extremal labelling for the *integral spum*.

(c) For the case of *integral radius*, note that $\mathrm{AP}\big(-2n+1, 4, n\big) \cup \mathrm{AP}\big(-4n+6, 4, 2n-3\big)$ and its negative $\mathrm{AP}\big(-2n+3, 4, n\big) \cup \mathrm{AP}\big(-4n+10, 4, 2n-3\big)$ are permissible labellings of $\mathcal{K}_n$, so that *integral radius* $\mathcal{K}_n \leq 4n-6$.

To show that this upper bound cannot be improved, we take two cases: (i) $1 \leq d \leq 3$, and (ii) $d \geq 4$. Recall that if L is an integral sum labelling, then $\max L = \max\{a+(n-1)d, 2a+(2n-3)d\}$ and $\min L = \min\{a, 2a+d\}$. We shall show that

$$\min L \geq -4n+6 \quad \text{and} \quad \max L \leq 4n-6$$

can only simultaneously hold when $d = 4$.

In case (i), we consider the subcases $d = 1$, $d = 2$, and $d = 3$ separately. We reperatedly use Theorem 1 to restrict admissible pairs (a, d).

When $d = 1$, by exception (i) in Theorem 1, either $a \leq -3n+4$ or $a \geq 2n-3$. The first gives $2a+1 \leq -6n+9 < -4n+6$, the second gives $2a+2n-3 \geq 6n-9 > 4n-6$.

When $d = 2$, by exception (ii) in Theorem 1, either $a \leq -4n+7$ or $a \geq 2n-5$. The first gives $2a+2 \leq -8n+16 < -4n+6$, the second gives $2a+2(2n-3) \geq 8n-16 > 4n-6$.

When $d = 3$, by exception (iii) in Theorem 1, either $a \leq -4n+6$ or $a \geq n-3$. The first gives $2a+3 \leq -8n+15 < -4n+6$, the second gives $2a+3(2n-3) \geq 7n-15 > 4n-6$.

For case (ii), we suppose $\min L \geq -4n+6$. Since $2a+d \geq \min L \geq -4n+6$, we have $\max L \geq (2a+d)+(2n-4)d \geq (-4n+6)+5(2n-4) = 6n-14 > 4n-6$ if $d > 4$, since $n > 4$. Hence we may assume $d = 4$. Note that the extremal labellings correspond to $2a+4 = -4n+6$ and to $2a+4 = -4n+10$. If $2a+4 = -4n+8$, then $a = -2n+2$ and $2a = 4(-n+1)$, which is impossible by exception (ii) in Theorem 1. Thus $2a+4 \geq -4n+12$, and so $\max L \geq (2a+4)+4(2n-4) \geq (-4n+12)+4(2n-4) > 4n-6$. This completes the claim for *integral radius* as also the claim for the extremal labelling for the *integral radius*.

$\square$

3. Star Graphs

In this section, we study sum labellings and integral sum labellings of star graphs. Star graphs are examples of complete bipartite graphs as well as paths. Harary [5] showed that the sum number $\sigma(\mathcal{K}_{1,n}) = 1$ for $n \geq 2$, and that the integral sum number $\zeta(\mathcal{K}_{1,n}) = 0$ for $n \geq 2$. For $n \geq 2$, we characterize sum labellings and integral sum labellings (Theorem 3) of $\mathcal{K}_{1,n}$. We show that with one exception for ordered pairs (a, d), every sum labelling of a star graph is a union of an n-term arithmetic progression and a singleton. There is no simple characterization of integral sum labellings of star graphs.

Singla et al. [10] have shown that for $n \geq 2$,

$$spum \; \mathcal{K}_{1,n} = 2n - 1, \qquad integral \; spum \; \mathcal{K}_{1,n} = 2n - 2$$

by providing a suitable sum labelling and proving that there is no integral sum labelling with a smaller difference than the one that yields *spum* (respectively, *integral spum*).

In this section, we deduce the results on *spum* and *integral spum* of $\mathcal{K}_{1,n}$, $n \geq 2$, from the characterization in Theorem 3. We also use this characterization to determine the integral radius $r(\mathcal{K}_{1,n})$, and characterize all integral labellings that yield the *spum*, *integral spum*, and *integral radius* of $\mathcal{K}_{1,n}$ (Theorem 4).

Theorem 3.

(a) *For $n \geq 2$, every sum labelling of $\mathcal{K}_{1,n}$ is of the form*

$$AP(a, d, n+1) \bigcup \{d\},$$

except when $a = \lambda d$, $\lambda \in [1, n-1]$.

(b) *For $n \geq 2$, every integral sum labelling of $\mathcal{K}_{1,n}$ is of the form*

$$S \cup \{0\},$$

where at most one of $k, -k \in S$ and $k, \ell \in S$, $k \neq \ell$, implies $k + \ell \notin S$.

Proof.   (a) Let $L = \{a_1, \ldots, a_{n+1}\} \cup \{b\}$ be a sum labelling of $\mathcal{K}_{1,n}$, where b is the label assigned to the isolated vertex. Without loss of generality, we may assume $1 \leq a_1 < a_2 < a_3 < \ldots < a_n$, and the label of the central vertex to be a_{n+1}. Since $a_n + a_{n+1} > \max\{a_n, a_{n+1}\}$, we must have $a_n + a_{n+1} = b$. Consider the n-term sequence

$$a_1 + a_{n+1} < a_2 + a_{n+1} < \cdots < a_n + a_{n+1}.$$

Since none of these terms can equal either a_1 or a_{n+1}, they must be the sequence

$$a_2 < a_3 < \cdots < a_n < b.$$

Hence $a_{i+1} - a_i = a_{n+1}$ for $i \in \{1, \ldots, n-1\}$. With $a_1 = a$ and $a_{n+1} = d$, the sequence is $AP(a, d, n+1) \cup \{d\}$.

Note that the vertex corresponding to the largest label is necessarily isolated, and that the vertex corresponding to label d is adjacent to each vertex other than the isolated vertex. It remains to check if vertices corresponding to labels $a + id$ and $a + jd$ are not adjacent, for each $i, j \in \{0, 1, \ldots, n-1\}$, $i \neq j$. This would require $2a + (i+j)d \notin L$. Since $2a + (i+j)d > d$, there only remains the possibility that $2a + (i+j)d = a + kd$, with $k \in \{0, \ldots, n\}$. This happens precisely when $a = \lambda d$, with $\lambda = k - (i+j)$. Note that $k - (i+j)$ must lie between $0 - (n-1) - (n-2)$ and $n - (0+1)$. However, the lower bound must be positive, since $a > 0$. Hence, $\lambda \notin [1, n-1]$, providing the exceptional cases.

(b) Let $L = \{a_1, \ldots, a_{n+1}\}$ be an integral sum labelling of $\mathcal{K}_{1,n}$. Without loss of generality, we may assume $a_1 < a_2 < a_3 < \ldots < a_n$, and the label of the central vertex to be a_{n+1}. If $a_{n+1} > 0$, then $a_n + a_{n+1} > a_n$ forces $a_n + a_{n+1} = a_{n+1}$. But then $a_n = 0$, and the corresponding vertex must be the central vertex. This is a contradiction. If $a_{n+1} < 0$, then $a_1 + a_{n+1} < a_1$ forces $a_1 + a_{n+1} = a_{n+1}$. But then $a_1 = 0$, and the corresponding vertex must be the central vertex. This is again a contradiction. Therefore, $a_{n+1} = 0$.

Let $S = \{a_1, \ldots, a_n\}$. If $a_i + a_j = a_k$, for some $i \neq j$, then the vertices corresponding to the labels a_i and a_j both have degrees greater than 1. Thus, $k, \ell \in S$, $k \neq \ell$, implies $k + \ell \notin S$.

If $k \in S$ and $-k \in S$, then the vertices corresponding to each label have degrees greater than 1 (since $k + (-k) \in L$). Hence, at most one of $k, -k \in S$. $\qquad \square$

As in the case with complete graphs, we use Theorem 3 to determine *spum*, *integral spum*, and *integral radius* of $\mathcal{K}_{1,n}$ for $n \geq 2$. We also succeed in characterizing the extremal labellings that yield these three parameters in the following theorem, although the arguments and computations for characterization of the extremal labellings for the *integral radius* is more difficult and lengthy in this case.

Theorem 4. *Let $n \geq 2$.*

(a) *spum $\mathcal{K}_{1,n} = 2n - 1$. Moreover, the only labelling that achieves the spum is*

$$AP(n, 1, n+1) \cup \{1\}.$$

(b) *integral spum $\mathcal{K}_{1,n} = 2n - 2$. Moreover, the only labelling (up to sign) that achieves the integral spum is*

$$AP(n-1, 1, n) \cup \{0\}.$$

(c) *integral radius $\mathcal{K}_{1,n} = \left\lfloor \frac{3n-2}{2} \right\rfloor$. Moreover, the only labelling (up to sign) that achieves the integral radius is*

$$AP\left(-(n-1), 1, \tfrac{n+1}{2}\right) \cup AP\left(n, 1, \tfrac{n-1}{2}\right) \cup \{0\}$$

when n is odd, and one of

$$AP\left(-(n-1), 1, \tfrac{n}{2}\right) \cup AP\left(n, 1, \tfrac{n}{2}\right) \cup \{0\},$$
$$AP\left(-n, 1, \tfrac{n+2}{2}\right) \cup AP\left(n+1, 1, \tfrac{n-2}{2}\right) \cup \{0\},$$
$$AP\left(-\tfrac{3n-4}{2}, 3, n\right) \cup \{0\},$$

when n is even. For $n = 4$, there is an additional extremal labelling, given by $\{-5, 0, 1, 2, 4\}$.

Proof. This follows from Theorem 3.

(a) Observe that $\{1, n, n + 1, \ldots, 2n\}$ is a permissible labelling of $\mathcal{K}_{1,n}$, so that *spum* $\mathcal{K}_{1,n}$ is bounded above by $2n - 1$.

To show this bound cannot be improved, consider any sum labelling L. Then $\max L - \min L = \max\{nd, a + (n - 1)d\}$ by Theorem 3, with some pairs (a, d) excluded. If $d \geq 2$, $\max L - \min L \geq 2n$. Hence we may assume $d = 1$.

When $d = 1$, we must have $a \geq n$ by Theorem 3. Thus, $\max L - \min L \geq a + n - 1 \geq 2n - 1$, with equality if and only if $a = n$ and $a + (n - 1)d \geq nd$. The latter condition is evident, and so *spum* $\mathcal{K}_{1,n} = 2n - 1$, with the only extremal labelling that achieves the *spum* given by $\mathrm{AP}(n, 1, n + 1) \cup \{1\} = \{1, n, n + 1, \ldots, 2n\}$.

This completes the claim for *spum*, as well as the claim for the extremal labelling for the *spum*.

(b) Observe that $\{0, n - 1, n, \ldots, 2n - 2\}$ is a permissible labelling of $\mathcal{K}_{1,n}$, so that *integral spum* $\mathcal{K}_{1,n}$ is bounded above by $2n - 2$.

To show this bound cannot be improved, consider any sum labelling $L = \{a_1, \ldots, a_n\} \cup \{0\}$, where $a_1 < \cdots < a_r < 0 < a_{r+1} < \cdots < a_n$ and $a_n - a_1 \leq 2n - 2$. We may assume, without loss of generality, that $a_n > -a_1$, since replacing L by $-L$ is a valid labelling.

Observe that at most one of the terms in

$$a_1 - a_r < a_1 - a_{r-1} < \cdots < a_1 - a_2 \tag{1}$$

is in L (because a_1 could possibly equal $2a_k$). Since each of these $r - 1$ terms lies between a_1 and 0, as do the $r - 1$ terms $a_2, \ldots, a_r$, we have $a_1 < -\big((r - 1) + (r - 2)\big) = -(2r - 3)$. When $r = 1$, we have the trivial sharper bound $a_1 < -(r - 1) = 0$.

Similarly, at most one of the terms in

$$a_n - a_{n-1} < a_n - a_{n-2} < \cdots < a_n - a_{r+1} \tag{2}$$

is in L (because a_n could possibly equal $2a_k$), and is distinct from the terms $-a_1, \ldots, -a_r$ also not in L. Since each of these terms lies between 0 and a_n, as do the terms $a_{r+1}, \ldots, a_{n-1}$ from L, we have $a_n > (n-r-1)+r+(n-r-2) = 2n - r - 3$.

Hence, $a_n - a_1 \geq (2n - r - 2) + (2r - 2) = 2n + r - 4$, so that $r \in \{0, 1, 2\}$.

Suppose $r = 2$. Thus, $a_n - a_1 \geq 2n - 2$, and since $a_n - a_1 \leq 2n - 2$, we have $a_n - a_1 = 2n - 2$. From $a_n > 2n - r - 3 = 2n - 5$ and $a_1 \leq -2$ (since $r = 2$), we have $a_n - a_1 \geq 2n - 2$. This forces $a_n = 2n - 4$ and $a_1 = -2$ (and $a_2 = -1$). From the argument in the previous paragraph, we know that the three sets

$$\{a_n - a_{n-1}, a_n - a_{n-2}, \ldots, a_n - a_{r+1}\}, \quad \{-a_1, \ldots, -a_r\}, \quad \{a_{r+1}, \ldots, a_{n-1}\}$$

are pairwise disjoint and lie in the interval $[1, a_n - 1] = [1, 2n - 5]$. The first of these sets has size $n - r - 1$ (if $a_n \neq 2a_k$ for any k) or $n - r - 2$ (if $a_n = 2a_k$ for some k), whereas the other two sets have size r and $n - r - 1$, respectively. Thus, their union has size $2n - r - 2$ or $2n - r - 3$. However, this union lies in the interval $[1, 2n - 5]$, forcing the second option, that the first set has size $n - r - 2$ and $a_n = 2a_k$ for some k. Now $a_n + a_1$ must lie in one of the three sets. If $a_n + a_1 = a_n - a_i$, then $a_i = -a_1$. Since at most one of $k, -k$ can belong to L, $a_n + a_1$ cannot belong to the first set. In order that the even integer $a_n + a_1$ lie in the second set, we must have $a_n + a_1 = -a_1$. But then $a_1 = -a_k$, and this is again impossible. Hence, $a_n + a_1$ cannot belong to the second set. Note that $a_n + a_1$ cannot lie in the third set since $k + \ell \notin L$ whenever $k \in L$ and $\ell \in L$ with $k \neq \ell$. This rules out the case $r = 2$.

When $r = 1$, as in the case $r = 2$, we argue that $a_n - a_1 = 2n - 2$ forces $a_n = 2n - 3$ and $a_1 = -1$. The three sets are pairwise disjoint and lie in the interval $[1, a_n - 1] = [1, 2n - 4]$, which forces the first of the three sets to have size $n - 3$ and for a_n to equal $2a_k$ for some k. The last statement is impossible, thereby ruling out the case $r = 1$.

Henceforth in this proof, we assume $r = 0$. Hence, $0 < a_1 < \cdots < a_n$, and so $a_n = 2n - 2$. By the conditions in Theorem 3, we must have at most one integer from each pair $\{k, 2n - 2 - k\}$ for $k \in \{1, \ldots, n-1\}$. Since we also need an additional $n - 1$ elements from these $n - 1$ pairs, we must choose exactly one integer from each pair. Thus, $n - 1 \in L$. Let k be the largest positive integer for which $a_k < n - 1$. Then $2n - 2 - a_i \notin L$ and $n - 1 + a_i \notin L$ for $1 \leq i \leq k$. If $2n - 2 - a_i = n - 1 + a_j$, $1 \leq i \leq j \leq k$, then $a_i + a_j = n - 1$. This is only possible if $i = j$, so that $k = 1$ and $a_1 = \frac{n-1}{2}$. But then $\frac{n-1}{2}, n, n + \frac{n-1}{2}$ all belong to L, which is not possible.

Hence $k = 0$, and so $L = \{n - 1, \ldots, 2n - 2\} \cup \{0\}$. This completes the claim for *integral spum* as also the claim for the extremal labelling for the *integral spum*.

(c) For the case of *integral radius*, note that $\text{AP}\left(-(n-1), 1, \lceil \frac{n}{2} \rceil\right) \cup \text{AP}\left(n, 1, \lfloor \frac{n}{2} \rfloor\right) \cup \{0\}$ and its negative, are permissible labellings of $\mathcal{K}_{1,n}$. For even n, we also note that both $\text{AP}\left(-n, 1, \frac{n+2}{2}\right) \cup \text{AP}\left(n+1, 1, \frac{n-2}{2}\right) \cup \{0\}$ and $\text{AP}\left(-\frac{3n-4}{2}, 3, n\right) \cup \{0\}$ and their negatives are permissible labellings of $\mathcal{K}_{1,n}$, and additionally, $\{-5, 0, 1, 2, 4\}$ is a permissible labelling of $\mathcal{K}_{1,4}$. Thus, the *integral radius* $\mathcal{K}_{1,n} \leq \lfloor \frac{3n-2}{2} \rfloor$.

Every integral labelling of $\mathcal{K}_{1,n}$ is of the form $L = \{a_1, \ldots, a_n\} \cup \{0\}$, with $a_1 < \cdots < a_r < 0 < a_{r+1} < \cdots < a_n$, $a_i + a_j \neq 0, a_k$ for $i \neq j$, by Theorem 3. Without loss of generality, let $r \leq \frac{n}{2}$, by replacing L by $-L$ if necessary.

Consider the sequence of $n - r - 1$ positive integers

$$a_n - a_{r+1}, a_n - a_{r+2}, \ldots, a_n - a_{n-1},$$

listed in increasing order. At most one of these integers or their negatives can belong to L (because a_n could possibly equal $2a_k$).

Let $R = \max\{-a_1, a_n\}$. Partition the set $\{1, \ldots, R\}$ into three classes as follows: (I) both $k, -k$ belong to L; (II) exactly one of $k, -k$ belongs to L; (III) neither k nor $-k$ belongs to L. The class (I) is empty, and the class (II) has exactly n elements by Theorem 3. The class (III) can exclude at most one element from the set $\{a_n - a_{r+1}, a_n - a_{r+2}, \ldots, a_n - a_{n-1}\}$, by the above discussion. Hence $R \geq 2n - r - 2 \geq \frac{3n-4}{2}$.

Assume $\frac{1}{2}a_i \in L$ for some $i \in \{1, \ldots, r\}$, $\frac{1}{2}a_n = a_k \in L$, and the set of elements in class (III) is contained in $\{a_n - a_{r+1}, a_n - a_{r+2}, \ldots, a_n - a_{n-1}\}$. Let $i \in \{1, \ldots, r\}$ be such that $\frac{1}{2}a_i = a_j \in L$. Then $0 < a_n - a_i < 2R$, and so $0 < a_k - a_j < R$. Note that $a_k - a_j$ is in class (III). Hence $a_k - a_j = a_n - a_\ell$ for some $\ell \in \{r+1, \ldots, n-1\}$. Using $a_n = 2a_k$ we arrive at the contradiction $a_j + a_k = a_\ell \in L$ with $j \neq k$.

CASE A. (n odd) When n is odd, $R \geq \frac{3n-3}{2}$, with equality if and only if $r = \frac{n-1}{2}$ and $\frac{1}{2}a_n = a_k \in L$. If $R = \frac{3n-3}{2}$, class (III) has $r - 1 = \frac{n-3}{2}$ elements. Hence, the set of elements in class (III) is $\{a_n - a_{r+1}, a_n - a_{r+2}, \ldots, a_n - a_{n-1}\} \setminus \{a_k\}$. By the argument in the previous paragraph, $\frac{1}{2}a_1 \notin L$.

Both the $(r-1)$-term sequences of positive integers

$$a_r - a_1, a_{r-1} - a_1, \ldots, a_2 - a_1,$$
$$a_r - a_1, a_r - a_2, a_{r-1} - a_2, \ldots, a_3 - a_2$$

are such that none of the elements or their negatives belong to L (because $\frac{1}{2}a_1 \notin L$ and $\frac{1}{2}a_2 \notin L$). Hence, both sequences represent the elements in class (III). Since both sequences are arranged in descending order, we must have

$$a_{i+1} - a_2 = a_i - a_1$$

for $i \in \{2, \ldots, r-1\}$. Thus, $a_{i+1} - a_i = a_2 - a_1$ for $i \in \{2, \ldots, r-1\}$. With $a_2 - a_1 = d$, this gives $\{a_1, \ldots, a_r\} = \mathrm{AP}(a_1, d, r)$. Thus, the set of elements in class (III) is the set $\{a_r - a_1, a_{r-1} - a_1, \ldots, a_2 - a_1\} = \{id : 1 \leq i \leq r-1\} = \mathrm{AP}(d, d, r-1)$.

Recall that $r = \frac{n-1}{2}$ and $\frac{1}{2}a_n = a_k \in L$. The sequence of $r - 1$ positive integers

$$a_n - a_{r+1}, \ldots, a_n - a_{k-1}, a_n - a_{k+1}, \ldots, a_n - a_{n-1}$$

is such that none of the elements or their negatives belong to L, and hence represent the elements in class (III). It follows that $\{a_{r+1}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\} = \mathrm{AP}(a_n, -d, r)$.

From $R = \max\{-a_1, a_n\} = \frac{3(n-1)}{2}$ and the fact that $a_n + a_1 \neq 0$, we now have $d(n-1) \leq a_n - a_1 < 3(n-1)$. Thus, $d = 1$ or 2.

Suppose $d = 2$. We note that the set $\{a_{r+1}, \ldots, a_n\}$ must exclude the differences $a_i - a_j$, and hence the integers $\pm 2, \pm 4, \pm 6, \ldots, \pm(n-1)$. This accounts for $\frac{n-1}{2}$ of the $\frac{3n-3}{2}$ pairs of the form $\{\pm k\}$, leaving $n - 1$ pairs of the form

$\pm k$ in L. Since $|L| = n$, $\pm k \in L$ for some k, and this is a contradiction. This contradiction implies $d = 1$.

Suppose $d = 1$. Since $a_k = \frac{1}{2}a_n > 0$, $a_k \geq a_{r+1}$. Now $k > r + 1$ leads to the imposibility that $a_{k-1} < a_k < a_{k+1}$ and $a_{k+1} - a_{k-1} = 1$. Hence, $k = r + 1$, so that $a_{r+1} = \frac{1}{2}a_n$ and $a_{r+2}, \ldots, a_n$ are r consecutive positive integers.

Since none of the a_i's are in class (III), we must have $a_r < -(r-1)$ and $a_{r+1} > r - 1$. Thus, both sets $\{-a_1, \ldots, -a_r\}$ and $\{a_{r+1}, \ldots, a_n\}$ are contained in $[r, 3r]$. Since the two sets must be disjoint by Theorem 3, the sets must partition $[r, 3r]$. It follows that either $a_{r+1} = r$ or $a_r = -r$. If $a_r = -r$, then $a_1 = -(2r-1)$, so that $a_{r+1} \geq 2r$. But $a_n \geq 4r > 3r = \frac{3(n-1)}{2}$. Hence, $a_{r+1} = r$, so that $a_n = 2r$ and $\{a_{r+1}, \ldots, a_n\} = \{r, \ldots, 2r\}$. This further implies $\{a_1, \ldots, a_r\} = \{-3r, \ldots, -(2r+1)\}$.

CASE B. (n even) When n is even, $R \geq \frac{3n-4}{2}$, with equality if and only if $r = \frac{n}{2}$ and $\frac{1}{2}a_n = a_k \in L$. If $R = \frac{3n-4}{2}$, class (III) has $r - 2 = \frac{n-4}{2}$ elements. Hence, the set of elements in class (III) is $\{a_n - a_{r+1}, a_n - a_{r+2}, \ldots, a_n - a_{n-1}\} \setminus \{a_k\}$. By an argument in the paragraph preceding Subcase A, $\frac{1}{2}a_1 \notin L$.

Now the sequence of $r - 1$ positive integers

$$a_r - a_1, a_{r-1} - a_1, \ldots, a_2 - a_1$$

is such that none of the elements or their negatives belong to L (because $\frac{1}{2}a_1 \notin L$). Hence, the sequence represents the elements in class (III). But this is impossible since the number of elements in class (III) is $r - 2$. This gives the lower bound $R \geq \frac{3n-2}{2}$, with equality if and only if either (i) $r = \frac{n-2}{2}$ and $\frac{1}{2}a_n = a_k \in L$, or (ii) $r = \frac{n}{2}$. If $R = \frac{3n-2}{2}$, class (III) has $\frac{n-2}{2}$ elements.

Subcase (i) ($r = \frac{n-2}{2}$ and $\frac{1}{2}a_n = a_k \in L$) Consider the sequence of $n-r-1 = \frac{n}{2}$ positive integers

$$a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_n - a_{r+1},$$

listed in increasing order. At most one of these integers or their negatives can belong to L (because $2a_{r+1}$ may belong to L). Since class (III) has $\frac{n-2}{2}$ elements, we must have $2a_{r+1} = a_j \in L$. If $a_k - a_{r+1} \notin \{0, a_{r+1}\}$, then $a_k - a_{r+1}$ belongs to class (III). But then $a_k - a_{r+1} = a_n - a_\ell$ for some $a_\ell \in L$. Since $a_n = 2a_k$, we arrive at the contradiction $a_k + a_{r+1} = a_\ell$. Therefore, $a_k = a_{r+1}$ or $a_k = 2a_{r+1} = a_j$.

Suppose $a_k = a_{r+1}$, and consider the two sequences of $\frac{n-2}{2}$ positive integers

$$a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_{n-1} - a_{r+1};$$
$$a_{r+3} - a_{r+2}, a_{r+4} - a_{r+2}, \ldots, a_n - a_{r+2},$$

each listed in increasing order. The elements of each sequence are in class (III), and so each sequence lists all elements of class (III). Hence, $a_i - a_{r+1} =$

$a_{i+1} - a_{r+2}$ for $i \in \{r+2, \ldots, n-1\}$. Thus, $a_{i+1} - a_i = a_{r+2} - a_{r+1}$. With $a_{r+2} - a_{r+1} = d$, this gives $\{a_{r+1}, \ldots, a_n\} = \text{AP}(a_{r+1}, d, r+2)$. Thus, the set of elements in class (III) is the set $\{a_{r+2}-a_{r+1}, a_{r+3}-a_{r+1}, \ldots, a_{n-1}-a_{r+1}\} = \{id : 1 \leq i \leq r\} = \text{AP}(d, d, r)$. Now $a_{r+1} = a_n - a_{r+1} = (r+1)d = \frac{1}{2}nd$, so that $a_n = 2a_{r+1} = nd \leq \frac{3n-2}{2}$. Thus, $d = 1$, $a_{r+1} = \frac{n}{2}$, and $\{a_{r+1}, \ldots, a_n\} = \text{AP}(\frac{n}{2}, 1, \frac{n+2}{2})$.

Since the integers in $\{1, \ldots, n\}$ belong to either class (III) or L, and since their negatives cannot lie in L, we must have $\{a_1, \ldots, a_r\} \subseteq \{-\frac{3n-2}{2}, \ldots, -(n+1)\}$. But the two sets have the same size, and so must be equal. Therefore,

$$L = \text{AP}(-\tfrac{3n-2}{2}, 1, \tfrac{n-2}{2}) \cup \text{AP}(\tfrac{n}{2}, 1, \tfrac{n+2}{2}) \cup \{0\}. \qquad (3)$$

Note that this labelling is the negative of the second labelling listed in this subcase. This completes the argument for the case $a_k = a_{r+1}$.

Suppose $a_k = 2a_{r+1}$. Note that $4a_{r+1} = a_n = \max L$. Partition the interval $I = [a_{r+1}, 4a_{r+1}]$ into $[a_{r+1}, 2a_{r+1})$, $[2a_{r+1}, 3a_{r+1})$, and $[3a_{r+1}, 4a_{r+1}]$. Hence, for each $b \in (a_{r+1}, 2a_{r+1})$, at most one of $b, b + a_{r+1}, b + 2a_{r+1}$ belongs to L. Since there are $n - r = \frac{n+2}{2}$ elements of L in the interval I and $a_{r+1} - 1$ choices for b, we have $(a_{r+1}-1)+3 \geq \frac{n+2}{2}$, so that $a_{r+1} \geq \frac{n-2}{2}$. On the other hand, since $4a_{r+1} = a_n \leq \frac{3n-2}{2}$, we also have $a_{r+1} \leq \frac{3n-2}{8}$. The two bounds for a_{r+1} can simultaneously hold only when $\frac{n-2}{2} \leq \frac{3n-2}{8}$, or when $n \leq 6$.

If $n = 6$, $a_{r+1} = 2$, so that the four positive integers in L consist of $2, 4, 8$, and $\ell \in (2, 8)$. Since $\ell \neq 6$, the only choices for ℓ are $3, 5, 7$. But now the two negative integers in L must be -1 and -7 (all other possibilities are eliminated either because $-k \notin L$ when $k \in L$ or because $k_1+k_2 \notin L$ whenever $k_1, k_2 \in L$ with $k_1 \neq k_2$). This already excludes $\ell = 7$, and $\ell = 3$ and $\ell = 5$ are excluded by the presence of $-1, 4$. We conclude that $n = 6$ does not provide a permissible labelling in this case.

If $n = 4$, $a_{r+1} = 1$, so that the three positive integers in L consist of $1, 2, 4$. Since $R = 5$, we must have $-5 \in L$. We easily verify that $\{-5, 0, 1, 2, 4\}$ is a permissible labelling of $\mathcal{K}_{1,4}$. This is the labelling given in the additional case.

Subcase (ii) $(r = \frac{n}{2})$ We may assume, without loss of generality, that $a_n = \frac{3n-2}{2}$ and $a_1 \geq -\frac{3n-4}{2}$, since $R = \frac{3n-2}{2}$ and $k \in L$ implies $-k \notin L$. For the rest of the proof, we consider the four cases arising out of whether or not $2a_{r+1}$ and $2a_{r+2}$ belong to L. We show that the elements $a_{r+1}, \ldots, a_n$ form an arithmetic progression, and deduce that the elements in class (III) also form an arithmetic progression, in each case.

Subsubcase (i) Suppose that $2a_{r+1} = a_k \in L$ and $2a_{r+2} = a_\ell \in L$, and consider

the two sequences of $\frac{n-4}{2}$ positive integers

$$a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_{k-1} - a_{r+1}, a_{k+1} - a_{r+1}, \ldots, a_n - a_{r+1};$$
$$a_{r+3} - a_{r+2}, a_{r+4} - a_{r+2}, \ldots, a_{\ell-1} - a_{r+2}, a_{\ell+1} - a_{r+2}, \ldots, a_n - a_{r+2}, a_n - a_{r+1}$$

each listed in increasing order. Since each term in each sequence is in class (III), which has $\frac{n-2}{2}$ terms, class (III) has an element, say b, which is not listed in the first sequence, and an element, say c, which is not listed in the second sequence.

We claim that $a_{k-1} - a_{r+1} < b < a_{k+1} - a_{r+1}$ and $a_{\ell-1} - a_{r+2} < c < a_{\ell+1} - a_{r+2}$.

Suppose $a_i - a_{r+1} < b < a_{i+1} - a_{r+1}$ and $a_j - a_{r+2} < c < a_{j+1} - a_{r+2}$. If $i < j$, comparing the two sequences both listed in increasing order yields $a_{i+3} - a_{i+1} = a_{r+2} - a_{r+1}$. Now $a_{i+2} - a_{i+1} < a_{i+3} - a_{i+1}$ and $a_{i+2} - a_{i+1}$ belongs to class (III), and this is impossible because $a_{i+3} - a_{i+1} = a_{r+2} - a_{r+1}$. If $i = j$, comparing the two sequences both listed in increasing order yields $a_{k+1} - a_{r+1} = a_{k+1} - a_{r+2}$, which is impossible. If $i > j$, comparing the two sequences both listed in increasing order yields $a_{j+1} - a_{r+1} = a_{j+1} - a_{r+2}$, which is impossible.

If $a_{k-1} - a_{r+1} < b < a_{k+1} - a_{r+1}$ and $a_j - a_{r+2} < c < a_{j+1} - a_{r+2}$, comparing the two sequences both listed in increasing order yields $a_{j+1} - a_{r+1} = a_{j+1} - a_{r+2}$, which is impossible.

If $a_i - a_{r+1} < b < a_{i+1} - a_{r+1}$ and $a_{\ell-1} - a_{r+2} < c < a_{\ell+1} - a_{r+2}$, comparing the two sequences both listed in increasing order yields $a_{i+3} - a_{i+1} = a_{r+2} - a_{r+1}$. Now $a_{i+2} - a_{i+1} < a_{i+3} - a_{i+1}$ and $a_{i+2} - a_{i+1}$ belongs to class (III), and this is impossible because $a_{i+3} - a_{i+1} = a_{r+2} - a_{r+1}$.

Thus, we have $a_{k-1} - a_{r+1} < b < a_{k+1} - a_{r+1}$ and $a_{\ell-1} - a_{r+2} < c < a_{\ell+1} - a_{r+2}$. Comparing the two sequences both listed in increasing order yields $a_k - a_{k-1} = a_{k-1} - a_{k-2} = \cdots = a_{r+2} - a_{r+1}$ and further $a_{k+2} - a_{k+1} = a_{k+3} - a_{k+2} = \cdots = a_{\ell-1} - a_{\ell-2} = a_{r+2} - a_{r+1}$ and $a_{\ell+1} - a_\ell = a_{\ell+2} - a_{\ell+1} = \cdots = a_n - a_{n-1} = a_{r+2} - a_{r+1}$. Therefore, we have $a_{j+1} - a_j = a_{r+2} - a_{r+1}$ for $j \neq k, \ell - 1$. We claim that this also holds for $j = k, \ell - 1$. To see this, note that

$$a_{k+1} - a_k, a_{k+2} - a_k, \ldots, a_n - a_k, a_n - a_{k-1}, \ldots, a_n - a_{r+1}$$

is an increasing sequence of $\frac{n-2}{2}$ terms in class (III). Hence $a_{k+1} - a_k = a_{r+2} - a_{r+1}$. Similarly, note that

$$a_\ell - a_{\ell-1}, a_{\ell+1} - a_{\ell-1}, \ldots, a_n - a_{\ell-1}, a_n - a_{\ell-2}, \ldots, a_n - a_{r+1}$$

is an increasing sequence of $\frac{n-2}{2}$ terms in class (III). Hence $a_\ell - a_{\ell-1} = a_{r+2} - a_{r+1}$ as well. Thus, we have shown that $\{a_{r+1}, \ldots, a_n\} = \mathrm{AP}(a_{r+1}, d, r)$, and so the set of elements in class (III) is the set $\{a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_n - a_{r+1}\} = \{id : 1 \leq i \leq r - 1\} = \mathrm{AP}(d, d, r - 1)$.

Subsubcase (ii) Suppose that $2a_{r+1} = a_k \in L$ and $2a_{r+2} \notin L$, and consider the two sequences of positive integers

$$a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_{k-1} - a_{r+1}, a_{k+1} - a_{r+1}, \ldots, a_n - a_{r+1};$$
$$a_{r+3} - a_{r+2}, a_{r+4} - a_{r+2}, \ldots, a_n - a_{r+2}, a_n - a_{r+1}$$

each listed in increasing order. Note that the first sequence has $\frac{n-4}{2}$ terms while the second sequence has $\frac{n-2}{2}$ terms. Since each term is in class (III), which has $\frac{n-2}{2}$ terms, class (III) has an element, say b, which is not listed in the first sequence.

If $b < a_{k-1} - a_{r+1}$, write $a_j - a_{r+1} < b < a_{j+1} - a_{r+1}$ with $j < k - 1$. Comparing the two sequences both listed in increasing order yields $a_{j+1} - a_j = a_j - a_{j-1} = \cdots = a_{r+2} - a_{r+1}$ and further $a_{j+1} - a_{r+1} = a_{j+3} - a_{r+2}$. Now $a_{j+2} - a_{j+1} < a_{j+3} - a_{j+1}$ and $a_{j+2} - a_{j+1}$ belongs to class (III), and this is impossible because $a_{j+3} - a_{j+1} = a_{r+2} - a_{r+1}$.

If $b > a_{k+1} - a_{r+1}$, then $a_{k+1} - a_{r+1} = a_{k+1} - a_{r+2}$ by comparing the two sequences both listed in increasing order. This is impossible.

Suppose $a_{k-1} - a_{r+1} < b < a_{k+1} - a_{r+1}$. Comparing the two sequences both listed in increasing order yields $a_k - a_{k-1} = a_{k-1} - a_{k-2} = \cdots = a_{r+2} - a_{r+1}$ and further $a_{k+2} - a_{k+1} = a_{k+3} - a_{k+2} = \cdots = a_n - a_{n-1}$. Since we also have $a_{k+2} - a_{k+1} = a_{r+2} - a_{r+1}$, we have $a_{j+1} - a_j = a_{r+2} - a_{r+1}$ for $j \neq k$. We claim that this also holds for $j = k$. To see this, note that

$$a_{k+1} - a_k, a_{k+2} - a_k, \ldots, a_n - a_k, a_n - a_{k-1}, \ldots, a_n - a_{r+1}$$

is an increasing sequence of $\frac{n-2}{2}$ terms in class (III). Hence $a_{k+1} - a_k = a_{r+2} - a_{r+1}$, proving the claim. Thus, we have shown that $\{a_{r+1}, \ldots, a_n\} = \mathrm{AP}(a_{r+1}, d, r)$, and so the set of elements in class (III) is the set $\{a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_n - a_{r+1}\} = \{id : 1 \leq i \leq r - 1\} = \mathrm{AP}(d, d, r - 1)$.

Subsubcase (iii) The case where $2a_{r+1} \notin L$ and $2a_{r+2} \in L$ can be dealt with in a manner analogous to the one given in Subsubcase (ii).

Subsubcase (iv) Suppose that $2a_{r+1} \notin L$ and $2a_{r+2} \notin L$, and consider the two sequences of $\frac{n-2}{2}$ positive integers

$$a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_{n-1} - a_{r+1}, a_n - a_{r+1};$$
$$a_{r+3} - a_{r+2}, a_{r+4} - a_{r+2}, \ldots, a_n - a_{r+2}, a_n - a_{r+1}$$

each listed in increasing order. As in Subcase (i), we deduce that $\{a_{r+1}, \ldots, a_n\} = \mathrm{AP}(a_{r+1}, d, r)$. Thus, the set of elements in class (III) is the set $\{a_{r+2} - a_{r+1}, a_{r+3} - a_{r+1}, \ldots, a_n - a_{r+1}\} = \{id : 1 \leq i \leq r - 1\} = \mathrm{AP}(d, d, r - 1)$.

Thus, in each case, we have shown that the elements $a_{r+1}, \ldots, a_n$ as well as the elements in class (III) form an arithmetic progression.

The sequence of positive integers

$$a_r - a_1, a_{r-1} - a_1, \ldots, a_2 - a_1$$

belong to class (III), except $a_k - a_1$ if $2a_k = a_1$. If $\frac{1}{2}a_1 \notin L$, then the $r-1$ positive integers listed in the sequence must coincide with the sequence $d, 2d, 3d, \ldots, (r-1)d$. Thus, $\{a_1, \ldots, a_r\} = \mathrm{AP}(a_1, d, r)$. If $\frac{1}{2}a_1 = a_k \in L$, then the $r-2$ positive integers $a_i - a_1$, with $i \neq k$, listed in the sequence must each be of the form id, $1 \leq i \leq r-1$. Since only one multiple of d from $\{d, 2d, 3d, \ldots, (r-1)d\}$ is missing from the sequence with $a_k - a_1$ removed, the difference between the consecutive terms $a_{k+1} - a_1$ and $a_{k-1} - a_1$ must be either d or $2d$. If this difference is d, then $a_k - a_{k-1} < (a_{k+1} - a_1) - (a_{k-1} - a_1) = d$. But $a_k - a_{k-1}$ is in class (III), and this is impossible. If this difference is $2d$, then $a_k - a_{k-1} = d$ since $a_k - a_{k-1}$ is in class (III). Then again $\{a_1, \ldots, a_r\} = \mathrm{AP}(a_1, d, r)$.

From

$$
\begin{aligned}
(r-1)d + 3 + (r-1)d \ &\leq\ (a_r - a_1) + (a_{r+1} - a_r) + (a_n - a_{r+1}) \\
&=\ a_n - a_1 \\
&\leq\ \tfrac{3n-2}{2} + \tfrac{3n-4}{2} \\
&=\ 3n - 3,
\end{aligned}
$$

we have $d \leq 3$.

Suppose $d = 1$. Then $a_{r+1} + \left(\frac{n}{2} - 1\right) = \frac{3n-2}{2}$, so that $a_{r+1} = n$. Hence, $\{a_{r+1}, \ldots, a_n\} = \left[n, \frac{3n-2}{2}\right]$, so that the set $\{a_1, \ldots, a_r\}$ must exclude both the set of their differences $\left[-\frac{n-2}{2}, -1\right]$ and their negatives $\left[-\frac{3n-2}{2}, -n\right]$. Thus, $\{a_1, \ldots, a_r\}$ must lie within the interval $\left[-(n-1), -\frac{n}{2}\right]$, which has size $\frac{n}{2} = r$. It follows that $\{a_1, \ldots, a_r\} = \left[-(n-1), -\frac{n}{2}\right]$. Thus, $\{a_1, \ldots, a_n\} = \mathrm{AP}\left(-(n-1), 1, \frac{n}{2}\right) \cup \mathrm{AP}\left(n, 1, \frac{n}{2}\right)$. This corresponds to the first extremal labelling in the theorem.

Suppose $d = 2$. We note that the set $\{a_1, \ldots, a_n\}$ must exclude the differences $a_i - a_j$, and hence the integers $\pm 2, \pm 4, \pm 6, \ldots, \pm(n-2)$. This accounts for $\frac{n-2}{2}$ of the $\frac{3n-2}{2}$ pairs of the form $\{\pm k\}$. Since L can contain at most one of $\pm k$ for each k, it follows that $\{a_1, \ldots, a_n\}$ consists of exactly one of $\pm k$ among the n remaining pairs. If $n \in L$, we must have $a_{r+1} = n$ since $n - 2 \notin L$. But then $a_n = a_{r+1} + 2\left(\frac{n}{2} - 1\right) = n + (n-2) > \frac{3n-2}{2}$. If $n \notin L$, then $-n \in L$, and again since $-(n-2) \notin L$ we must have $a_r = -n$. But now $a_1 = a_r - 2\left(\frac{n}{2} - 1\right) = -n - (n-2) < -\frac{3n-2}{2}$. Thus, there is no extremal labelling corresponding to this case.

Suppose $d = 3$. Then $a_{r+1} + 3\left(\frac{n}{2} - 1\right) = \frac{3n-2}{2}$, so that $a_{r+1} = 2$. Also $a_1 = a_r - 3\left(\frac{n}{2} - 1\right) \leq -1 - \frac{3(n-2)}{2} = -\frac{3n-4}{2}$, so that $a_1 = -\frac{3n-4}{2}$. Hence $a_r =$

$a_1 + 3(r-1) = -\frac{3n-4}{2} + \frac{3(n-2)}{2} = -1$. Thus, $\{a_1, \ldots, a_n\} = \mathrm{AP}(-\frac{3n-4}{2}, 3, n)$. This corresponds to the third extremal labelling in the theorem.

This completes the claim for *integral radius*, as well as the claim for the extremal labelling for the *integral radius*.

$\square$

4. Complete Symmetric Bipartite Graphs

In this section, we study sum labellings and integral sum labellings of complete symmetric bipartite graphs, those for which the two partite sets have the same size. Hartsfield and Smyth [7] showed that the sum number $\sigma(\mathcal{K}_{n,n}) = 2n - 1$ for $n \geq 2$, and Yan and Liu [11] showed that the integral sum number $\zeta(\mathcal{K}_{n,n}) = 2n - 1$ for $n \geq 2$. For $n \geq 2$, we characterize sum labellings and integral sum labellings (Theorem 5) of $\mathcal{K}_{n,n}$. We show that with six exceptions for ordered pairs (a, d), every integral sum labelling of a complete symmetric bipartite graph is a union of two (disjoint) n-term arithmetic progressions and a $(2n - 1)$-term arithmetic progression.

Singla et al. [10] have shown that for $n \geq 2$,

$$spum\ \mathcal{K}_{n,n} = 7n - 7, \qquad integral\ spum\ \mathcal{K}_{n,n} = 7n - 7$$

by providing a suitable sum labelling and proving that there is no integral sum labelling with a smaller difference than the one that yields *spum* (respectively, *integral spum*).

In this section, we deduce the results on *spum* and *integral spum* of $\mathcal{K}_{n,n}$, $n \geq 2$, from the characterization in Theorem 5. We also use this characterization to determine the integral radius $r(\mathcal{K}_{n,n})$, and characterize all integral labellings that yield the *spum*, *integral spum*, and *integral radius* of $\mathcal{K}_{n,n}$ (Theorem 6).

Theorem 5. *For $n \geq 2$, every integral sum labelling of $\mathcal{K}_{n,n}$ is of the form*

$$AP(a, d, n) \cup AP(b, d, n) \cup AP(a + b, d, 2n - 1)$$

except when

(i) $a, b = \lambda d$, $\lambda \in [-3n + 3, 2n - 2] \cap \mathbb{Z}$,
(ii) $2a - b, 2b - a = \lambda d$, $\lambda \in [-2n + 3, n - 2] \cap \mathbb{Z}$,
(iii) $a - b = \lambda d$, $\lambda \in [-2n + 3, 2n - 3] \cap \mathbb{Z}$,
(iv) $a + b = \lambda d$, $\lambda \in [-4n + 5, 2n - 3] \cap \mathbb{Z}$,
(v) $2a, 2b = \lambda d$, $\lambda \in [-3n + 3, n - 1] \cap \mathbb{Z}$, *or*
(vi) $2a + b, 2b + a = \lambda d$, $\lambda \in [-4n + 5, n - 2] \cap \mathbb{Z}$.

Proof. Let $L = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\} \cup \{c_1, \ldots, c_{2n-1}\}$ be an integral sum labelling of $\mathcal{K}_{n,n}$, where the graph induced by the labels $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ form a biclique and the graph induced by the labels $C = \{c_1, \ldots, c_{2n-1}\}$ is an independent set. Without loss of generality, we may assume $a_1 < a_2 < a_3 < \ldots < a_n$, $b_1 < b_2 < b_3 < \ldots < b_n$, and $c_1 < c_2 < c_3 < \ldots < c_{2n-1}$. We denote the vertices induced by $A \cup B \cup C$ by $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_{2n-1}$, with $\ell(u_i) = a_i$, $\ell(v_i) = b_i$ and $\ell(w_i) = c_i$.

Yan and Liu [11] showed that the set $A \cup B$ is sum free. We claim that this implies the terms in A, B, C are each in AP. Consider the $(2n-1)$-term sequence:

$$a_1 + b_1 < a_1 + b_2 < \ldots < a_1 + b_n < a_2 + b_n < \ldots < a_n + b_n.$$

Each of these terms is in C, and so these terms must be the sequence $c_1, \ldots, c_{2n-1}$.

Now consider the $(2n-1)$-term sequence:

$$a_1 + b_1 < a_2 + b_1 < \ldots < a_2 + b_{n-1} < a_3 + b_{n-1} < \ldots < a_n + b_{n-1} < a_n + b_n.$$

Each of these terms is in C, and so these terms must be the sequence $c_1, \ldots, c_{2n-1}$.

Comparing terms in the two sequences yields $a_1 + b_k = a_2 + b_{k-1}$ for $k \in \{2, \ldots, n\}$ and $a_k + b_n = a_{k+1} + b_{n-1}$ for $k \in \{2, \ldots, n-1\}$. Thus the terms in A and in B are in AP, with the same common difference. The first sequence yields

$$c_k = \begin{cases} a_1 + b_k & \text{if } 1 \le k \le n; \\ a_{k-n+1} + b_n & \text{if } n+1 \le k \le 2n-1. \end{cases}$$

Hence, with $a_1 = a$, $b_1 = b$ and $a_2 - a_1 = b_n - b_{n-1} = d$, we get $c_k = a + b + kd$ for $k \in \{0, \ldots, 2n-2\}$. This proves that the terms in C are in AP.

Thus $L = \mathrm{AP}(a, d, n) \cup \mathrm{AP}(b, d, n) \cup \mathrm{AP}(a + b, d, 2n - 1)$, for some choice of $a, b \in \mathbb{Z}$ and $d \in \mathbb{N}$.

Since the terms in L are distinct, we must have $a + id \ne b + jd$ with $i, j \in \{0, \ldots, n-1\}$. Thus we must exclude $b - a = (i - j)d$, with i, j from the sets as given above. With $i, j \in \{0, \ldots, n-1\}$ we get $i - j \in \{-(n-1), \ldots, n-1\}$. We must also have $a + id \ne a + b + kd$ with $i \in \{0, \ldots, n-1\}$ and $k \in \{0, \ldots, 2n-2\}$. Thus we must exclude $b = (i - k)d$ with $i - k \in \{-(2n-2), \ldots, n-1\}$. Interchanging the roles of a and b implies that we must also exclude $a = (j - k)d$ with $j - k \in \{-(2n-2), \ldots, n-1\}$.

Since the set $A \cup B$ is sum free, $a_i + b_j \in C$. This translates to the requirement $(a + id) + (b + jd) = a + b + (i + j)d \in \{a + b + kd : 0 \le k \le 2n - 2\}$, and hence imposes no restrictions on a, b, d since $0 \le i + j \le 2n - 2$.

We must also ensure that $a_i + a_j, b_i + b_j, c_i + c_j, a_i + c_j, b_i + c_j \notin L$.

The first of these yields $(a + id) + (a + jd) = 2a + (i + j)d \ne a + kd, b + kd, a + b + \ell d$ with $i, j, k \in \{0, \ldots, n-1\}$, $i \ne j$, and $\ell \in \{0, \ldots, 2n - 2\}$. Thus we must exclude $a = (k - i - j)d$ in the first subcase, $2a - b = (k - i - j)d$ in the second subcase, and $a - b = (\ell - i - j)d$ in the third subcase, with i, j, k, ℓ from the sets given above. In the first and second subcases, $k - i - j \in \{-(2n - 3), \ldots, n - 2\}$; in the third subcase, $\ell - i - j \in \{-(2n - 3), \ldots, 2n - 3\}$.

The second of these may be obtained by interchanging the roles of a and b. Thus we must exclude $b, 2b - a = (k - i - j)d$ with $k - i - j \in \{-(2n - 3), \ldots, n - 2\}$, and $b - a = (\ell - i - j)d$ with $\ell - i - j \in \{-(2n - 3), \ldots, 2n - 3\}$.

The third of these yields $(a + b + id) + (a + b + jd) = 2a + 2b + (i + j)d \neq a + kd, b + kd, a + b + \ell d$ with $i, j, \ell \in \{0, \ldots, 2n - 2\}$, $i \neq j$, and $k \in \{0, \ldots, n - 1\}$. Thus we must exclude $2a + b = (k - i - j)d$ in the first subcase, $a + 2b = (k - i - j)d$ in the second subcase, and $a + b = (\ell - i - j)d$ in the third subcase, with i, j, k, ℓ from the sets given above. In the first and second subcases, $k - i - j \in \{-(4n - 5), \ldots, n - 2\}$; in the third subcase, $\ell - i - j \in \{-(4n - 5), \ldots, 2n - 3\}$.

The fourth of these yields $(a + id) + (a + b + jd) = 2a + b + (i + j)d \neq a + kd, b + kd, a + b + \ell d$ with $i, k \in \{0, \ldots, n - 1\}$, and $j, \ell \in \{0, \ldots, 2n - 2\}$. Thus we must exclude $a + b = (k - i - j)d$ in the first subcase, $2a = (k - i - j)d$ in the second subcase, and $a = (\ell - i - j)d$ in the third subcase, with i, j, k, ℓ from the sets given above. In the first and second subcases, $k - i - j \in \{-(3n - 3), \ldots, n - 1\}$; in the third subcase, $\ell - i - j \in \{-(3n - 3), \ldots, 2n - 2\}$.

The last of these may be obtained by interchanging the roles of a and b. The first subcase is unchanged. Thus we must exclude $2b = (k - i - j)d$ with $k - i - j \in \{-(3n - 3), \ldots, n - 1\}$ and $b = (\ell - i - j)d$ with $\ell - i - j \in \{-(3n - 3), \ldots, 2n - 2\}$.
This completes the proof. $\qquad\square$

As in the previous two sections, we use Theorem 5 to determine *spum, integral spum*, and *integral radius* of $\mathcal{K}_{n,n}$ for $n \geq 2$. We also characterize the extremal labellings that yield these three parameters in the following theorem.

Theorem 6. *Let $n \geq 2$.*

(a) *spum $\mathcal{K}_{n,n} = 7n - 7$. Moreover, the only labelling that achieves the spum is*

$$AP(3n - 3, 1, n) \bigcup AP(5n - 5, 1, n) \bigcup AP(8n - 8, 1, 2n - 1).$$

(b) *integral spum $\mathcal{K}_{n,n} = 7n - 7$. Moreover, the only labelling (up to sign) that achieves the integral spum is*

$$AP(3n - 3, 1, n) \bigcup AP(5n - 5, 1, n) \bigcup AP(8n - 8, 1, 2n - 1).$$

(c) *integral radius $\mathcal{K}_{n,n} = 6n - 5$. Moreover, the only labelling (up to sign) that achieves the integral radius is*

$$AP(-6n + 7, 4, n) \bigcup AP(2n - 1, 4, n) \bigcup AP(-4n + 6, 4, 2n - 1).$$

Proof. This follows from Theorem 5. Observe that $AP(3n - 3, 1, n) \cup AP(5n - 5, 1, n) \cup AP(8n - 8, 1, 2n - 1)$ is a permissible labelling of $\mathcal{K}_{n,n}$, so that both *spum* $\mathcal{K}_{n,n}$ and *integral spum* $\mathcal{K}_{n,n}$ are bounded above by $(10n - 10) - (3n - 3) = 7n - 7$.

To show this bound cannot be improved, we need to make separate arguments for sum labellings and integral sum labellings. We may assume, without loss of generality, that $a < b$.

(a) If L is a sum labelling, then $\max L - \min L = (a + b + (2n-2)d) - a = b + (2n-2)d$ by Theorem 5, with some triples (a, b, d) excluded. If $d \geq 4$, $\max L - \min L > 7n - 7$. Hence we may assume $d \in \{1, 2, 3\}$.

When $d = 3$, we show that $b > 3n - 3$. Note that a, b cannot assume integers of the form $3t$ in the interval $[1, 6n - 6]$, by exception (i). Thus each of a, b must be of the form $3t+1$ or $3t+2$ in the interval $[1, 6n-6]$. If $a \equiv b \pmod 3$, then $b - a = 3t$ where $1 \leq t \leq 2n - 3$, which is not possible by exception (iii). Otherwise $a \equiv -b \pmod 3$, and then $b + a = 3t$ with $t \in [1, 2n - 3]$, which is not possible by exception (iv). Hence $b > 3n - 3$, and so $\max L - \min L = b + 3(2n - 2) > 7n - 7$.

Thus there is no extremal labelling when $d = 3$.

When $d = 2$, we show that $b > 4n - 4$. Note that a, b cannot assume even values in the interval $[1, 4n - 4]$ by exception (i). Thus a, b may possibly take only odd values in the interval $[1, 4n - 4]$. If $a = 2s+1, b = 2t+1 \in [1, 4n-4]$, then $b - a = 2(t - s)$ with $t - s \in [1, 2n - 3]$, which contradicts exception (iii). Hence $b > 4n - 4$, and so $\max L - \min L = b + 2(2n - 2) > 7n - 7$.

Thus there is no extremal labelling when $d = 2$.

When $d = 1$, we show that $b \geq 5n - 5$, with equality if and only if $a = 3n - 3$. From exception (i), $a \geq 2n - 1$, and from exception (ii), $2a - b \leq -2n + 2$ or $2a - b \geq n - 1$. In the first case, $b \geq 2a + 2n - 2 \geq 2(2n - 1) + 2n - 2 > 5n - 5$. There remains the case $2a - b \geq n - 1$, which together with $b - a \geq 2n - 2$ from exception (iii), gives $a \geq 3n - 3$. Therefore $b \geq a + 2n - 2 \geq 5n - 5$, with equality if and only if $a = 3n - 3$. Thus $\max L - \min L = b + (2n-2)d \geq 7n - 7$, with equality if and only if $b = 5n - 5$.

This completes the claim for *spum* as also the claim for the extremal labelling for the *spum*.

(b) Since the case $a > 0$ is covered by part (a), and $a \neq 0$, we may assume throughout this Case that $a < 0$. If L is an integral sum labelling, then $\max L - \min L \geq (a + b + (2n-2)d) - (a + b) = (2n-2)d$ by Theorem 5, with some triples (a, b, d) excluded. If $d \geq 4$, $\max L - \min L > 7n - 7$. Hence we may assume $d \in \{1, 2, 3\}$.

When $d = 3$, we show that at least one of $a < -4(n-1)$, $b > n-1$ must hold. Suppose, to the contrary, that $a \geq -4(n - 1)$ and $b \leq n - 1$. Note that a, b cannot assume integers of the form $3t$ in the interval $[-4(n - 1), n - 1]$, by exception (i). Thus each of a, b must be of the form $3t+1$ or $3t+2$ in the interval $[-4(n - 1), n - 1]$. If $a \equiv b \pmod 3$, then $b - a = 3t$ where $1 \leq t \leq \frac{5}{3}(n - 1)$, which is not possible by exception (iii). Otherwise $a \equiv -b \pmod 3$, and then $b + a = 3t$ with $-\frac{1}{3}(8n - 9) \leq t \leq \frac{1}{3}(n - 2)$, which is not possible by exception (iv). Hence at least one of $a < -4(n - 1)$, $b > n - 1$ must hold. But then $\max L - \min L \geq (b + (n-1)d) - (a + b) = -a + 3(n-1) > 7n - 7$ if $a < -4(n-1)$ and $\max L - \min L \geq (a + b + (2n - 2)d) - a = b + 3(2n - 2) > 7n - 7$ if

$b > n - 1$.

Thus there is no extremal labelling when $d = 3$.

When $d = 2$, we show that at least one of $a < -5(n-1)$, $b > 3(n-1)$ must hold. Suppose, to the contrary, that $a \geq -5(n-1)$ and $b \leq 3(n-1)$. Note that a, b lie in the interval $[-5(n-1), 3(n-1)]$ and must be odd, by exception (i). Now $b - a \geq 4n - 4$ by exception (iii) and the fact that $b - a$ is even. Thus $b \geq -(n-1)$, and so $a + b = 2t$ where $-3(n-1) \leq t \leq \frac{1}{2}(3n-4)$, which is not possible by exception (iii). Hence at least one of $a < -5(n-1)$, $b > 3(n-1)$ must hold. But then $\max L - \min L \geq (b + (n-1)d) - (a+b) = -a + 2(n-1) > 7n - 7$ if $a < -5(n-1)$ and $\max L - \min L \geq (a + b + (2n-2)d) - a = b + 2(2n-2) > 7n - 7$ if $b > 3(n-1)$.

Thus there is no extremal labelling when $d = 2$.

When $d = 1$, we show that there is no extremal labelling when $b > 0$. To do this, we show that at least one of $a < -6(n-1)$, $b > 5(n-1)$ must hold. Suppose, to the contrary, that $a \geq -6(n-1)$ and $b \leq 5(n-1)$. Then $a < -3(n-1)$ and $b > 2(n-1)$, by exception (i), and so $-4(n-1) < a + b < 2(n-1)$. This contradicts exception (iv). Hence at least one of $a < -6(n-1)$, $b > 5(n-1)$ holds, and $\max L - \min L \geq (b + (n-1)d) - (a+b) = -a + n - 1 > 7n - 7$ if $a < -6(n-1)$ and $\max L - \min L \geq (a + b + (2n-2)d) - a = b + 2n - 2 > 7n - 7$ if $b > 5(n-1)$.

Therefore any extremal labelling in this case must have $b < 0$. From exception (i), we must have $a < b < -3(n-1)$. But then all labels in L are negative, and applying Case I to $-L$ we obtain the desired result.

This completes the claim for *integral spum* as also the claim for the extremal labelling for the *integral spum*.

(c) For the case of *integral radius*, note that $\mathrm{AP}(-6n+7, 4, n) \cup \mathrm{AP}(2n-1, 4, n) \cup \mathrm{AP}(-4n+6, 4, 2n-1)$ and its negative $\mathrm{AP}(2n-3, 4, n) \cup \mathrm{AP}(-6n+5, 4, n) \cup \mathrm{AP}(-4n+2, 4, 2n-1)$ are permissible labelling of $\mathcal{K}_{n,n}$, so that *integral radius* $\mathcal{K}_{n,n} \leq 6n - 5$.

Recall that if L is an integral sum labelling, then $\max L = \max\{b + (n-1)d, a + b + (2n-2)d\}$ and $\min L = \min\{a, a+b\}$. We shall show that

$$\min L \geq -6n + 5 \quad \text{and} \quad \max L \leq 6n - 5$$

can only simultaneously hold when $d = 4$.

Thus, we may assume $-6n + 5 \leq a < b \leq 6n - 5 - (n-1)d$ and $-6n + 5 \leq a + b \leq 6n - 5 - (2n-2)d$. If $d \geq 7$, then

$$14n - 14 \leq (a + b + (2n-2)d) - (a+b) \leq \max L - \min L$$
$$\leq (6n - 5) - (-6n + 5) = 12n - 10.$$

Hence we may assume $d \in \{1, 2, 3, 4, 5, 6\}$. We repeatedly use Theorem 5 to restrict admissible triples (a, b, d).

  – When $d = 1$, both a, b lie in $[-6n + 5, -3n + 2] \cup [2n - 1, 5n - 4]$ by
    exception (i), and $b - a \geq 2(n - 1)$ by exception (iii). If both $a, b \in$
    $[-6n + 5, -3n + 2]$, then $a \leq -5(n - 1)$, giving rise to the impossibility
    $\min L \leq a + b \leq -8(n-1)$. If both $a, b \in [2n-1, 5n-4]$, then $b \geq 4(n-1)$,
    giving rise to the impossibility $\max L \geq a + b + 2(n - 1) \geq 8(n - 1)$.
    We may therefore suppose that $a \in [-6n + 5, -3n + 2]$ and $b \in [2n - 1, 5n - 4]$. If $a = -6n + 5$ and $b = 2n - 1$, then $a + 2b = -2n + 3$,
    which contradicts exception (vi). If $a = -3n + 2$ and $b = 5n - 4$, then
    $2a + b = -n$, which also contradicts exception (vi). If the ordered pair
    $(a, b) \neq (-6n+5, 2n-1)$ or $(-3n+2, 5n-4)$, then $a+b \in [-4n+5, 2n-3]$,
    which contradicts exception (iv).

  – When $d = 2$, both a, b lie in $[-6n + 5, 4n - 3] \setminus 2\mathbb{Z}$, by exception (i). But
    then $a + b \in [-6n + 5, 2n - 1] \cap 2\mathbb{Z}$, which contradicts exception (iv).

  – When $d = 3$, both a, b lie in $[-6n + 5, 3n - 2] \setminus 3\mathbb{Z}$, by exception (i), and
    $a + b \in [-6n + 5, 1] \setminus 3\mathbb{Z}$, by exception (iv). Hence $a \equiv b \pmod 3$; write
    $b - a = 3t$, $t \geq 2n - 2$, by exception (iii). Therefore $b = a + 3t \geq 1$, and
    so $-6n + 7 \leq a + 2b = (a + b) + b \leq 3n - 1$. Since $3 \mid (a + 2b)$, we have a
    contradiction to (vi).

  – When $d = 4$, both a, b lie in $[-6n + 5, 2n - 1] \setminus 2\mathbb{Z}$, by exception (v),
    and $a + b \in [-6n + 5, -2n + 3] \setminus 4\mathbb{Z}$, by exception (iv). Hence $a \equiv b$
    $\pmod 4$; write $b - a = 4t$. Note that $t \geq 2n - 2$, by exception (iii), and
    $t \leq 2n - 2$ since $b - a \leq (2n - 1) - (-6n + 5)$. Thus, $t = 2n - 2$. Now
    $b = a + 4t \geq 2n - 3$ implies $b \in \{2n - 3, 2n - 1\}$ and $a \in \{-6n + 5, -6n + 7\}$.

  – When $d = 5$, both a, b lie in $[-6n + 5, n] \setminus 5\mathbb{Z}$, by exception (i), and
    $a + b \in [-6n+5, -4n+5] \setminus 5\mathbb{Z}$, by exception (iv). Now $b - a \in [1, 7n-5] \setminus 5\mathbb{Z}$,
    by exception (iii), $-12n + 10 \leq 2a + b = a + (a + b) \leq -3n + 5$ and
    $-12n + 10 \leq a + 2b = b + (a + b) \leq -3n + 5$. By exception (vi),
    $5 \nmid (2a + b)$ and $5 \nmid (a + 2b)$. Thus, $a \not\equiv tb \pmod 5$ for $t \in \{0, 1, 2, 3, 4\}$,
    and this is impossible.

  – When $d = 6$, both a, b lie in $[-6n + 5, 1] \setminus 3\mathbb{Z}$, by exception (v), and
    $a + b \in [-6n+5, -6n+7] \setminus 6\mathbb{Z}$, by exception (iv). Now $b - a \in [1, 6n-4] \setminus 6\mathbb{Z}$,
    by exception (iii), $-12n + 10 \leq 2a + b = a + (a + b) \leq -6n + 8$ and
    $-12n + 10 \leq a + 2b = b + (a + b) \leq -6n + 8$. By exception (vi),
    $6 \nmid (2a + b)$ and $6 \nmid (a + 2b)$. Thus, since $3 \nmid (a + b)$, $a \not\equiv tb \pmod 6$ for
    $t \in \{0, 1, 2, 3, 4, 5\}$, and this is impossible.

This completes the claim for *integral radius* as also the claim for the extremal
labelling for the *integral radius*.

$\square$

5. Concluding Remarks

We also attempted to study the characterization of sum labellings and integral sum labellings of some other classes of graphs, viz., the complete bipartite graphs $\mathcal{K}_{m,n}$, paths $\mathcal{P}_n$, and cycles $\mathcal{C}_n$, but without much success. In the case of $\mathcal{K}_{m,n}$, $m \neq 1, n$, the sum number $\sigma(\mathcal{K}_{m,n})$ is given by a rather forbidding formula (see [3, p. 230]), and the integral sum number $\zeta(\mathcal{K}_{m,n})$ only conditionally (see [3, p. 232]). All this does not suggest a characterization is possible in the general case for complete bipartite graphs. The *spum* and *integral spum* of paths and cycles were studied by the second author (see [10]), but the results were not encouraging enough to suggest that a characterization of sum labelling or integral sum labellings is feasible.

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