

Minimum flow rate of Taylor cone-jets of highly viscous liquids

Anupam Choubey¹, Abhishek Kumar Singh² and Supreet Singh Bahga¹ 

¹Department of Mechanical Engineering, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 110016, India

²Department of Mechanical Engineering, Bennett University, Greater Noida, Uttar Pradesh, 201310, India

Corresponding author: Supreet Singh Bahga, bahga@mech.iitd.ac.in

(Received 8 April 2025; revised 9 March 2026; accepted 10 March 2026)

We present an experimental and theoretical investigation of steady Taylor cone-jetting of highly viscous liquids at the minimum flow rate required for steady jetting. To achieve a steady cone-jet, the viscous liquid is flown through a conducting needle under the action of a strong electric field acting between the needle and a flat collector plate. Experiments reveal that the minimum flow rate and the corresponding jet diameter depend on both the needle diameter and the electrical conductivity of the liquid. Subsequently, we used the experimental measurements to formulate correlations among the minimum flow rate, the needle diameter and the physical properties of the liquid, including the electrical conductivity. To elucidate the underlying physics and experimental observations, we performed an order-of-magnitude analysis. The scaling analysis reveals that in the cone region, the viscous and interfacial tension forces are of comparable magnitude, while in the current transfer region, the viscous and electrostatic suction forces are the dominant resisting and driving forces, respectively. Subsequently, we theoretically derived the scaling relations for the minimum flow rate and the corresponding jet diameter by considering the balance of forces at the cone-tip and in the current transfer region. The empirical and theoretical scaling laws for the minimum flow rate and the corresponding jet diameter agree well for highly viscous liquids with electrical conductivity spanning over four orders of magnitude. Lastly, we present the limits that describe the regime for which the minimum flow rate depends on the needle diameter, and the derived scaling laws are applicable.

Key words: electrokinetic flows, microfluidics, electrohydrodynamic effects

1. Introduction

When a liquid meniscus, held at the tip of a conducting needle, is subjected to a strong electric field E , the liquid meniscus deforms and attains a conical shape. The strong electric field of $\mathcal{O}(10^6 \text{ Vm}^{-1})$ is achieved by applying a potential difference between the needle tip and a planar electrode placed downstream of the needle. The conical meniscus so obtained is known as the Taylor cone (Taylor 1964). To obtain a Taylor cone, typically, a leaky dielectric liquid (Saville 1997) is used as the working liquid and injected through the fine conducting needle with quiescent air as the surrounding dielectric medium. To sustain the Taylor cone, the liquid must be fed continuously at a certain flow rate Q under the action of the electric field E due to the applied electric potential difference.

Depending on the magnitude of the electric field E and the imposed flow rate Q , various jetting modes are observed, namely, the dripping, microdripping, pulsating jet, steady cone-jet, tilted jet and multi-jet (Cloupeau & Prunet-Foch 1989; Lee *et al.* 2013). Among these modes, the steady cone-jet mode, characterised by a fine jet that is emitted from the apex of the stable conical meniscus, has been studied in great detail (Zeleny 1914; Taylor 1964; Fernández de La Mora 2007; Singh, Srivastava & Bahga 2022). The formation of a steady jet requires strong electric forces to balance the opposing interfacial tension force and accelerate the liquid in the downstream direction. These strong electric forces result from the coupling of the electric field with the free and polarisation charges on the liquid surface (Gañán-Calvo 1997b; Higuera 2003).

Over the past few decades, the steady cone-jet mode has found numerous applications in various domains, including electrospraying (Gañán-Calvo *et al.* 2018; Rosell-Llompart, Grifoll & Loscertales 2018), electrospinning (Yarin 2003) and electrohydrodynamic jet (EHD-jet) printing (Onses *et al.* 2015; Choubey, Dubey & Bahga 2023; Choubey & Bahga 2024). This is due to the ability of steady cone-jet mode to produce continuous jets, which may break up into uniformly sized droplets further downstream. The jet diameter of a steady cone-jet in electrospinning and EHD-jet printing or the droplet diameter in electrospraying primarily depends on the imposed flow rate along with the physical properties of the liquid (Gañán-Calvo 1997b). However, the steady cone-jet mode can be obtained for a limited range of the electric field due to the applied electric potential difference and the imposed flow rate of the liquid, as depicted in figure 1.

On the electric field flow rate ($E-Q$) operating diagram, as shown in figure 1, the steady Taylor cone-jet forms for a narrow range of the electric field. This narrow ($E-Q$) region in the two-dimensional space is bounded by the lowest value of the flow rate, as shown in figure 1, known as the minimum flow rate, Q_{min} , below which a steady cone-jet mode can no longer be sustained (Bober & Chen 2011; Chen 2011; Gañán-Calvo *et al.* 2013). The order of magnitude of the threshold electric field, E_{min} , required for the formation of a conical meniscus attached to a needle can be estimated by balancing Maxwell stresses with capillary stresses (Zeleny 1935; Taylor 1964; Smith 1986; Basaran & Scriven 1990; Yarin, Koombhongse & Reneker 2001). When Taylor's self-similar solution (Taylor 1964) is matched to a meniscus attached to a needle of finite outer diameter d_n , this balance yields a characteristic field scale $E_{min} \sim \sqrt{\gamma/(\epsilon_0 d_n)}$ (Pantano, Gañán-Calvo & Barrero 1994; Chen 2011), where γ is the interfacial tension and ϵ_0 is the permittivity of the surrounding medium. This field corresponds to a static equilibrium threshold, below which a conical meniscus cannot be sustained (Pantano *et al.* 1994). It is important to emphasise that this threshold field does not ensure stable cone-jet operation. The dynamic stability of the cone-jet depends jointly on the applied electric field and the issued flow rate (Chen 2011). At the minimum flow rate condition, steady cone-jet operation is observed only when the applied field is of the order of the threshold value. For $E \ll E_{min}$, the system operates in

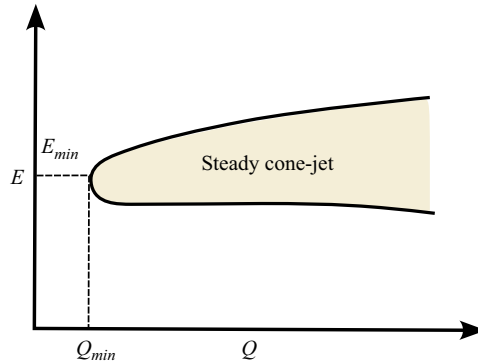


Figure 1. Schematic representation of the operating regime of a steady Taylor cone-jet. At the minimum flow rate Q_{min} , the electric field due to the applied electric potential difference must be equal to E_{min} to get a steady cone-jet. For flow rates above Q_{min} , a steady cone-jet can be obtained for a limited range of E around E_{min} . However, for flow rates below Q_{min} , steady cone-jetting is not possible. Instead, a pulsating cone-jet is observed at and around E_{min} for flow rates below Q_{min} .

the dripping regime, whereas for $E \gg E_{min}$, a multi-jet mode is observed. For $E \lesssim E_{min}$, a pulsating jet is observed (Chen 2011). Thus, for steady cone-jetting, the imposed flow rate must be greater than or equal to the minimum flow rate. However, this is a necessary but not sufficient condition. In addition, the electric field due to the applied potential difference must lie within the stability range of the steady cone-jet mode. As illustrated in figure 1, the steady cone-jetting occurs for $Q \geq Q_{min}$ only over a finite interval of electric field around the threshold value, where the threshold value is of $\mathcal{O}(\sqrt{\gamma/(\epsilon_0 d_n)})$.

The width of the deposited feature in electrospinning and EHD-jet printing, and the droplet diameter in electrospraying depend on the diameter of the jet emitted from the tip of the Taylor cone (Qian, Lan & Zhang 2018). It is well established that the diameter of the jet emitted in the steady cone-jet mode or the diameter of the droplets obtained in electrospraying is a strong function of flow rate and decreases upon reducing the injection flow rate (Cloupeau & Prunet-Foch 1989; Gañán-Calvo 1997b; Gañán-Calvo 2004). Thus, the minimum flow rate is important as it sets the limiting condition for the smallest obtainable size of continuous jets or droplets from the steady cone-jetting of a particular liquid (Ponce-Torres *et al.* 2018). In applications such as electrospinning and EHD-jet printing, typically, cone-jets of viscous Newtonian (Cho *et al.* 2024) or polymeric liquids (Choubey *et al.* 2023; Choubey & Bahga 2024) are employed near the minimum flow rate condition to deposit micrometer or sub-micrometre sized continuous features. Therefore, the estimation of the minimum flow rate is of fundamental and practical relevance.

Fernandez De La Mora *et al.* (1990), De La Mora & Loscertales (1994) and De La Mora (1996) suggested that for liquids with relatively high conductivity ($\kappa \gtrsim 10^2 \mu\text{S m}^{-1}$), the minimum flow rate of steady cone-jetting is independent of the viscosity of the liquid and the needle diameter. In particular, the minimum flow rate scales as $Q_{min} \sim \beta Q_0 \sim \beta \gamma \epsilon_0 / (\rho \kappa)$. Here, β , ρ and κ denote the electrical permittivity ratio, density and the conductivity of the liquid, respectively. However, this scaling relation for the minimum flow rate lacked a rigorous theoretical basis.

Later, Gañán-Calvo *et al.* (2013) provided the proper theoretical background for such a scaling relation for the minimum flow rate. They argued that the minimum flow rate is independent of the geometrical parameters, such as the needle diameter (d_n), due to the locality of the current transfer region, and depends solely on the physical properties of the liquids. To derive the scaling laws, Gañán-Calvo *et al.* (2013) considered the balance of

dominant forces in the current transfer region. In particular, they suggested two scenarios. In one case, the viscous force is responsible for opposing the formation of the jet at the minimum flow rate condition, known as the viscous limit case. In the other case, the polarisation force stalls the jet, known as the polarisation limit case. Gañán-Calvo *et al.* (2013) argued that liquids having $\beta Re_0 < 1$ follow the viscous limit case and, in this scenario, the minimum flow rate scales as $Q_{min} \sim Q_0 Re_0^{-1}$. Here, Re_0 represents the electrohydrodynamic Reynolds number which is expressed as $Re_0 = [\gamma^2 \varepsilon_0 \rho / (\mu^3 \kappa)]^{1/3}$. However, liquids that satisfy the $\beta Re_0 \gtrsim 1$ criterion follow the polarisation limit case and the minimum flow rate scales as $Q_{min} \sim \beta Q_0$.

Building on the work of Gañán-Calvo *et al.* (2013), Ponce-Torres *et al.* (2018) performed linear stability analysis on the liquid of the base state, described by the steady cone-jetting, to determine the minimum flow rate of steady cone-jets. To validate their numerical results, they used existing experimental data on the minimum flow rate of liquids satisfying the $\beta Re_0 < 1$ and $\beta Re_0 > 1$ criteria. In addition, Ponce-Torres *et al.* (2018) performed new experiments for liquids satisfying the $\beta Re_0 \sim 1$ criterion. Their study suggests that if the ratio of the needle diameter d_n to the reference scale for the jet diameter $d_0 = [\gamma \varepsilon_0^2 / (\rho \kappa^2)]^{1/3}$ is less than 80, Q_{min} depends on the needle diameter. However, a suitable explanation for such dependence on the needle diameter and scaling law for minimum flow rate in such scenarios was not provided. Ponce-Torres *et al.* (2018) acknowledged that the scaling laws provided by Gañán-Calvo *et al.* (2013) are in good agreement with their numerical results and experimental observation for $d_n/d_0 \gtrsim 80$. Other experimental studies investigating the influence of needle design on the minimum flow rate in electrospraying and electrohydrodynamic jetting have explored a variety of needle geometries and materials (Lozano & Martínez-Sánchez 2005; Castro & Fernández De La Mora 2009; Fernández-Martínez *et al.* 2025; Gañán-Calvo & López-García 2025). These studies demonstrate that the minimum flow rate in electrospraying and electrohydrodynamic jetting is influenced not only by the liquid properties and operating conditions, but also by the tip geometry and material of the needle. In particular, the use of dielectric and bevelled needles has been reported to substantially reduce the minimum flow rate, enabling steady jetting of low-conductivity Newtonian and viscoelastic solutions (Fernández-Martínez *et al.* 2025). Such configurations have consequently facilitated high-resolution printing using the same working solutions (Fernández-Martínez *et al.* 2025).

Subsequent to Ponce-Torres *et al.* (2018), in a study with salt-doped propylene carbonate (PC) and tributylphosphate (TBP), Gamero-Castaño & Magnani (2019) showed that the dimensionless minimum flow rate $Q_{min}/Q_0 = f(Re_0, \beta)$ and is independent of the needle diameter. Unlike Gañán-Calvo *et al.* (2013), who derived scaling relations based on the balance of forces in the current transfer region, Gamero-Castaño & Magnani (2019) adopted an approach based on the energy balance in the cone-to-jet transition region to obtain the scaling relations for the minimum flow rate. Furthermore, they suggested that the energy loss due to viscous dissipation at the cone-to-jet transition region is the main reason for the instability near the minimum flow rate. They argued that near the minimum flow rate, the work done by the dominant driving force is not sufficient to compensate for the energy loss taking place in the cone-to-jet transition region due to viscous dissipation. Given the significance of the viscous dissipation, Gamero-Castaño & Magnani (2019) concluded that the minimum flow rate of liquids with $\beta Re_0 < 1$, $\beta Re_0 \gtrsim 1$ and $\beta Re_0 \gg 1$ depends on Re_0 . In particular, they observed that the minimum flow rate increases upon decreasing Re_0 and increasing β .

We note that the liquids considered by Gañán-Calvo *et al.* (2013), Ponce-Torres *et al.* (2018) and Gamero-Castaño & Magnani (2019) that satisfy the $\beta Re_0 < 1$ condition

have electrical conductivity κ of $\mathcal{O}(10^3 - 10^5 \mu\text{S m}^{-1})$ and dynamic viscosity μ of $\mathcal{O}(1 - 10 \text{ mPa s})$. Such high values of electrical conductivity ($Re_0 \sim \kappa^{-1/3}$) ensured that $\beta Re_0 < 1$ even though their dynamic viscosity μ ($Re_0 \sim \mu^{-1}$) is relatively low. However, in these cases, the Ohnesorge number defined as $Oh = \mu / \sqrt{\rho \gamma d_n}$ remains less than unity, $Oh < 1$. Despite extensive work on the steady cone-jetting of less viscous liquids, limited studies have focused on the steady cone-jetting at very low flow rates for liquids with high dynamic viscosity (Ku *et al.* 2001; Ku & Kim 2003). Notably, Scheideler & Chen (2014) were the only ones to report experimental work on the minimum flow rate of highly viscous liquids with dynamic viscosity μ of $\mathcal{O}(10^2 - 10^3 \text{ mPa s})$ and electrical conductivity κ of $\mathcal{O}(10^2 - 10^4 \mu\text{S m}^{-1})$ such as salt-doped glycerol solutions. In addition, they used needles with a blunt end for their minimum flow rate experiments. We note that for the liquids considered by Scheideler & Chen (2014), the Ohnesorge number is greater than unity $Oh > 1$, suggesting a distinct regime of steady cone-jetting that differs from the previously studied cases satisfying $\beta Re_0 < 1$ and $Oh < 1$. This distinction suggests that steady cone-jetting behaviour of highly viscous liquids ($Oh > 1$) satisfying $\beta Re_0 < 1$ may not follow the same scaling laws as less viscous counterparts having $\beta Re_0 < 1$ but $Oh < 1$. While existing scaling laws for the low viscosity liquids predict Q_{min} based solely on the liquid's properties, the experimental observation of Scheideler & Chen (2014) indicates that, for highly viscous liquids, Q_{min} also depends on the geometrical parameters, specifically on the needle diameter even when needles with a blunt end are employed for electrospraying or electrohydrodynamic jetting experiments.

In contrast to the findings of Gañán-Calvo *et al.* (2013), Ponce-Torres *et al.* (2018) and Gamero-Castaño & Magnani (2019) for liquids with $\beta Re_0 < 1$, the experiments of Scheideler & Chen (2014) demonstrated that the minimum flow rate strongly depends on the needle diameter even when $d_n/d_0 \gtrsim 80$ and the needle tip is blunt, while remaining independent of the electrical conductivity of the liquid. In particular, Scheideler & Chen (2014) observed a power law dependence of the minimum flow rate on the needle diameter, specifically $Q_{min} \sim d_n^{1.45 \pm 0.18}$. Subsequently, Scheideler & Chen (2014) proposed a scaling law for the minimum flow rate as $Q_{min} \sim \gamma d_n^2 / \mu$. Although the scaling law approximately captures the power law dependence on the needle diameter, it overestimates the minimum flow rate by two to three orders of magnitude when compared with the measured values of Q_{min} reported in the same work.

Furthermore, in their cone-jetting experiments with viscous liquids, the range of electrical conductivity κ , considered by Scheideler & Chen (2014), was limited to $\mathcal{O}(10^2 - 10^4 \mu\text{S m}^{-1})$. Consequently, their experiments did not capture the dependence of Q_{min} on κ , resulting in an incorrect scaling law for the minimum flow rate, which is independent of the electrical conductivity of the viscous liquid. These contrasting findings have resulted in two groups of thought regarding the minimum flow rate of liquids satisfying the $\beta Re_0 < 1$ criterion. One of them suggests that Q_{min} depends only on the physical properties of the liquid, including κ , and is independent of the geometrical parameters such as d_n . The other argues that Q_{min} is independent of κ but strongly depends on d_n , specifically $Q_{min} \sim d_n^2$.

Therefore, based on the previous discussion, we conclude that the minimum flow rate of cone-jets of liquids with $\beta Re_0 < 1$ is still not fully understood. Consequently, an extensive investigation is required across a wide range of needle diameters and electrical conductivity to understand the parameters that influence the minimum flow rate. Moreover, the current literature also lacks experimentally validated scaling laws for the minimum flow rate and the associated quantities, such as the jet diameter at the minimum flow rate.

1.1. Scope of the present work

In this work, we present theoretical scaling laws for the minimum flow rate of the Taylor cone-jet and the jet diameter at the minimum flow rate of highly viscous liquids, along with their experimental validation. For the minimum flow rate experiments, we employ highly viscous liquids with electrical conductivity spanning over four orders of magnitude and stainless steel needles with a blunt end. Specifically, we show that the minimum flow rate scales as $Q_{min} \sim Q_0(d_n/d_0)^{8/5} Re_0^{7/5}$ and the jet diameter at the minimum flow rate scales as $d \sim d_0(d_n/d_0)^{3/5} Re_0^{2/5}$. These scaling relations show excellent agreement with the experimental data for viscous liquids with electrical conductivity varying over four orders of magnitude. In addition, we present the limits of the regime in which these scaling laws are valid.

We begin by presenting the details of the experimental set-up, methodology and the data in § 2. Section 3 presents dimensional analysis and empirical correlations for the minimum flow rate, and the jet diameter at the minimum flow rate. Section 4 presents the theoretical scaling analysis and validation of the derived scaling relations with our experimental data and that obtained from the literature. In § 5, we discuss the validity of the assumptions based on which the scaling laws were derived. Subsequently, we discuss the limiting conditions under which the minimum flow rate of highly viscous liquid follows the derived scaling laws and shows dependence on the needle diameter. Lastly, § 6 provides the concluding remarks.

2. Experiments and results

In this section, we present the experimental methodology and the measurements to estimate the minimum flow rate for steady cone-jetting. We also present the measurements of the jet diameter and the total current carried by the jet at the minimum flow rate condition.

2.1. Experimental set-up

Figure 2 shows the schematic of the experimental set-up used for steady cone-jetting experiments. To measure the minimum flow rate, we employed a needle-plate configuration, wherein the needle was mounted on a vertical translation stage and a fine wire-mesh collector was positioned below the needle at a controllable distance (H) from the needle tip, as shown in figure 2. A high-precision syringe pump (KD Scientific Legato 100, USA) was used to feed a viscous liquid with the desired flow rate Q to a needle with an outer diameter d_n . A DC high-voltage power supply (Keithley-2290-10, USA) was used to apply a potential difference V between a needle and the collector. The current I carried by the cone-jet was measured using a custom-built ammeter with precision operational amplifiers (op-amps). The total current carried by the cone-jet was directed to the inverting terminal of the op-amp (OPA227P, Texas Instruments), while the voltage drop across a 10 M Ω shunt resistor was monitored and measured using a multimeter (Hewlett Packard 34401a, precision of 10^{-6} mV). Subsequently, the total current carried by the jet was calculated as $I = V_{sh}/R_{sh}$, where R_{sh} is the resistance of the shunt resistor and V_{sh} is the measured voltage drop across the shunt resistor. To ensure high accuracy in the current measurement, the non-inverting terminal of the op-amp was grounded through another precision op-amp (MCP-601, Texas Instruments). To visualise and capture the steady-state images of the cone-jet, a stereo-zoom microscope (Radical Scientific, 0.8 – 4.5 $\times$ zoom), equipped with a CMOS camera (Radical Scientific, Procam HS-5) and a collimated LED light source (Thorlabs, USA), was employed. To measure the asymptotic jet diameter,

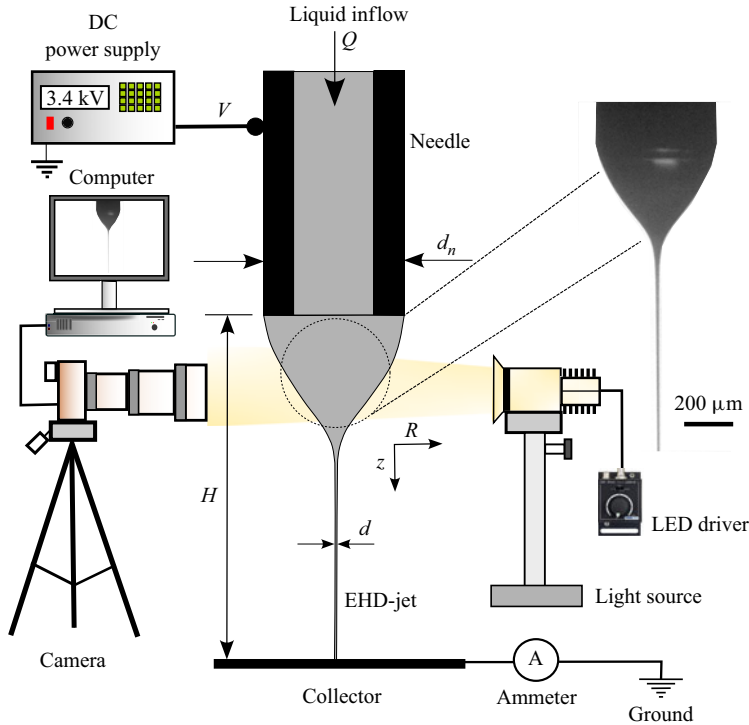


Figure 2. Schematic representation of the experimental set-up employed for steady cone-jetting of viscous leaky dielectric liquids. A stainless steel needle with an outer diameter d_n and a blunt end is fixed at a distance H from the collector. The highly viscous liquid is fed to the stainless steel needle at a flow rate of Q , and a potential difference of V is maintained between the needle tip and the collector. Due to the application of a strong electric field $\mathcal{O}(10^6 \text{ V m}^{-1})$, the meniscus of the liquid gets stretched and a fine jet issues from the cone tip. Simultaneously, the jet is visualised through a horizontally mounted microscope, and images of the jet are captured using a CMOS camera for further analysis. An ammeter, connected between the collector and the ground, measures the total current carried by the jet. The snapshot shows a steady cone-jet of pure glycerol issuing out from a stainless steel needle with an outer diameter of $d_n = 500 \mu\text{m}$ at a flow rate of $Q = 10 \mu\text{l min}^{-1}$, while a potential difference of $V = 3.4 \text{ kV}$ is maintained across a gap of $H = 5 \text{ mm}$ between the needle and the collector.

images of a segment of the jet were captured at a distance of $0.5H$ from the needle tip using a separate CMOS camera (Basler acA640–750 μm) equipped with a zoom lens ($0.7 - 4.5\times$ zoom). The raw images were processed to remove the background in MATLAB.

2.2. Materials and experimental procedure

To quantify the dependence of the minimum flow rate on the needle diameter, we employed five stainless steel needles, each having a flat end and a length of 25 mm, with outer diameters of $d_n = 300, 400, 500, 700$ and $1000 \mu\text{m}$. To observe the effect of the stand-off distance on the minimum flow rate of each liquid, we varied H between 3 and 18 mm. For minimum flow rate experiments, we used pure glycerol (Sigma Aldrich) and glycerol doped with varying concentrations of sodium chloride (CDH, India) to achieve a wide range of electrical conductivity spanning four orders of magnitude. Note that the liquids used in the experiments wet the stainless steel surface and, hence, the meniscus is pinned at the outer edge of the needle as shown in figure 2. The electrical conductivities of the doped glycerol solutions were measured using a conductivity meter (Jenway 4510,

Liquids	ρ [kg m ⁻³]	γ [mN m ⁻¹]	κ [μS m ⁻¹]	Q_0 [× 10 ⁻² nl s ⁻¹]	d_0 [× 10 ⁻² μm]	I_0 [nA]	Re_0 [× 10 ⁻³]	βRe_0
Solution I	1260	63.0	4.8	9281.4	555.3	5.3	15.0	0.60
Solution II	1260	63.0	55.0	810.0	109.2	5.3	6.7	0.27
Solution III	1260	63.0	430.0	103.6	27.2	5.3	3.4	0.14
Solution IV	1260	63.0	3430.0	12.9	6.9	5.3	1.7	0.07

Table 1. Values of density (ρ), interfacial tension (γ) and electrical conductivity (κ) of the working liquids measured at 20 °C. Also tabulated are the reference scales for flow rate (Q_0), jet diameter (d_0) and the total current carried by jet (I_0), along with the corresponding values of the electrohydrodynamic Reynolds number (Re_0) and βRe_0 . The Ohnesorge number, defined as $Oh = \mu / \sqrt{\rho \gamma d_n}$ for solutions I–IV are in the range of 5.0 – 9.1 corresponding to the needle diameters employed in the present study, which vary between 300 and 1000 μm.

UK) immediately after the measurement of Q_{min} . The densities ρ of the liquids were measured by weighing a known volume (10 ml) of each solution using a precision weighing balance (A&D Weighing, FZ–300 GD, precision of 1 mg). The interfacial tension γ of the liquid–air interface was measured using the pendant drop method. Table 1 provides the measured properties of the liquids and reference scales for flow rate, jet diameter and total current denoted as $Q_0 = \gamma \varepsilon_0 / (\rho \kappa)$, $d_0 = [\gamma \varepsilon_0^2 / (\rho \kappa^2)]^{1/3}$ and $I_0 = \sqrt{\gamma^2 \varepsilon_0 / \rho}$ (Gañán-Calvo *et al.* 1994), respectively, and the corresponding values of $Re_0 = \rho Q_0 / (\mu d_0)$; a detailed discussion on these reference scales is provided in § 3. The values of other relevant physical properties, such as dynamic viscosity (μ) and dielectric constant (β), were considered to be the same as those of pure glycerol and were obtained from the literature (Scheideler & Chen 2014) as $\mu = 1.4$ Pa s and $\beta = 41$, respectively. For each solution, a fresh 5 ml syringe, a tubing and five needles were used to avoid contamination. Before commencing the experiment, the entire system was thoroughly flushed with the working solution to remove any trapped air bubbles.

To determine the minimum flow rate for a specific solution, we performed experiments for a large number of (V , Q) combinations. We chose flow rates specifically around the Q_0 value, such that steady cone-jetting of the liquid was possible. Out of these large (V , Q) data sets, the smallest flow rate at which steady jetting could be observed was considered the minimum flow rate and the corresponding value of the applied potential was considered the threshold potential. Each experiment was repeated five times to ascertain the repeatability of the experiments. The minimum flow rate values reported here are the mean values for the corresponding operating conditions. Similarly, the reported values of the total current carried by the jet and the jet diameter at the minimum flow rate are also the mean values for the corresponding operating conditions. The random uncertainties in the measurement of the minimum flow rate and the total current carried by the steady jet were estimated using t -distribution (Bahga 2021). The total uncertainties in the measurement of the minimum flow rate and the total current were dominated by random uncertainty, which were less than 5 % at a 95 % confidence level. However, the total uncertainty in the measurement of the jet diameter was the result of the random uncertainty, estimated using t -distribution and systematic uncertainty, influenced by the effective pixel size of the camera sensor (1 μm). The total uncertainty in diameter measurements, obtained by combining systematic and random uncertainties, was less than 5 % of the mean jet diameter at a 95 % confidence level. Note that all experiments were performed at an ambient temperature range of 20 °C–22 °C and a relative humidity range of 30 %–40 %.

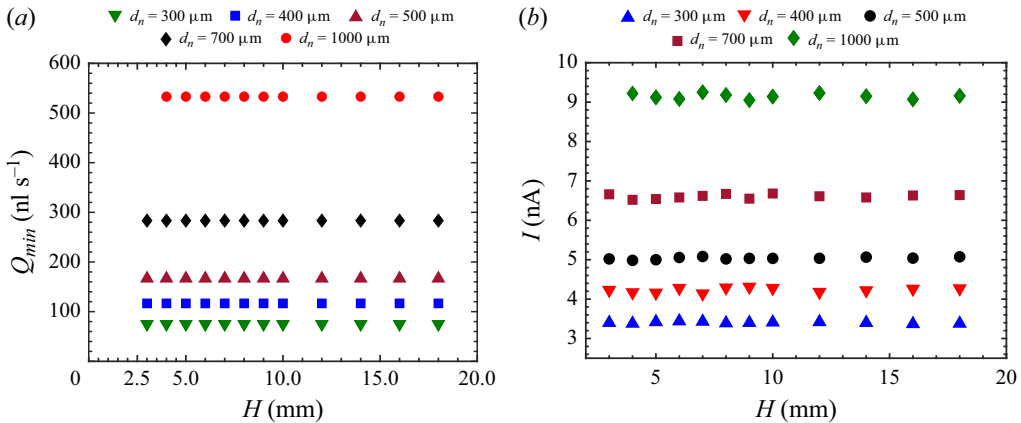


Figure 3. Variation of (a) the minimum flow rate (Q_{min}) necessary for steady cone-jetting of solution I and (b) the total current carried by the jet at Q_{min} with the stand-off distance (H). We infer that both minimum flow rate Q_{min} and the total current carried by the jet I at Q_{min} are independent of stand-off distance H .

2.3. Experimental results and discussion

2.3.1. Measurements of minimum flow rate of viscous liquids

In this section, we present the experimental data for pure glycerol (Solution I) and salt-doped glycerol solutions (Solutions II–IV) to elucidate the effect of stand-off distance, needle diameter and electrical conductivity of a viscous liquid on the minimum flow rate essential for steady cone-jetting.

To elucidate the effect of stand-off distance on the minimum flow rate, we performed experiments with solution I, varying H from 3 to 18 mm for a given needle. We then repeated the procedure for all the needles and measured the corresponding values of the minimum flow rate. Figure 3(a) shows the variation of the minimum flow rate with the stand-off distance for all the needles. From figure 3(a), we note that Q_{min} is independent of the stand-off distance H for a given needle diameter. However, the threshold potential corresponding to the minimum flow rate, that is, the minimum potential difference required for the onset of steady EHD-jetting, increases with increasing stand-off distance for all needle diameters. This trend suggests that, for a given needle diameter, the threshold electric field acting on the working liquid, near the needle tip, remains nearly the same for the range of H considered in the present study. This observation is consistent with our earlier observation of Q_{min} being independent of H . The exact dependence of the threshold potential on the geometrical parameters is discussed in detail in § 2.3.3.

However, Q_{min} monotonically increases with the needle diameter for all stand-off distances. For each case, we also measured the total current I carried by the jet at the minimum flow rate condition. Figure 3(b) shows the variation of I with H for solution I for all five needles. From figure 3(b), we note that the total current I , similar to Q_{min} , is also independent of H for a given needle diameter and increases with increasing needle diameter. Therefore, we can conclude that stand-off distance H does not affect the minimum flow rate necessary for steady cone-jetting of a viscous liquid.

Next, to investigate the effect of the needle diameter and the electrical conductivity of the viscous liquids on the minimum flow rate essential for steady cone-jetting, we kept the stand-off distance fixed at $H = 5$ mm. Figure 4(a–d) show images of the cone-jets corresponding to solutions I–IV for $d_n = 400 \mu\text{m}$ with minimum flow rate values of 116.7, 41.7, 25.0 and 11.7 nl s⁻¹, respectively. From figure 4(a–d), we infer that the minimum flow rate and the corresponding jet diameter decrease with an increase in the

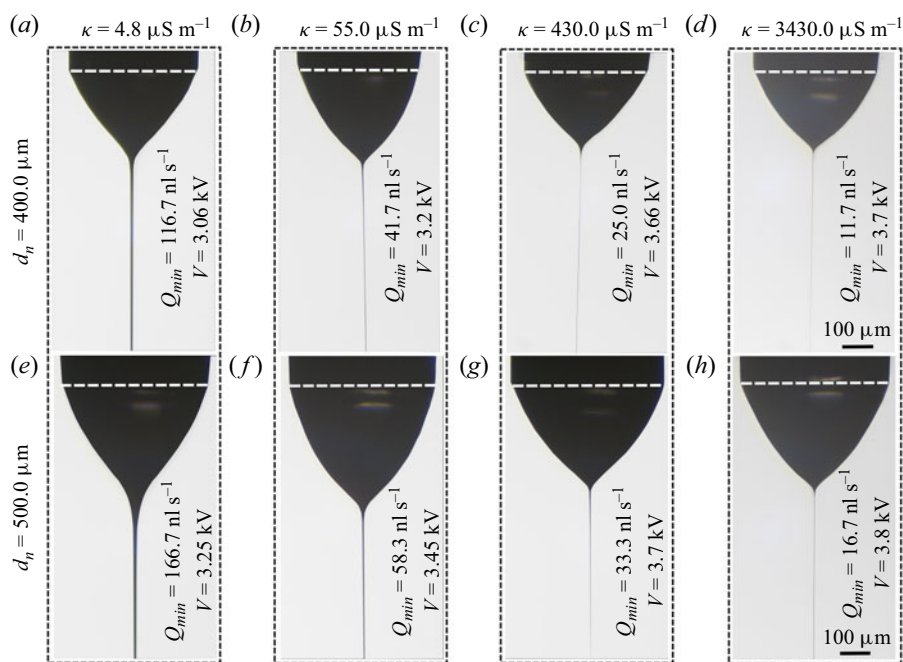


Figure 4. Snapshots of steady Taylor cone-jets of solutions I–IV at the minimum flow rate condition. Steady state cone-jet images of solutions I–IV when flown through needles with outer diameters of $d_n = (a-d)$ 400 and $(e-h)$ 500 μm . The minimum flow rates of steady Taylor cone-jets for panels $(a-d)$ and $(e-h)$ are 116.7, 41.7, 25.0 and 11.7 nl s^{-1} , and 166.7, 58.3, 33.3 and 16.7 nl s^{-1} , respectively. From these observations, we infer that the minimum flow rate of Taylor cone-jets of highly viscous leaky dielectric liquid decreases with the increase in the electrical conductivity of the liquids for a particular needle and increases with the increase in the needle diameter for a given solution. Moreover, the jet diameter, similar to the minimum flow rate, decreases with the increases in the electrical conductivity of the liquid for a given needle diameter, and increases with the increase in the needle diameter for a given solution. For $(a-d)$, the potential differences maintained between the needle tip and the collector necessary for steady cone-jetting are 3.3, 3.5, 3.7 and 3.8 kV, and 3.4, 3.7, 3.7 and 3.8 kV for panels $(e-h)$, respectively. These potential differences, when changed slightly, resulted in pulsating jets in the respective experiments.

electrical conductivity of the liquid for a fixed needle diameter. Figure 4(e–h) show the images of cone-jets corresponding to solutions I–IV for $d_n = 500 \mu\text{m}$ with minimum flow rate values of 166.7, 58.3, 33.3 and 16.7 nl s^{-1} , respectively. Similar to steady jetting for a needle with $d_n = 400 \mu\text{m}$, the minimum flow rate and the corresponding jet diameter decrease with an increase in the liquid conductivity for the needle with $d_n = 500 \mu\text{m}$.

Next, to understand the effect of the needle diameter on the minimum flow rate of a viscous liquid, we refer to figure 3(a). Through figure 3(a), we have already shown that the minimum flow rate of solution I increases with the increase in the needle diameter for a given stand-off distance. To verify whether this trend holds for the other liquids, we compare the minimum flow rate values of steady cone-jetting across all the needle diameters while keeping the electrical conductivity fixed. We repeat this process for all four liquids. The comparison of figure 4(a–d) with figure 4(e–h) suggests that the minimum flow rate and the corresponding jet diameter increase with increasing needle diameter. Therefore, the increase in the minimum flow rate upon increasing d_n is consistent for all four liquids, which have electrical conductivities spanning over four orders of magnitude.

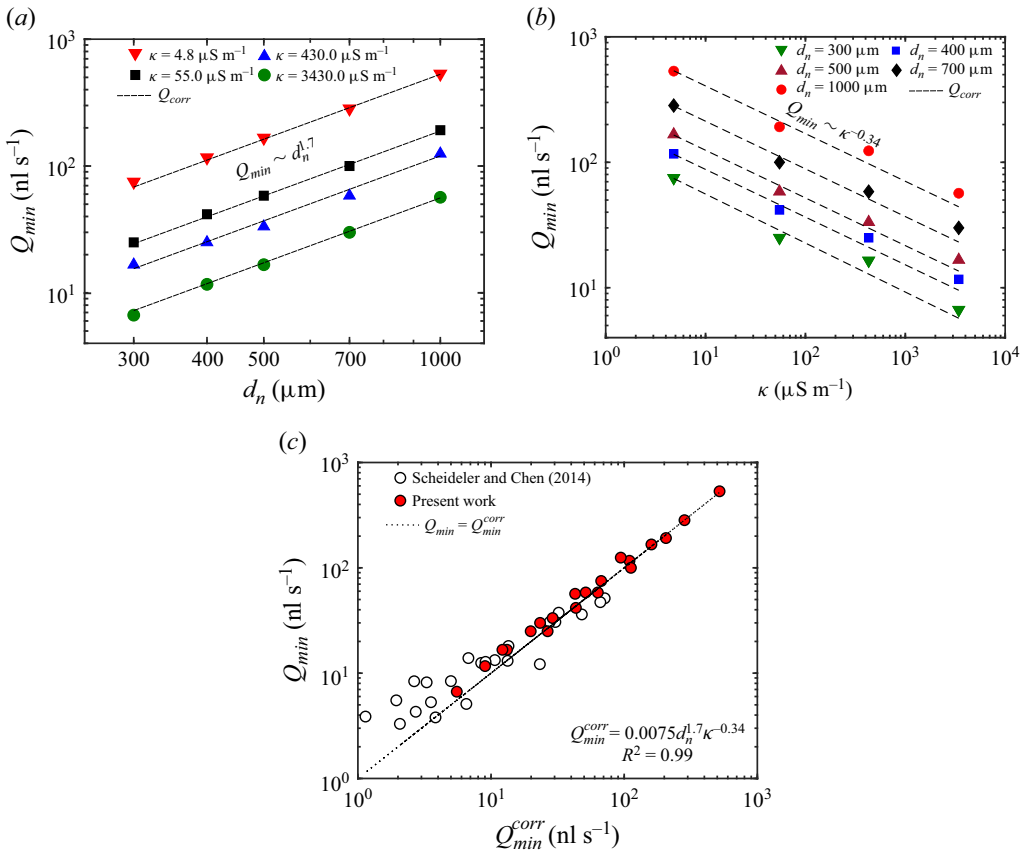


Figure 5. Variation of the minimum flow rate (Q_{min}) in steady cone-jetting of highly viscous leaky dielectric liquids with the outer diameter of the needles (d_n) and electrical conductivity of the liquids (κ). (a) Variation of Q_{min} with d_n on a log-log plot for solutions I–IV. The experimental data of Q_{min} for each of the four liquids closely follow the scaling law $Q_{min} \sim d_n^{1.7}$. (b) Variation of Q_{min} with κ of the four liquids used in the cone-jetting experiment for all the five needle diameters in a log-log plot. In this case, the same experimental data of Q_{min} aligns well with the scaling law $Q_{min} \sim \kappa^{-0.34}$ for each needle diameter. (c) Minimum flow rate value predicted by the empirical correlation Q_{min}^{corr} (2.1), which is in excellent agreement with the experimental data of the present work and that of Scheideler & Chen (2014).

To further explore the dependence of the minimum flow rate on the needle diameter, in figure 5(a), we plot the measured values of the minimum flow rate of all four liquids against the needle diameter on a logarithmic scale. The experimental data suggest that the minimum flow rate increases linearly with the needle diameter on a log-log plot, indicating a power-law variation of Q_{min} on d_n with a positive exponent. Moreover, figure 5(a) also indicates that for a given needle diameter, the minimum flow rate decreases with increasing electrical conductivity of the liquid.

To obtain the scaling of the minimum flow rate on the electrical conductivity of a solution, we plot the same data of Q_{min} in figure 5(b) against the electrical conductivity of all four liquids on a logarithmic scale. We infer from the plot that the minimum flow rate decreases linearly with the conductivity of the liquid on a log-log plot, suggesting a power-law variation of Q_{min} on κ but with a negative exponent.

Now that the dependence of the minimum flow rate on the needle diameter and the electrical conductivity of the liquid has been established, we perform a multi-variable

regression analysis (Bahga 2021) with the experimental data of the present work to obtain the respective exponents, and subsequently the dimensional correlation for the minimum flow rate in terms of d_n and κ , given as

$$Q_{min}^{corr} = 0.0075 d_n^{1.7} \kappa^{-0.34}. \quad (2.1)$$

Note that Q_{min}^{corr} , d_n and κ in (2.1) are in nl s^{-1} , μm and $\mu\text{S m}^{-1}$, respectively. The correlation, given by (2.1), suggests that the minimum flow rate scales as $Q_{min} \sim d_n^{1.7}$ across liquids with varying electrical conductivities. This observation is consistent with the measurements of Scheideler & Chen (2014), who reported a scaling of the form $Q_{min} \sim d_n^{1.45 \pm 0.18}$ for salt-doped glycerol solutions. Moreover, the empirical correlation also suggests that $Q_{min} \sim \kappa^{-0.34}$ for all the needle diameters. We note that due to the weak dependence of the minimum flow rate on the electrical conductivity of the liquids, specifically $Q_{min} \sim \kappa^{-0.34}$, Scheideler & Chen (2014), who performed cone-jetting experiments for a narrow range of electrical conductivity, incorrectly concluded that the minimum flow rate is independent of the electrical conductivity of the viscous liquid.

Next, we plot the minimum flow rate values predicted by the empirical correlation in the same plot where the experimental data are already overlayed, as shown in figures 5(a) and 5(b). In these figures, the symbols represent the experimental data, whereas the dashed lines represent the predicted values of the minimum flow rate using the empirical correlation (2.1). To further establish the efficacy of the empirical correlation, in figure 5(c), we plot the minimum flow rate values obtained from the empirical correlation against the experimental data of both the present work and Scheideler & Chen (2014) on a logarithmic scale. From figure 5(c), we infer that the experimental data collapse on the dashed line, indicating a one-to-one correspondence.

Building on the present work, in § 2.3.2, we present the empirical scaling for the jet diameter at the minimum flow rate condition with d_n and κ . Subsequently, in § 2.3.3, we also present the scaling for the onset potential essential for steady cone-jetting at the minimum flow rate condition.

2.3.2. Jet diameter at the minimum flow rate

The jet diameter of a steady cone-jet is an important quantity for applications such as electrospraying and EHD-jet printing because it influences the size of the monodispersed droplets or the resolution of the deposited patterns. In the present study, we observe that cone-jets attain an asymptotic jet diameter far downstream of the cone-tip, within the precision of our measurements. This behaviour is consistent with the ballistic regime described by Gañán-Calvo (2025), which is particularly favourable for steady cone-jets of highly viscous liquids such as those used in the present study. Hereafter, the jet diameter refers to the asymptotic diameter measured downstream of the cone-tip.

Prior work on electrospraying and steady cone-jetting has shown that the jet diameter is positively correlated with the liquid flow rate. Moreover, from figure 4, we note that the jet diameter depends on the needle diameter and the electrical conductivity of the viscous liquid. In particular, the jet diameter decreases with increasing electrical conductivity and increases with the needle diameter for a given liquid. It is important to note that due to the high electrical conductivities of solutions III and IV, the corresponding jet diameters were below the resolution of the imaging system used in the present study. Consequently, the jet diameters for these two liquids could not be measured through imaging.

To explore the dependence of the jet diameter on the needle diameter, in figure 6(a), we plot the measured values of the jet diameter for solutions I and II against the outer diameter of the five needles employed in our experiments. The experimental data suggest that the

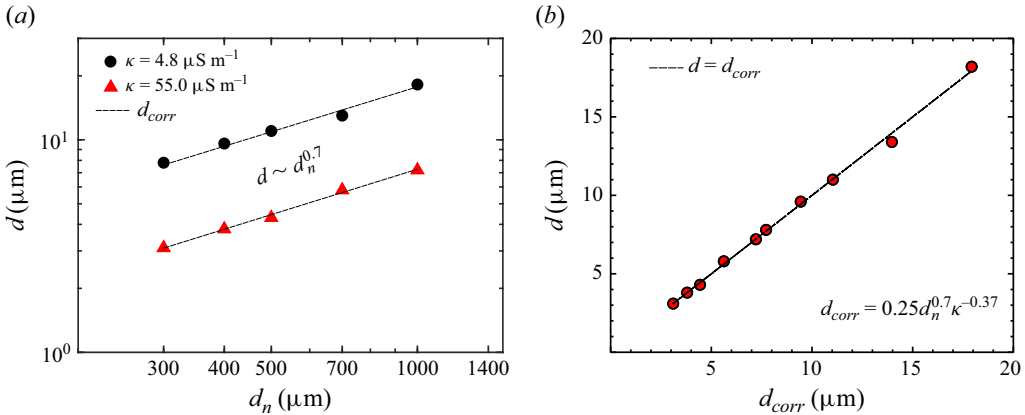


Figure 6. (a) Variation of the jet diameter d at the minimum flow rate condition with the outer diameter of the needles (d_n) for solutions I and II. (b) Jet diameter predicted by the proposed correlation d_{corr} (2.2), which aligns well with the experimental observation.

jet diameter at the minimum flow rate increases monotonically with the needle diameter. From figure 6(a), we also note that the jet diameter at Q_{min} decreases with increasing electrical conductivity of the liquid. Therefore, the jet diameter exhibits a qualitatively similar dependence on the needle diameter and the electrical conductivity, as observed for the minimum flow rate. As a result, the jet diameter, similar to the minimum flow rate, can be represented as $d = cd_n^a\kappa^b$, where a , b and c are constants to be determined using multivariable regression analysis (Bahga 2021). The resulting empirical correlation for the jet diameter is given by

$$d_{corr} = 0.25d_n^{0.7}\kappa^{-0.37}. \quad (2.2)$$

Here, d_{corr} , d_n and κ are in μm and $\mu\text{S m}^{-1}$, respectively. Although the dimensional correlation (2.2) is based on the jet diameter data of two liquids (Solutions I and II) only, we will justify the applicability of the correlation in §§ 4.4 and 4.5 using theoretical scaling analysis.

Moreover, the experimentally measured jet diameter, when plotted against the predicted values in figure 6(b), collapses on the 45° line indicating a one-to-one correspondence. Therefore, from figures 6(a) and 6(b), we infer that the empirical correlation given by (2.2) provides an excellent prediction of the jet diameter.

2.3.3. Onset potential at the minimum flow rate

To ensure steady cone-jetting at the minimum flow rate condition, the applied potential has to be equal to the critical value known as the onset potential V_{onset} . At V_{onset} , flow rates at and above Q_{min} provide steady cone-jetting, whereas flow rates below Q_{min} result in a pulsating cone-jet (Bober & Chen 2011; Chen 2011). Thus, for a given solution and geometrical configuration, the minimum flow rate alone has significance only if reported along with V_{onset} (Bober & Chen 2011; Chen 2011). A good initial approximation of V_{onset} is therefore essential for locating Q_{min} of steady cone-jetting of a solution. In § 2.3.1, we present the experiments with solution I to show that Q_{min} is independent of H . For the same set of experiments, we also recorded the onset potential at which steady cone-jetting was obtained. Figure 7(a) shows the variation of onset potential V_{onset} with varying needle-to-collector distance H for all five needles that are employed in the present work for solution I. This figure shows that V_{onset} monotonically increases with increasing H

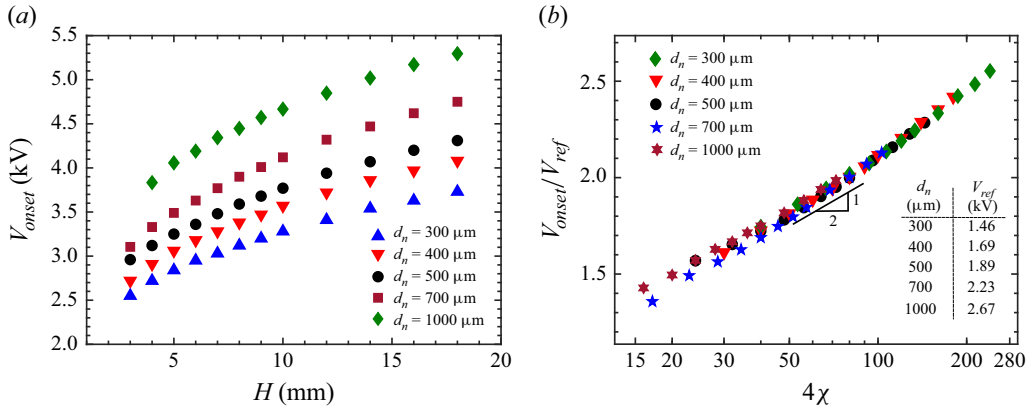


Figure 7. (a) Variation of the onset potential essential for steady cone-jetting at the minimum flow rate (Q_{min}) condition with the geometrical parameters of a cone-jetting set-up such as H and d_n for solution I. (b) Linear variation of dimensionless onset potential $\tilde{V}_{onset}$ with 4χ on a semi-log plot, which suggests a logarithmic dependence.

for a given value of d_n and also increases with the increasing d_n for a given H . To further elucidate the dependence of V_{onset} on H and d_n , we scale the data represented in figure 7(a) with the reference potential $V_{ref} = (\gamma d_n / \epsilon_0)^{1/2}$ (Zeleny 1935) and plot against the aspect ratio $\chi = (H/d_n)$ in a semi-log plot, as shown in figure 7(b). From figure 7(b), we observe that all the data points collapse to a single curve showing that the dimensionless onset potential $\tilde{V}_{onset} = (V_{onset}/V_{ref})$ varies logarithmically with the aspect ratio χ .

The logarithmic dependence of $\tilde{V}_{onset}$ on χ can be established by deriving the relationship for the electric field at the needle tip, $E = E(V_{onset}, d_n, H)$. From the perspective of electric field distribution, the needle tip and the protruding liquid meniscus jointly can be approximated as a point electrode facing a flat counter electrode. The corresponding electric field at the tip of a needle for a point-plane geometry can be expressed as (Loeb *et al.* 1941; Smith 1986),

$$E \sim \frac{V_{onset}}{d_n \ln(4\chi)}. \quad (2.3)$$

Next, the potential required to set in the cone-jet mode at the minimum flow rate condition can be approximated by substituting the electric field E in (2.3) with Taylor's electrostatic field, $E \sim (\gamma / (\epsilon_0 d_n))^{1/2}$ (Taylor 1964). Upon substitution, we get

$$\frac{V_{onset}}{d_n \ln(4\chi)} \sim \left(\frac{\gamma}{\epsilon_0 d_n} \right)^{1/2}, \quad (2.4)$$

which shows that

$$\tilde{V}_{onset} \sim \ln(4\chi). \quad (2.5)$$

Hence, for a given liquid, $\tilde{V}_{onset}$ varies logarithmically with χ and for a given geometrical configuration, V_{onset} depends only on the interfacial tension of the liquid. From figure 7(b), we also note that the slope of the line that shows a logarithmic dependence of $\tilde{V}_{onset}$ on χ has a value of 0.5, which is in agreement with the findings of the previous studies (Loeb *et al.* 1941; Smith 1986; Gañán-Calvo *et al.* 1994; Pantano *et al.* 1994). Therefore, we can

rewrite (2.5) as

$$\tilde{V}_{onset} = 0.5 \ln(4\chi). \quad (2.6)$$

Thus, (2.6) provides an estimate of the onset potential required to establish steady cone-jetting of highly viscous liquids for a given geometrical configuration at the minimum flow rate condition, and will be useful in applications such as EHD-jet printing, electrospinning and electrospraying while operating at or near Q_{min} .

3. Dimensional analysis and empirical correlation

We here present a dimensional analysis to describe the relevant dimensionless groups that influence the minimum flow rate necessary for steady cone-jetting of a highly viscous liquid. Thereafter, we reformulate the empirical correlations for the minimum flow rate and the jet diameter in terms of the identified dimensionless groups. Although, the dimensional empirical correlations for Q_{min} and d in terms of the needle diameter (d_n) and the electrical conductivity (κ) of the liquid have been proposed in §§ 2.3.1 and 2.3.2, their validity is limited to salt-doped glycerol solutions only. Therefore, empirical correlations in the dimensionless form are necessary to extend their applicability to steady cone-jetting of any Newtonian highly viscous liquid.

The dimensional quantities that essentially govern the steady cone-jetting can be grouped into the following three categories.

- (i) The operating parameters, such as the flow rate of the viscous liquid Q and the potential difference maintained between the needle tip and the collector V .
- (ii) The geometrical parameters, such as the outer diameter of the needle d_n , and the distance between the needle tip and the collector H .
- (iii) The physical properties of the liquid, such as density ρ , dynamic viscosity μ , electrical conductivity κ , interfacial tension γ and electrical permittivity of the liquid ε_i , and the properties of the surrounding medium (Gañán-Calvo 2004; Singh *et al.* 2022).

Typically, the surrounding medium is quiescent air, which allows one to exclude its dynamical effects on the system, and the only important parameter characterising its influence is the electrical permittivity $\varepsilon_0 = 8.854 \times 10^{-12} \text{ Fm}^{-1}$. However, the applied potential V is excluded from the dimensional analysis because, for a given solution, needle diameter and stand-off distance, the minimum flow rate of steady cone-jetting can be realised for a unique value of the applied potential, V_{onset} . Therefore, the applied voltage is not considered as an independent variable and is excluded from the list of parameters that influence Q_{min} (Bober & Chen 2011).

In the steady cone-jet mode, the conduction current dominates in the cone region near the tip of the needle, where the flow speed and the surface charge density on the liquid–air interface are low. However, the surface convection current dominates in the jet segment, where the conduction resistance is too high due to the small values of the jet diameter (Higuera 2003). In the intermediate region, which connects the cone and the fine jet, the conduction and surface convection currents are of the same order of magnitude (Gañán-Calvo 1997b, 1999; Gañán-Calvo *et al.* 2013). Gañán-Calvo (1997b) showed that in the steady cone-jet mode, the jet emission for leaky dielectric liquids relies on the balance of forces in this intermediate region. This intermediate region is termed as the current transfer region and the characteristic size of this region is much smaller than that of the cone-jetting set-up (Gañán-Calvo *et al.* 2013). Accordingly, Gañán-Calvo *et al.* (2013)

excluded geometrical parameters such as stand-off distance H and needle diameter d_n from the list of parameters affecting Q_{min} . We have also shown through figure 3(a) that the minimum flow rate of steady cone-jetting of a viscous liquid is independent of the stand-off distance H . Therefore, we do not consider H to influence the minimum flow rate. However, our experimental observation indicates that the minimum flow rate of a viscous liquid in steady cone-jetting does depend on the needle diameter d_n . In particular, we observed a dependence of the form $Q_{min} \sim d_n^{1.7}$, as shown in figure 5(a). Thus, the minimum flow rate Q_{min} ejected in the steady cone-jet mode can be represented as a function of seven dimensional parameters as

$$Q_{min} = F(\rho, \gamma, \mu, \kappa, d_n, \varepsilon_i, \varepsilon_0). \quad (3.1)$$

Using the Buckingham Π theorem (Evans 1972), we can group the dimensional quantities into four independent dimensionless groups as

$$\tilde{Q}_{min} = \frac{Q_{min}}{Q_0}, \quad \tilde{d}_n = \frac{d_n}{d_0}, \quad Re_0 = \frac{\rho Q_0}{\mu d_0} = \left(\frac{\gamma^2 \varepsilon_0 \rho}{\mu^3 \kappa} \right)^{1/3} \quad \text{and} \quad \beta = \frac{\varepsilon_i}{\varepsilon_0}. \quad (3.2)$$

Here, Q_0 and d_0 are defined by

$$Q_0 = \frac{\gamma \varepsilon_0}{\rho \kappa} \quad \text{and} \quad d_0 = \left(\frac{\gamma \varepsilon_0^2}{\rho \kappa^2} \right)^{1/3}. \quad (3.3a,b)$$

The reference scales for the flow rate Q_0 and the jet diameter d_0 in the current transfer region, defined by (3.3a) and (3.3b), respectively, can be obtained by order-of-magnitude analysis, as shown by Higuera (2003). Briefly, the balance between the inertia and the surface tension forces in the inertia-dominated jetting results in the departure of the surface from the steady Taylor cone. Mathematically, an order-of-magnitude balance between the two forces yields $\rho Q_0^2/d_0^4 = \gamma/d_0$, which, on further simplification, leads to the expression $d_0 = (\rho Q_0^2/\gamma)^{1/3}$. This expression is the inertial scale for the jet diameter, provided by Fernandez De La Mora *et al.* (1990). Similarly, the balance between the surface tension force (γ/d_0) and the force due to normal electric stress ($\varepsilon_0 E_0^2$) gives the scale of the electric field as $E_0 = \sqrt{\gamma/(\varepsilon_0 d_0)}$. Subsequently, the scales for the electric potential and the charge density are obtained as $V_0 = E_0 d_0 = \sqrt{\gamma d_0/\varepsilon_0}$ and $\sigma_0 = \varepsilon_0 E_0$. Next, the balance between the bulk conduction ($I_b = \kappa d_0^2 E_0$) and surface convection current ($I_s = \sigma_0 Q_0 d_0/d_0^2 = \varepsilon_0 E_0 Q_0/d_0$) at the current transfer region yields $d_0 = (\varepsilon_0 Q_0/\kappa)^{1/3}$. Finally, we equate the two expressions of d_0 , $d_0 = (\rho Q_0^2/\gamma)^{1/3}$ and $d_0 = (\varepsilon_0 Q_0/\kappa)^{1/3}$, to get the reference scales for flow rate (Q_0) and jet diameter (d_0) in terms of the physical properties of the liquids, as given by (3.3a) and (3.3b), respectively. For liquids with low to moderate electrical permittivity (β), the dependence of minimum flow rate on electrical permittivity can be neglected (Scheideler & Chen 2014). Therefore, we can express the relationship between the dimensionless groups in (3.2) as

$$\tilde{Q}_{min} = G(\tilde{d}_n, Re_0). \quad (3.4)$$

The functional dependence of $\tilde{Q}_{min}$ on $\tilde{d}_n$ and Re_0 described in (3.4) can be obtained empirically or by mathematical modelling. To obtain a dimensionless empirical correlation for the minimum flow rate, we use the dimensional empirical correlation (2.1) and subsequently substitute $d_n = \tilde{d}_n d_0$ and $\kappa = \gamma^2 \varepsilon_0 \rho / (Re_0^3 \mu^3)$ from (3.2). The substitution

yields

$$\begin{aligned}\tilde{Q}_{min} Q_0 &= 0.0075 (\tilde{d}_n d_0)^{1.7} \left(\frac{\gamma^2 \varepsilon_0 \rho}{Re_0^3 \mu^3} \right)^{-0.34}, \\ \tilde{Q}_{min} &= 0.3 \tilde{d}_n^{1.7} Re_0^{1.4}.\end{aligned}\quad (3.5)$$

Similar to $\tilde{Q}_{min}$, the dimensionless jet diameter $\tilde{d} = d/d_0$ at the minimum flow rate condition can also be expressed as a function of $\tilde{d}_n$ and Re_0 . We write

$$\tilde{d} = G'(\tilde{d}_n, Re_0). \quad (3.6)$$

Next, to get a dimensionless empirical correlation for the jet diameter at the minimum flow rate condition, we substitute $d_n = \tilde{d}_n d_0$ and $\kappa = \gamma^2 \varepsilon_0 \rho / (Re_0^3 \mu^3)$ obtained from (3.2) into the dimensional empirical correlation for the jet diameter, given by (2.2). The substitution yields

$$\begin{aligned}\tilde{d} d_0 &= 0.25 (\tilde{d}_n d_0)^{0.7} \left(\frac{\gamma^2 \varepsilon_0 \rho}{Re_0^3 \mu^3} \right)^{-0.37}, \\ \tilde{d} &= 0.6 \tilde{d}_n^{0.7} Re_0^{0.5}.\end{aligned}\quad (3.7)$$

For a steady cone-jet, the inverse of electrohydrodynamic Reynolds number, Re_0^{-1} , signifies the relative importance of the viscous force with respect to the inertia and the interfacial tension forces, respectively, in the current transfer region. From table 1, we note that Re_0^{-1} calculated based on the reference scales of flow rate Q_0 and jet diameter d_0 indicates the dominance of viscous force over inertia and interfacial tension forces, respectively. Note that the inverse of the electrohydrodynamic Reynolds number denotes the electrohydrodynamic Ohnesorge number, defined as $Oh_0 = \mu / \sqrt{\rho \gamma d_0}$ (Nganguia *et al.* 2016). We also highlight that for the present study and that of Scheideler & Chen (2014), Q_{min} shows a dependence on d_n even when $\tilde{d}_n \gtrsim 80$. This is in contrast to the observation of Ponce-Torres *et al.* (2018) since the liquids considered in the present study and that of Scheideler & Chen (2014) are highly viscous in nature.

Therefore, the proposed dimensionless empirical correlations, given by (3.5) and (3.7), successfully capture the importance of the viscous force, needle diameter and the electrical conductivity of the liquid. These correlations can now be employed to provide accurate estimates of the minimum flow rate and the corresponding jet diameters for steady cone-jetting of highly viscous Newtonian liquids.

Having established the dependence of the dimensionless minimum flow rate $\tilde{Q}_{min}$ and the jet diameter $\tilde{d}$ on the dimensionless needle diameter $\tilde{d}_n$ and the electrohydrodynamic Reynolds number Re_0 of the liquid, in the following sections, we try to provide a theoretical basis for the scaling behaviours observed for $\tilde{Q}_{min}$ and $\tilde{d}$. We also note that the empirical correlations suggest that with increasing value of $\tilde{d}_n$, the minimum flow rate $\tilde{Q}_{min}$ and corresponding jet diameter $\tilde{d}$ increase. However, these empirical correlations are valid within a bounded regime defined by lower and upper limits. In the subsequent theoretical analysis, specifically in § 5, we explicitly identify these bounds within which the empirical correlation for the minimum flow rate holds and beyond which the dependence of minimum flow rate on the needle diameter no longer applies.

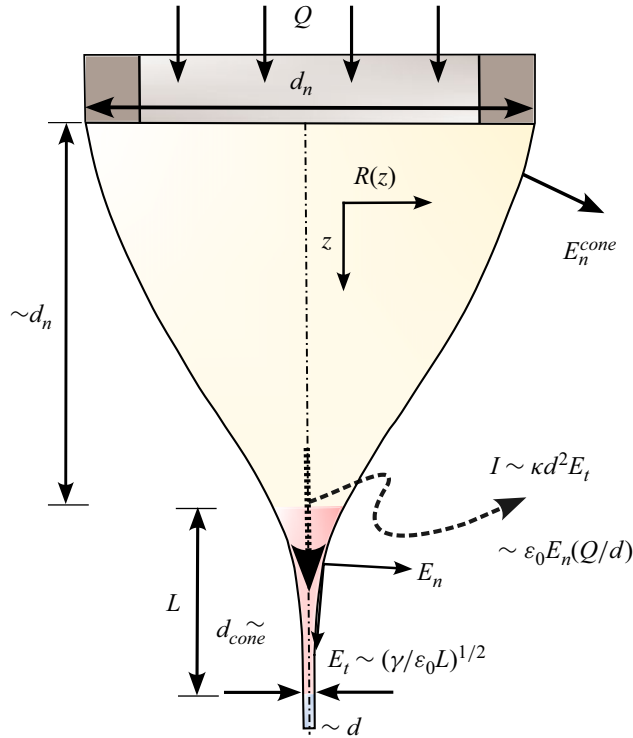


Figure 8. Schematic representation of the Taylor cone-jet of a highly viscous liquid at the minimum flow rate configuration. The steady cone-jet can be divided into three regions, viz. the cone region, the fully developed jet region and the intermediate current transfer region, which connects the cone and the jet.

4. Scaling analysis

4.1. Governing equations

In § 3, based on the experimental data and the dimensional analysis, we obtained the empirical correlations for the minimum flow rate and the corresponding jet diameter for Taylor cone-jetting of highly viscous liquids. However, to provide a physical reasoning underlying such a relationship, here, we derive similar scaling laws based on the governing equations describing steady cone-jets. To this end, we model a viscous liquid flowing through a metallic nozzle of outer diameter d_n with a flow rate of Q under the effect of the electric field E due to the applied electric potential difference, as illustrated in figure 8. For the highly viscous liquids used in the present work, the Debye layer thickness, λ_D , is significantly smaller than the jet diameter, with $\lambda_D \sim \mathcal{O}(0.1\text{--}1\text{ nm})$. In steady EHD-jets at the minimum flow rate condition, despite of a thin electric double layer, the effective charged region in the jet may still be large due to advection. Nevertheless, within the purview of the present study, we note that the dominant charged transport remains localised near the interface and the bulk may be treated effectively as electro-neutral. Accordingly, the leaky-dielectric model (Saville 1997) provides a consistent reduced description of the force balances at the cone-tip and in the current transfer region. We note that outside this regime, the leaky-dielectric model may not hold and a fully coupled electrokinetic model may be required for solving the electrohydrodynamic jetting problem as discussed by López-Herrera *et al.* (2023). Therefore, to describe the steady cone-jetting of the viscous liquid, we assume the viscous liquid to behave as a leaky-dielectric (Saville 1997).

For a leaky-dielectric liquid, at the minimum flow rate condition, the residence time in the current transfer region is greater than or, at best, comparable to the charge relaxation time (Gañán-Calvo *et al.* 2013; Ponce-Torres *et al.* 2018). That is, $\tau_e/\tau_h \lesssim 1$ (Gañán-Calvo *et al.* 2013; Ponce-Torres *et al.* 2018), where $\tau_e = \beta\epsilon_0/\kappa$ is the charge relaxation time scale and $\tau_h = Ld^2/Q$ is the convective time scale. Here, d and L denote the characteristic jet diameter and the length of the current transfer region, as illustrated in figure 8. The convective time scale can be considered as the residence time in the current transfer region. Thus, for a leaky-dielectric liquid, the condition for fast charge relaxation can be expressed as

$$\frac{\tau_e}{\tau_h} = \frac{\beta\epsilon_0 Q}{Ld^2\kappa} \lesssim 1. \quad (4.1)$$

For a leaky-dielectric liquid satisfying (4.1), all the charges in the bulk of the liquid are quickly conducted to the liquid surface. Therefore, the free charge is present only on the jet surface, and there are no free charges present in the bulk solution and the surrounding medium. We note that the electric field, E , inside and outside of the jet, satisfies (Saville 1997)

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = 0. \quad (4.2)$$

Because the electric field is irrotational, the tangential components of the electric field corresponding to the inner and the outer fluids at the surface of the jet are equal, hereafter denoted as E_t . Moreover, following Gauss's law, the normal components of the electric fields (E_n^{out} , E_n^{in}) at the jet surface can be related to the surface charge density σ as $\sigma = \epsilon_0 E_n^{out} - \epsilon_i E_n^{in}$.

For a leaky-dielectric liquid, we have $\epsilon_i E_n^{in} \ll \epsilon_0 E_n^{out}$ (Gañán-Calvo 1997b). Consequently, the expression for surface charge density reduces to $\sigma \approx \epsilon_0 E_n^{out}$. For simplification, hereafter, we denote E_n^{out} as E_n . Hence, the expression for surface charge density can be rewritten as $\sigma \approx \epsilon_0 E_n$. The interaction between the surface charge and the components of the electric field results in tangential (σE_t) and normal ($\epsilon_0[E_n^2 + (\beta - 1)E_t^2]$) electric stresses on the liquid surface. These electric stresses, along with the stresses due to interfacial tension force, viscous force and inertia, describe Taylor cone-jetting.

Considering rapid charge relaxation and radial momentum diffusion, the steady cone-jetting of a viscous leaky-dielectric liquid can be approximated as a quasi-one-dimensional (quasi-1-D) flow problem. The quasi-1-D governing equations (Feng 2002) for mass and momentum conservation can be written respectively as

$$Q = \pi R^2 v \quad (4.3)$$

and

$$\frac{d}{dz} \left(\frac{\rho v^2}{2} \right) = \rho g + \frac{3}{R^2} \frac{d}{dz} \left(\mu R^2 \frac{dv}{dz} \right) + \frac{2\sigma E_t}{R} + \frac{d}{dz} \left[\frac{\sigma^2}{2\epsilon_0} - \frac{\gamma}{R} + \frac{\epsilon_0 (\beta - 1) E_t^2}{2} \right]. \quad (4.4)$$

We neglect the body force due to gravity in (4.4), as the Bond number $Bo = \rho g R^2/\gamma \ll 1$. Next, we substitute $v = Q/\pi R^2$ from (4.3) and $\sigma \approx \epsilon_0 E_n$ in (4.4) to get the modified momentum equation as (Gañán-Calvo 2004),

$$\underbrace{\frac{d}{dz} \left(\frac{\gamma}{R} \right)}_{\text{I}} + \underbrace{\frac{d}{dz} \left(\frac{1}{2\pi^2} \frac{\rho Q^2}{R^4} \right)}_{\text{II}} + \underbrace{\frac{6\mu Q}{\pi R^2} \frac{d}{dz} \left(\frac{1}{R} \frac{dR}{dz} \right)}_{\text{III}} = \underbrace{\frac{2\epsilon_0 E_t E_n}{R}}_{\text{IV}} + \underbrace{\frac{d}{dz} \left[\frac{\epsilon_0 E_n^2}{2} \right]}_{\text{V}} + \underbrace{\frac{d}{dz} \left[\frac{\epsilon_0 (\beta - 1) E_t^2}{2} \right]}_{\text{VI}}. \quad (4.5)$$

Here, R , v , E_n and E_t denote the radius of the jet at an axial location z , the area-averaged axial velocity of the liquid, and the normal and tangential components of the local electric field acting on the liquid surface, respectively. The three terms on the left-hand side of (4.5), denoted as I, II and III, correspond to the axial components of the interfacial tension force, the liquid inertia and the viscous force per unit volume, respectively. Similarly, the three terms on the right-hand side of (4.5), denoted as IV, V and VI, represent the axial components of the tangential electrostatic surface force, and the electrostatic suction and polarization forces per unit volume, respectively. The terms on the left side of (4.5) restrict, while those on the right side of (4.5) drive the cone-to-jet transition (Gañán-Calvo 2004).

In the presence of an external electric field, charges within the liquid migrate quickly to the liquid's free surface, eventually modifying the local electric field. Historically, two contrasting models have been proposed to explain the dominant mechanism of charge transport. Melcher & Warren (1971) proposed a model based on dominant bulk conduction, whereas Turnbull (2002) emphasised surface convection as the primary contributor. These two different views led to the distinct formulations of the current balance. Recognising the limitations of both the simplified approaches, Gañán-Calvo (1994) was the first to present a formulation that incorporates both conduction and convection contributions, as represented in (4.6). This modified formulation was further used by Gañán Calvo (1997a) in the study of electrified jets. Therefore, the migration of charges to the free surface occurs through bulk conduction, followed by surface convection along the jet and the total current I carried by the electrified jet consists of bulk conduction (I_{cond}) and the surface convection (I_{conv}) currents (Gañán-Calvo 1994; Gañán Calvo 1997a; Ponce-Torres *et al.* 2018; Singh *et al.* 2022). Although the magnitudes of the conduction and surface convection currents vary along the axial direction, the total current I , given by (4.6), remains constant throughout the length of the jet (Singh *et al.* 2023). Therefore, the equation for conservation of charge at any axial location along the length of a steady cone-jet gives the total current I , which can be expressed as

$$I = \pi R^2 \kappa E_t + 2\pi \sigma R v. \quad (4.6)$$

The first and second terms on the right-hand side of (4.6) represent the conduction (I_{cond}) and the convection (I_{conv}) currents, respectively.

Now that we are equipped with the necessary governing equations that describe an electrified jet, we perform scaling analysis for the minimum flow rate required for steady cone-jetting of highly viscous leaky-dielectric liquid. At this point, it is pertinent to revisit the existing scaling laws for Q_{min} provided in the literature. This will help differentiate the jetting regimes for highly viscous liquids presented in this work from those documented in the literature. We note that in the subsequent discussions, specifically in §§ 4.2 and 4.4, the quasi-1-D model (4.5) has not been used for any quantitative estimation of the different force terms; rather, it is employed solely to identify the scaling of the different driving and resisting forces.

4.2. Existing scaling laws

In this section, we revisit the work of Gañán-Calvo *et al.* (2013) and Scheideler & Chen (2014) before deriving the scaling laws for highly viscous liquids.

4.2.1. Insights from the work of Gañán-Calvo *et al.*

Gañán-Calvo *et al.* (2013) assumed that the stability of the cone-jet mode at the minimum flow rate condition is primarily dependent on the local flow arising in the current transfer region. Furthermore, the characteristic length of the current transfer region is much smaller

than those characterising the cone-jetting set-up, i.e. $L \ll d_n$ and $L \ll H$. Because of the locality of the current transfer region, Gañán-Calvo *et al.* reasoned that the stability of the cone-jet mode at the minimum flow rate condition does not depend on the length scales associated with the cone-jetting set-up.

Next, they suggested that, in the current transfer region, the balance of the primary supporting and resistant forces is essential to achieve and sustain a steady jet. Using d and L as the reference scales for the radial and axial lengths of the current transfer region, respectively, the axial momentum (4.5) suggests the scaling for all the resisting and the supporting forces in the current transfer region as follows.

(i) Resisting forces: {I : $\gamma d^{-1} L^{-1}$, II : $\rho Q^2 d^{-4} L^{-1}$, III : $\mu Q d^{-2} L^{-2}$ }.

(ii) Supporting forces: {IV : $\varepsilon_0 E_n E_t d^{-1}$, V : $\varepsilon_0 E_n^2 L^{-1}$, VI : $\varepsilon_0 \beta E_t^2 L^{-1}$ }.

In particular, Gañán-Calvo *et al.* argued that in the current transfer region among the supporting forces, the polarisation force is, at best, comparable to the electrostatic suction force, whereas the tangential electric force is significantly smaller than the electrostatic suction force throughout the current transfer region. On the other hand, among the opposing forces, the viscous force is, at best, comparable to the inertia term and the interfacial tension force is negligibly small. Therefore, Ganan-Calvo *et al.* considered inertia and electrostatic suction forces to be the primary resisting and driving forces at the current transfer region. Subsequently, they derived the scaling relations for jet diameter and the length of the current transfer region as

$$d \sim d_0 \left(\frac{Q}{Q_0} \right)^{1/2} \quad \text{and} \quad L \sim d_0 \left(\frac{Q}{Q_0} \right), \quad (4.7a,b)$$

by considering the balance of these forces in the current transfer region.

Furthermore, Gañán-Calvo *et al.* (2013) argued that at the minimum flow rate condition, there could be two scenarios for which steady jet emission is not possible.

(i) The viscous forces may become comparable with the inertia and resist the jet emission before the polarisation force, referred to as the viscous limit. In the viscous limit, the minimum flow rate and the jet diameter at the minimum flow rate scale as

$$Q_{min} \sim Q_0 Re_0^{-1} \quad \text{and} \quad d \sim d_0 Re_0^{-1/2}, \quad (4.8a,b)$$

respectively.

(ii) The polarisation force may become negative beyond the cone-jet transition region and oppose the driving force first, termed as the polar limit. In the polar limit, the minimum flow rate and the jet diameter at the minimum flow rate scale as

$$Q_{min} \sim Q_0 \beta \quad \text{and} \quad d \sim d_0 \beta^{1/2}, \quad (4.9a,b)$$

respectively.

From (4.8a) and (4.9a), Ganan-Calvo *et al.* noted that the ratio of the minimum flow rate for the viscous limit case to the minimum flow rate for the polar limit case equals to βRe_0 . Therefore, they concluded that the term βRe_0 dictates which limit out of the viscous and polar limits will govern the minimum flow rate of steady cone-jetting. In particular, for $\beta Re_0 < 1$, the viscous limit governs Q_{min} and for $\beta Re_0 > 1$, the polar limit governs Q_{min} .

Solution	ρ [kg m ⁻³]	κ [mS m ⁻¹]	γ [mN m ⁻¹]	μ [mPa s]	β	Q_0 [$\times 10^{-2}$ nl s ⁻¹]	d_0 [nm]	Re_0 [$\times 10^{-3}$]	βRe_0	Oh
3-ethylene glycol	1134	5.04	45.4	36.6	23.77	7.03	49.80	43.75	1.04	0.51
3-ethylene glycol	1134	15.80	45.4	36.6	23.77	2.24	23.25	29.90	0.71	0.51
4-ethylene glycol	1137	7.38	44.8	45.6	20.5	4.72	38.42	30.67	0.63	0.64
4-ethylene glycol	1137	37.00	44.8	45.6	20.5	0.943	13.11	17.92	0.37	0.64
EMI-Im	1520	880	34.9	34.0	3.0	0.023	1.32	7.79	0.023	0.47

Table 2. Physical properties of the liquids used by Gañán-Calvo *et al.* (2013) that satisfy the $\beta Re_0 < 1$ criterion. Also tabulated are the reference scales for the flow rate Q_0 and jet diameter d_0 , along with the corresponding values of the electrohydrodynamic Reynolds number Re_0 , βRe_0 and Ohnesorge number (Oh). The Ohnesorge number ($Oh = \mu / \sqrt{\rho \gamma d_n}$) reported in the table is evaluated based on the diameter of the needle, $d_n = 100 \mu\text{m}$, employed in the study of Gañán-Calvo *et al.* (2013).

To validate the proposed scaling laws, in their study, Ganan-Calvo *et al.* considered liquids that satisfy both $\beta Re_0 \lesssim 1$ and $\beta Re_0 > 1$ conditions. However, our focus here is exclusively on liquids that meet the $\beta Re_0 < 1$ criterion, which are tabulated in table 2. Among all liquids considered by Ganan-Calvo *et al.*, 3-ethylene glycol, 4-ethylene glycol and 1-ethyl-3-methylimidazolium bis(trifluoromethylsulfonyl) imide (EMI-Im), which have electrical conductivities of $\mathcal{O}(10^3 - 10^5 \mu\text{S m}^{-1})$, satisfy the condition $\beta Re_0 < 1$, as shown in table 2. Subsequently, to validate their scaling laws for the minimum flow rate, Ganan-Calvo *et al.* plotted the experimental data for the flow rate on a two-dimensional (2-D) regime plot, with $Q_{min}/(\beta Q_0)$ along the ordinate and $(\beta Re_0)^{-1}$ on the abscissa, as shown in figure 9. In their case, for $\beta Re_0 < 1$, the experimental data for the minimum flow rate lay just above the $(Q_{min}/Q_0)Re_0 = 1$ line, validating the derived scaling law for the minimum flow rate, $Q_{min} \sim Q_0 Re_0^{-1}$. Moreover, as we are dealing with the minimum flow rate, in figure 9, we only highlighted the lowest value of the flow rate that has been reported in figure 2 of Gañán-Calvo *et al.* (2013).

An alternative explanation of the minimum flow rate $Q_{min} \sim Q_0 Re_0^{-1}$ in the viscous limit can be given by considering the ratio of τ_μ and τ_h . Here, τ_μ represents the viscous diffusion time scale across the length of the current transfer region. Using (4.7), we show that

$$\frac{\tau_\mu}{\tau_h} \sim \frac{\rho L^2 Q}{\mu L d^2} \sim \frac{Q_{min} Re_0}{Q_0} \gtrsim 1. \quad (4.10)$$

Therefore, $(Q_{min}/Q_0)Re_0 \gtrsim 1$ is equivalent to stating that at the minimum flow rate condition, the viscous diffusion time scale, $\tau_\mu \sim \rho L^2/\mu$, across the length of the current transfer region is greater than or comparable to the characteristic residence time scale, $\tau_h \sim L d^2/Q$.

Alternatively, the limit can be expressed in terms of the ratio of inertia force density (F_{int} , term II of (4.5)) to the viscous force density (F_{visc} , term III of (4.5)) in the current transfer region. Using the respective scales of the force densities and (4.7), we get

$$\left(\frac{F_{int}}{F_{visc}} \right) \sim \frac{\rho Q^2 d^{-4} L^{-1}}{\mu Q d^{-2} L^{-2}} \sim \frac{\rho Q L}{\mu d^2} \sim \frac{Q_{min} Re_0}{Q_0} \gtrsim 1. \quad (4.11)$$

Therefore, the assumption of Ganan-Calvo *et al.* that inertia is the dominant resisting force in the current transfer region is justified for the liquids listed in table 2.

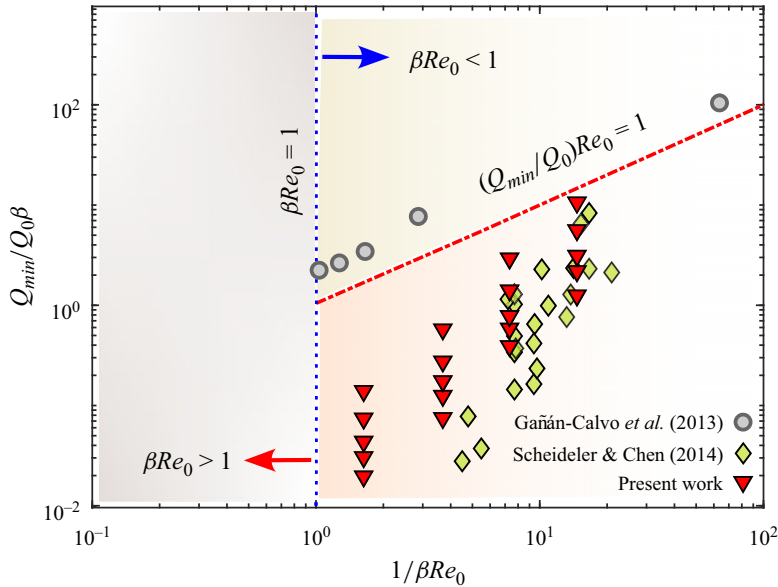


Figure 9. A two-dimensional regime plot for minimum flow rate of different liquids on a $Q_{min}/(Q_0\beta)$ versus $1/(\beta Re_0)$ plot. The regime plot is divided into two regions $\beta Re_0 < 1$ and $\beta Re_0 > 1$ by the vertical boundary line (in blue colour), $\beta Re_0 = 1$. The region $\beta Re_0 < 1$ is further divided by the slanted boundary line (in red colour), $(Q_{min}/Q_0)Re_0 = 1$, into two sub-regions $(Q_{min}/Q_0)Re_0 > 1$ and $(Q_{min}/Q_0)Re_0 < 1$. The minimum flow rate data from Scheideler & Chen (2014) and present work clearly fall within the unexplored $(Q_{min}/Q_0)Re_0 < 1$ region. The grey hollow circular symbols represent the minimum flow rate data gathered from literature (figure 2 of Gañán-Calvo *et al.* 2013), which fall in the $(Q_{min}/Q_0)Re_0 > 1$ region close to the boundary 2. The details of the plotted experimental data are provided in table 2, in table 4 and in table 6.

Similarly, if we compare the electrostatic suction force density (F_{es} , term V of (4.5)) with the polarisation force density (F_{pol} , term VI of (4.5)) in the current transfer region, we note that

$$\left(\frac{F_{pol}}{F_{es}}\right) \sim \frac{\varepsilon_0 \beta E_t^2 L}{L \varepsilon_0 E_n^2} \sim \frac{\beta E_t^2}{E_n^2} \sim \frac{\beta d^2}{L^2} \sim \frac{\beta Q_0}{Q}. \quad (4.12)$$

For the $\beta Re_0 < 1$ case, the minimum flow rate $Q_{min} \sim Q_0 Re_0^{-1}$ (see (4.8a)) and $(F_{pol}/F_{es}) \sim \beta Re_0 < 1$ at the minimum flow rate condition. Therefore, the assumption of Gañán-Calvo *et al.* that at the minimum flow rate condition, in the current transfer region, the electrostatic suction force is the dominant driving force is justified for the liquids listed in table 2.

We note that the criteria $(Q_{min}/Q_0)Re_0 \gtrsim 1$ and $\beta Re_0 < 1$ can also be expressed as

$$\alpha_\rho \gtrsim \alpha_\mu^{1/4} \quad \text{and} \quad \frac{\alpha_\rho}{(\beta - 1)} > 1, \quad (4.13)$$

respectively, where $\alpha_\rho = Q_{min}/Q_0$ and $\alpha_\mu = \alpha_\rho/Re_0^3$ (Gañán-Calvo 2004). These terms are widely used in the literature on Taylor cone-jets. We also note that (4.13) is the well-known criteria for the inertia electrostatic suction regime (IE regime) (Gañán-Calvo 2004). Thus, the liquids considered by Gañán-Calvo *et al.* (2013) which satisfy the $\beta Re_0 < 1$ criterion follow IE scaling.

4.2.2. Insights from the work of Scheideler and Chen

In their study on the minimum flow rate of steady cone-jetting, Scheideler & Chen (2014) considered highly viscous liquids such as salt-doped glycerol solutions with κ of $\mathcal{O}(10^2 - 10^4 \mu\text{S m}^{-1})$. They observed that $Q_{\min}$ decreases with increasing viscosity of the liquid and increases with the increasing needle diameter. In particular, $Q_{\min} \sim d_n^{1.45 \pm 0.18}$. Scheideler & Chen (2014) also concluded that $Q_{\min}$ is independent of the electrical conductivity of the viscous liquid. Based on their experimental observation, Scheideler & Chen (2014) argued that for highly viscous liquids, $Q_{\min}$ must depend on the dynamic viscosity of the liquid and the diameter of the needle used in the cone-jetting experiments. In particular, Scheideler & Chen (2014) referred to the scaling law, $Q_{\min} \sim \beta Q_0$, proposed by Fernández de La Mora (2007) and advocated for a modified scaling law that could better account for their experimental findings. To support their experimental observation and derive the suitable scaling law, Scheideler & Chen (2014) proposed that the viscous diffusion time scale based on the needle diameter, $\tau_\mu \sim \rho d_n^2 / \mu$, should become comparable to or smaller than the charge relaxation time scale $\tau_e \sim \beta \varepsilon_0 / \kappa$, which can be written as

$$\begin{aligned} \tau_\mu &\lesssim \tau_e, \\ \frac{\rho d_n^2}{\mu} &\lesssim \frac{\beta \varepsilon_0}{\kappa}. \end{aligned} \quad (4.14)$$

Subsequently, upon substituting $\beta \varepsilon_0 / \kappa$ from (4.14) into the minimum flow rate scaling, $Q_{\min} \sim \beta Q_0$, Scheideler & Chen (2014) obtained the new scaling for $Q_{\min}$ as

$$Q_{\min} \sim \frac{\beta \gamma \varepsilon_0}{\rho \kappa} \sim \frac{\gamma \rho d_n^2}{\mu \rho} \sim \frac{\gamma d_n^2}{\mu}. \quad (4.15)$$

We note that the scaling, given by (4.15), includes dynamic viscosity and the needle diameter. In particular, $Q_{\min} \sim \mu^{-1}$ and $Q_{\min} \sim d_n^2$. However, the assumption (4.14) made by Scheideler & Chen (2014) itself is not satisfied by most of the liquids considered in their study. As a result, the measured value of the minimum flow rate and that predicted by the scaling law (4.15) differ by several orders-of-magnitude (see table 6). In particular, the scaling obtained by Scheideler & Chen (2014), (4.15), is independent of κ . However, in § 2.3.1, through experiments, we have shown that $Q_{\min} \sim \kappa^{-0.34}$ for the salt-doped glycerol solutions with electrical conductivity spanning over four orders of magnitude, $\kappa \sim \mathcal{O}(10^0 - 10^3 \mu\text{S m}^{-1})$. This difference in observations is attributed to the relatively narrow range of electrical conductivity of the viscous liquids considered by Scheideler & Chen (2014) in their study.

To find out the regime in which the experimental data of the present work and that of Scheideler & Chen (2014) fall, we plot the experimental data in figure 9. As depicted in figure 9, we observe that all the experimental data of Scheideler & Chen (2014) and the present work satisfy the $\beta Re_0 < 1$ criterion, but lie much below the $(Q_{\min}/Q_0)Re_0 = 1$ line, suggesting a distinct jetting regime for highly viscous liquids.

4.3. Dominant forces

We note that the cone-jetting regime for $\beta Re_0 < 1$ and $(Q_{\min}/Q_0)Re_0 < 1$ has received little attention in the literature. We also note that the condition $(Q_{\min}/Q_0)Re_0 < 1$ can also be expressed as

$$\alpha_\rho < \alpha_\mu^{1/4}. \quad (4.16)$$

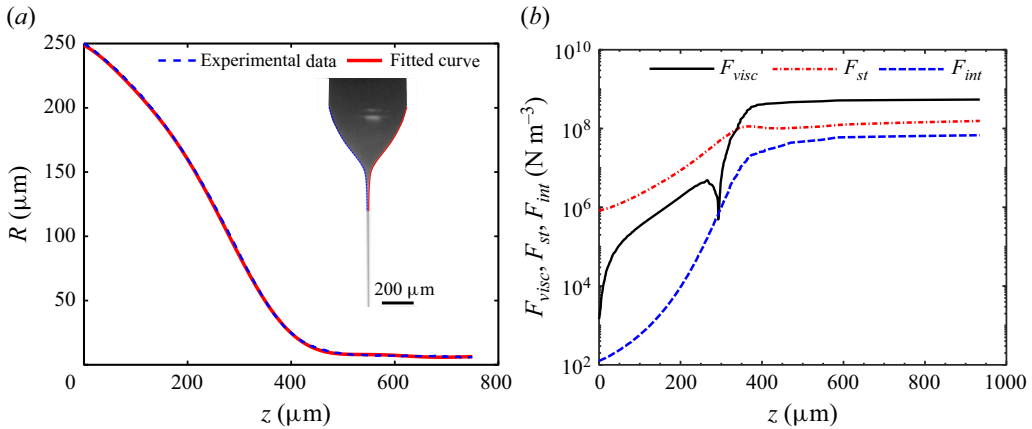


Figure 10. Prediction of various forces in the jet through experimentally measured jet radius. (a) Variation of jet radius R with the axial direction z for solution I, when flown through a needle with outer diameter $d_n = 500 \mu\text{m}$ at the minimum flow rate condition. The steady jetting could be sustained at the minimum flow rate of $Q_{min} = 10 \mu\text{l min}^{-1}$, while a potential difference of $V = 3.4 \text{ kV}$ was maintained across a gap of $H = 5 \text{ mm}$ between the needle and the collector plate. (b) Variation of resistive force densities, F_{st} , F_{int} and F_{visc} , with the axial direction z . We note that at the cone tip, the viscous force and the interfacial tension forces are of the same order of magnitude, and beyond the cone tip, the viscous force dominates.

This is one of the criteria for the viscous electrostatic suction (VE) scaling, defined by Gañán-Calvo (2004). Therefore, (4.16) suggests that the viscous force is the dominant resisting force in the current transfer region. To further support our assumption, we extract R versus z data from the snapshot of a steady cone-jet of solution I. The steady jet, as shown in the inset of figure 10(a), was sustained at the minimum flow rate of $Q_{min} = 10 \mu\text{l min}^{-1}$, while a potential difference of $V = 3.4 \text{ kV}$ was maintained across a gap of $H = 5 \text{ mm}$ between the needle and the collector. Since the experimental data are discrete and noisy, we fit the data set with a Fourier series, ensuring goodness of fit, to get a smooth and continuous curve for the jet radius, as shown in figure 10(a).

Subsequently, we use the data set for jet radius to evaluate and plot the axial components of the interfacial tension force (F_{st}), the liquid inertia (F_{int}) and the viscous force density (F_{visc}), corresponding to terms I–III of (4.5), respectively, against z . Figure 10(b) shows the estimated force densities based on measured $R(z)$. From figure 10(b), we infer that at the cone tip, F_{st} and F_{visc} are of the same orders of magnitude, and F_{int} is less than both F_{visc} and F_{st} at least by an order of magnitude. Moreover, in the current transfer region, F_{visc} is the dominant resisting force.

Next, we try to determine the dominant driving force in the current transfer region. The driving forces in (4.5) are the tangential electric force (F_{el}), electrostatic suction (F_{es}) and polarisation force (F_{pol}), corresponding to terms IV–VI of (4.5), respectively. Previous work from the literature (Gañán-Calvo 1999; Gañán-Calvo *et al.* 2013) suggests that in the current transfer region, the force due to electric shear stress is of the same order of magnitude as that due to the electrostatic suction. This is due to the difference in the areas of surfaces where these stresses act. The electric shear stress acts on the lateral surface of the steady cone-jet, whereas the electrostatic suction force acts along the cross-section of the jet (Ponce-Torres *et al.* 2018). However, the polarisation force per unit volume, $\beta E_t^2 L^{-1}$, can at most be of the same order of magnitude as the electrostatic suction force per unit volume, $\epsilon_0 E_n^2 L^{-1}$, as $E_t \ll E_n$ in the current transfer region. Therefore, we consider electrostatic suction to be the dominant driving force in the current transfer region.

In summary, for our cone-jetting experiments and that of Scheideler & Chen (2014), the current transfer region is formed by the balance of viscous and electrostatic suction forces, whereas at the cone tip, among the resisting forces, the viscous and interfacial tension forces are equally important. Having identified the dominant resisting and driving forces responsible for steady cone-jetting of a highly viscous liquid, we now derive the theoretical scaling laws for the minimum flow rate and the jet diameter at the minimum flow rate condition.

4.4. Derivation of the scaling laws

To derive the scaling law for the minimum flow rate of highly viscous liquids, we consider the balance of dominant forces at the cone-tip and the current transfer region. First, we consider the balance of forces at the cone tip. As shown in figure 8, the cone is characterised by its height, which is of $\mathcal{O}(d_n)$ and by the diameter of the tip of the cone, denoted by d_{cone} . In the cone region, the primary driving force is the electrostatic force, while the dominant resisting forces are the interfacial and viscous terms. For steady Taylor cone formation for highly viscous leaky-dielectric liquids, the viscous, interfacial tension and electrostatic forces should be of the same order of magnitude at the cone-tip. Therefore, scaling the corresponding terms in the momentum (4.5), at the cone tip, we obtain

$$\frac{\gamma}{d_{cone}^2} \sim \frac{\mu Q}{d_n d_{cone}^3} \sim \frac{\varepsilon_0 E_n^2}{d_n}, \quad (4.17)$$

which yields

$$d_{cone} \sim \frac{\mu Q}{\gamma d_n} \quad \text{and} \quad E_n \sim \left(\frac{\gamma d_n}{\varepsilon_0 d_{cone}^2} \right)^{1/2}. \quad (4.18a,b)$$

The equilibrium of the electric forces in the cone yields Taylor's solution, which allows us to calculate the characteristic value of the tangential component of the electric field (Gañán-Calvo *et al.* 2013) as

$$E_t \sim \left(\frac{\gamma}{\varepsilon_0 L} \right)^{1/2}. \quad (4.19)$$

Next, we consider the balance of forces in the current transfer region. As discussed in § 4.3, we assume that in the current transfer region, the viscous force is the dominant resistant force, whereas the electrostatic suction force is the dominant driving force. Thus, the balance between the electrostatic suction and viscous force in the current transfer region yields

$$\frac{\mu Q}{d^2 L^2} \sim \frac{\varepsilon_0 E_n^2}{L}. \quad (4.20)$$

The irrotationality of the electric field, (4.2), provides us with the reference scales for the normal component of the electric field and the surface charge density as $E_n \sim E_t L/d$ and $\sigma \sim \varepsilon_0 E_n \sim \varepsilon_0 E_t L/d$, respectively. We also note that in the current transfer region, the conduction and surface convection currents are of comparable magnitude (Gañán-Calvo 2004; Gañán-Calvo *et al.* 2013; Singh *et al.* 2022). Therefore, from (4.3) and (4.6), we can write $\kappa d^2 E_t \sim \sigma (Q/d^2)d$. Using the scaling of surface charge density $\sigma \approx \varepsilon_0 E_n$, we can further simplify the expression as $\kappa d^2 E_t \sim (\varepsilon_0 E_t L Q/d^3)d$, which yields the scaling for the length of the current transfer region as

$$L \sim \frac{\kappa d^4}{\varepsilon_0 Q}. \quad (4.21)$$

Equations (4.19)–(4.21) and $E_n \sim E_t L/d$ constitute four relations for E_n , E_t , L and d . Eliminating E_n and E_t from these relations yields

$$d \sim d_0 \left(\frac{Q}{Q_0} \right)^{3/8} Re_0^{-1/8} \quad (4.22a)$$

and

$$L \sim d_0 \left(\frac{Q}{Q_0} \right)^{1/2} Re_0^{-1/2}. \quad (4.22b)$$

At the minimum flow rate condition, the jet diameter at the cone-tip, d_{cone} given by (4.18a), is expected to be of the same order of magnitude as the diameter of the jet in the current transfer region, d , given by (4.22a). Therefore, considering $d_{cone} \sim d$ at the minimum flow rate condition, we get

$$\frac{\mu Q_{min}}{\gamma d_n} \sim d_0 \left(\frac{Q_{min}}{Q_0} \right)^{3/8} Re_0^{-1/8}, \quad (4.23)$$

which, upon further simplification, yields

$$Q_{min} \sim d_n^{8/5} \kappa^{-2/5} \left(\frac{\gamma^7 \varepsilon_0^2}{\mu^7} \right)^{1/5}. \quad (4.24)$$

The scaling relation in the dimensionless form can be expressed as

$$\begin{aligned} \frac{Q_{min}}{Q_0} &\sim \left(\frac{d_n}{d_0} \right)^{8/5} Re_0^{7/5}, \\ \tilde{Q}_{min} &\sim \tilde{d}_n^{8/5} Re_0^{7/5}. \end{aligned} \quad (4.25)$$

Next, from the regression on the experimental data, we get the fitting parameter as 0.5 for the dimensionless minimum flow rate $\tilde{Q}_{min}$. Therefore, (4.25) can be rewritten as

$$\tilde{Q}_{min} = 0.5 \tilde{d}_n^{8/5} Re_0^{7/5}. \quad (4.26)$$

Note that the scaling for Q_{min} , given by (4.24), closely captures $Q_{min} \sim d_n^{1.7}$ and $Q_{min} \sim \kappa^{-0.34}$ scaling observed in our experiments. Moreover, (4.24) also suggests that Q_{min} decreases with the increase in the viscosity of the liquid, which is in line with the observation of Scheideler & Chen (2014).

Also, the scaling laws for d and L at the minimum flow rate can be obtained by substituting the expression of Q_{min} , given by (4.25), in (4.22a) and (4.22b), respectively. Upon substitution, we get

$$d \sim d_0 \left(\frac{d_n}{d_0} \right)^{3/5} Re_0^{2/5} \sim d_n^{3/5} \kappa^{-2/5} \left(\frac{\gamma^3 \varepsilon_0^3}{\mu} \right)^{2/5} \quad (4.27a)$$

and

$$L \sim d_0 \left(\frac{d_n}{d_0} \right)^{4/5} Re_0^{1/5} \sim d_n^{4/5} \kappa^{-3/5} \left(\frac{\gamma \varepsilon_0^3}{\mu} \right)^{1/5}. \quad (4.27b)$$

Similar to the minimum flow rate, the regression on the experimental data provides the fitting parameter as 0.75 for the dimensionless jet diameter. Therefore, we can rewrite (4.27a) as

$$\tilde{d} = 0.75 \tilde{d}_n^{3/5} Re_0^{2/5}. \quad (4.28)$$

From (4.27a), we also note that $d \sim d^{0.6}$ and $d \sim \kappa^{-0.4}$, which agree well with the $d \sim d_n^{0.7}$ and $d \sim \kappa^{-0.37}$ scaling observed in our experiments. We note that our scaling for the minimum flow rate differs from that of Gañán-Calvo *et al.* (2013) because we consider viscous force to be the primary resisting force instead of inertia in the current transfer region, as discussed in § 4.3. Additionally, our analysis also assumes that the viscous and interfacial tension forces are equally important in the cone region, which results in the dependence of Q_{min} on the needle diameter d_n .

4.5. Validation of the scaling relations

We now validate these scaling laws by comparing the calculated values of the minimum flow rate and the jet diameter, based on the derived scaling laws, with the experimentally measured values. To differentiate between the experimental data and the value obtained from the theoretical expression, we denote experimentally measured values of the minimum flow rate and jet diameter as Q_{min}^{exp} and d_{exp} , respectively. Similarly, we denote the values of the minimum flow rate and the jet diameter obtained from the derived scaling laws by Q_{min}^{theory} and d_{theory} , respectively. In addition, Q_{min}^{corr} and d_{corr} denote the minimum flow rate and the corresponding jet diameter values obtained from the empirical correlations.

To validate the scaling law for the minimum flow rate, given by (4.26), we first compare the dimensionless minimum flow rate values calculated based on the derived scaling law (4.26), with the experimental data and with that obtained from the empirical correlation (3.5). The experimental data consist of minimum flow rate values provided by Scheideler & Chen (2014) and those measured in the present work. As depicted in figure 11(a), for comparison, we plot the values of the dimensionless minimum flow rate obtained from the derived expression along the abscissa and the corresponding experimental data of the minimum flow rate (represented by the circular symbol) and the minimum flow rate values obtained from the correlation (represented by the triangular symbol) along the ordinate on a log–log scale. The dashed line in figure 11(a) represents the line with unity slope. From figure 11(a), we infer that all the data points collapse to the dashed line, indicating a close match among experimental data, empirical correlation and the theoretical scaling law.

Next, we compare the dimensionless jet diameter for liquids I and II calculated from the derived scaling law, given by (4.28), with the experimentally measured jet diameter and that obtained from the empirical correlation, given by (3.7). It is important to note that due to the high electrical conductivities of solutions III and IV, the corresponding jet diameters were below the resolution of the imaging system used in the current study. Consequently, the jet diameters for these two liquids could not be measured. Therefore, to validate the scaling law, in figure 11(b), we plot the measured dimensionless jet diameter data against the values obtained from the theoretical scaling (4.28) and that obtained from the empirical correlation (3.7) for solutions I and II on a log–log scale. The dashed line in figure 11(b) represents the line with unity slope. From figure 11(b), we infer that the dimensionless jet diameter values obtained from the derived expression closely align with the experimentally measured values (represented by hollow symbols) and that predicted

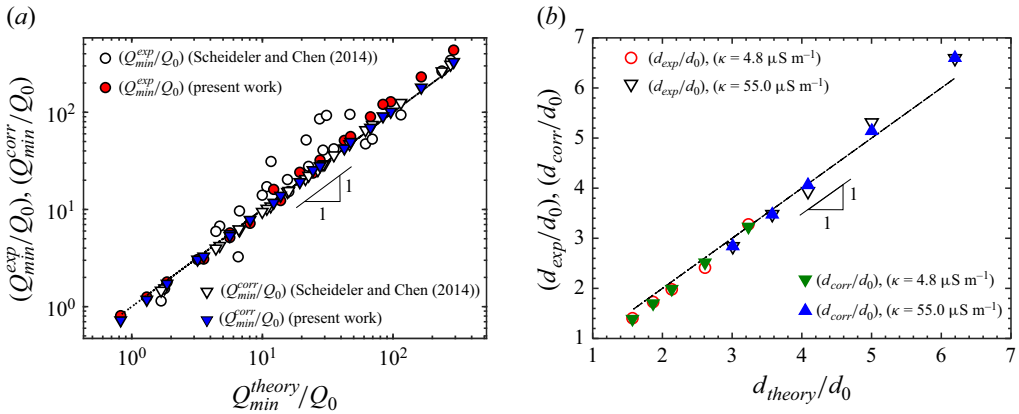


Figure 11. Validation of the theoretical scaling laws for the minimum flow rate (Q_{min}) and the jet diameter (d) at the minimum flow rate. (a) Comparison of experimental measurement of Q_{min} of the present work and that of Scheideler & Chen (2014) with those predicted by theoretical scaling law ((4.26)). Also shown is the comparison between the Q_{min} predicted by the empirical correlation (3.5) and the theoretical scaling law. (b) Comparison of the experimentally measured jet diameter d with those predicted by the theoretical scaling law (4.28). In the same plot, we also compare the jet diameter values predicted by the empirical correlation (3.7) with the theoretical scaling laws. As observed in panels (a) and (b), the theoretical scaling laws for the minimum flow rate and the jet diameter agree well with the corresponding experimental data.

by the empirical correlation (represented by solid symbols). This validates the theoretical scaling laws for the minimum flow rate and the jet diameter at the minimum flow rate.

5. Limits of validity of the derived scaling relations

Having validated the derived scaling laws for the minimum flow rate and the jet diameter at Q_{min} , it is important to define the limits within which these scaling laws are applicable. In this section, we first discuss the validity of the important assumptions made for the theoretical scaling analysis and then the corresponding limits describing the regimes in which the scaling laws are applicable. Therefore, we revisit the assumptions made in §§ 4.1 and 4.3 for the scaling analysis of the minimum flow rate of a steady cone-jet of a viscous leaky-dielectric liquid.

5.1. Validation of the assumptions

5.1.1. Fast charge relaxation

The theoretical scaling analysis presented in § 4.4 assumes that the characteristic time scale for charge relaxation $\tau_e \sim \beta\epsilon_0/\kappa$ is comparable to or smaller than the characteristic residence time in the current transfer region, $\tau_h \sim Ld^2/Q$, as represented in (4.1). Therefore, the condition $\tau_e/\tau_h \lesssim 1$ is crucial for the formation of a charge layer on the liquid surface, ensuring steady cone-jetting. By substituting the expressions for d and L from (4.22a) and (4.22b) into (4.1), we obtain

$$\frac{\tau_e}{\tau_h} \sim \frac{\beta\epsilon_0 Q}{Ld^2\kappa} \sim \frac{\beta Re_0}{\left(\frac{QRe_0}{Q_0}\right)^{1/4}}. \quad (5.1)$$

In table 3, we present the values of (5.1) corresponding to all four liquids at the minimum flow rate condition. From table 3, we note that the rapid charge relaxation criterion, $\tau_e/\tau_h \lesssim 1$, is satisfied by all liquids. More importantly, as shown by Ponce-Torres *et al.*

κ [$\mu\text{S m}^{-1}$]	Q_0 [$\times 10^{-2} \text{ nl s}^{-1}$]	Re_0 [$\times 10^{-3}$]	d_n [μm]	Q_{min}^{exp} [nl s^{-1}]	τ_e/τ_h , (F_{pol}/F_{es}) _{ct}	(F_{int}/F_{visc}) ct	(F_{int}/F_{visc}) cone	(F_{st}/F_{visc}) ct	(F_{st}/F_{visc}) cone
4.8	9281.4	15.0	300.0	75.0	1.81	0.04	0.01	0.21	1.6
			400.0	116.7	1.62	0.05	0.01	0.20	1.6
			500.0	166.7	1.48	0.07	0.01	0.19	1.6
			700.0	283.3	1.30	0.10	0.02	0.18	1.6
			1000.0	533.0	1.11	0.16	0.04	0.17	1.5
55.0	810.0	6.7	300.0	25.0	0.71	0.05	0.02	0.13	1.1
			400.0	41.7	0.62	0.08	0.03	0.12	1.1
			500.0	58.3	0.57	0.10	0.04	0.12	1.1
			700.0	100.0	0.50	0.15	0.06	0.11	1.1
			1000.0	191.7	0.42	0.25	0.10	0.10	1.0
430.0	103.6	3.4	300.0	16.7	0.28	0.11	0.05	0.08	0.6
			400.0	25.0	0.25	0.15	0.07	0.08	0.6
			500.0	33.3	0.24	0.19	0.10	0.08	0.7
			700.0	60.0	0.20	0.30	0.15	0.07	0.6
			1000.0	125.0	0.17	0.51	0.27	0.07	0.6
3430.0	12.9	1.7	300.0	6.7	0.12	0.16	0.11	0.06	0.5
			400.0	11.7	0.11	0.25	0.16	0.05	0.4
			500.0	16.7	0.10	0.32	0.22	0.05	0.4
			700.0	30.0	0.09	0.50	0.36	0.05	0.4
			1000.0	56.7	0.07	0.80	0.61	0.04	0.4

Table 3. Details of the needle diameter and the conductivity of the liquids used in the present work. Additionally, it includes the experimental data for the minimum flow rate and the values of τ_e/τ_h , (F_{int}/F_{visc}), (F_{st}/F_{visc}) and (F_{pol}/F_{es})_{ct}. From these values, we infer that viscous force is the dominant resisting force and electrostatic suction is the dominant driving force in the current transfer region. Moreover, viscous and interfacial tension forces are of the same order of magnitude at the cone-tip.

(2018), the scaling relations for the normal component of the electric field (E_n), surface charge density (σ) and the electrostatic force terms remain valid even when $\tau_e \sim \tau_h$. Therefore, in the present study, the assumption of fast charge relaxation is justified and the associated scaling analysis remains valid.

Subsequently, setting $\tau_e/\tau_h \lesssim 1$ in (5.1) at $Q = Q_{min}$, we obtain a limit on Q_{min} for which rapid charge relaxation can be assumed,

$$Q_{min} \gtrsim \beta^4 Q_0 Re_0^3. \quad (5.2)$$

Therefore, $Q_{min} = \beta^4 Q_0 Re_0^3$ represents the first constraint for the minimum flow rate and flow rates above this limiting value will satisfy the rapid charge relaxation criterion.

5.1.2. Viscous force is the dominant resisting force

In § 4.3, we assumed that among interfacial tension, inertia and viscous forces, the viscous force is the dominant resisting force in the current transfer region. To justify the assumption, we now compare the viscous force with the inertia and the interfacial tension force in the current transfer region. Using the derived scaling relations for d and L given by (4.22a) and (4.22b), we can write in the current transfer region,

$$\left(\frac{F_{int}}{F_{visc}} \right)_{ct} \sim \frac{\rho L^2 Q}{\mu L d^2} \sim \left(\frac{Q Re_0}{Q_0} \right)^{3/4} \quad (5.3a)$$

and

$$\left(\frac{F_{st}}{F_{visc}}\right)_{ct} \sim \frac{\gamma d^2 L^2}{d L \mu Q} \sim \frac{Re_0^{1/2}}{\left(\frac{Q Re_0}{Q_0}\right)^{1/8}}. \quad (5.3b)$$

Next, in table 3, we present the values of $(F_{int}/F_{visc})_{ct}$ and $(F_{st}/F_{visc})_{ct}$ for all four liquids, based on these relations at the minimum flow rate condition. From table 3, we note that the conditions $(F_{int}/F_{visc})_{ct} < 1$ and $(F_{st}/F_{visc})_{ct} < 1$ are satisfied. Thus, our assumption of viscous force being the dominant resisting force in the current transfer region, while deriving the scaling laws, is justified.

Consequently, setting $(F_{int}/F_{visc})_{ct} \lesssim 1$ in (5.3a) at $Q = Q_{min}$, we obtain

$$Q_{min} \lesssim Q_0 Re_0^{-1}. \quad (5.4)$$

Therefore, we note that $Q_{min} = Q_0 Re_0^{-1}$ represents the second constraint for the minimum flow rate and for flow rates below this limiting value, the viscous force dominates inertia in the current transfer region.

Similarly, setting $(F_{st}/F_{visc})_{ct} \lesssim 1$ in (5.3b) at the minimum flow rate condition, we get

$$Q_{min} \gtrsim Q_0 Re_0^3. \quad (5.5)$$

Therefore, $Q_{min} = Q_0 Re_0^3$ represents the third constraint for the minimum flow rate and for flow rates above this limiting value, the viscous force dominates interfacial tension force in the current transfer region.

While deriving the scaling laws, we also considered that at the cone tip, among the resisting forces, the viscous and the interfacial tension forces are of the same order of magnitude and are greater than inertia by at least one order of magnitude. To verify these assumptions, we compare the forces at the cone-tip. We also assumed that the jet diameters at the cone-tip and in the current transfer region are of the same order of magnitude. Therefore, using the derived scaling law for jet diameter d , given by (4.22a), we can write at the cone-tip,

$$\left(\frac{F_{int}}{F_{visc}}\right)_{cone} \sim \frac{\rho Q d_n}{\mu d_{cone}^2} \sim \left(\frac{Q Re_0}{Q_0}\right) \left(\frac{d_n}{d_0}\right) \left(\frac{d_0}{d_{cone}}\right)^2 \sim \left(\frac{d_n Re_0}{d_0}\right) \left(\frac{Q Re_0}{Q_0}\right)^{1/4}, \quad (5.6a)$$

and

$$\left(\frac{F_{st}}{F_{visc}}\right)_{cone} \sim \frac{\gamma d_n d_{cone}}{\mu Q} \sim \left(\frac{Q Re_0}{Q_0}\right)^{-1} \left(\frac{d_n Re_0^2}{d_0}\right) \left(\frac{d_{cone}}{d_0}\right) \sim \left(\frac{d_n Re_0}{d_0}\right) \frac{Re_0^{1/2}}{\left(\frac{Q Re_0}{Q_0}\right)^{5/8}}. \quad (5.6b)$$

In table 3, we present the values of $(F_{int}/F_{visc})_{cone}$ and $(F_{st}/F_{visc})_{cone}$ at the minimum flow rate condition, based on the derived relations for all four liquids. From table 3, we note that $(F_{int}/F_{visc})_{cone} < 1$ and $(F_{st}/F_{visc})_{cone} \sim 1$. Therefore, our assumption of viscous force and interfacial tension forces being of the same order of magnitude at the cone-tip is justified.

5.1.3. Electrostatic suction force is the dominant driving force

In § 4.3, we assume that in the current transfer region, the forces due to tangential electric stress and electrostatic suction are of the same order of magnitude. Additionally, we argued that in the current transfer region, the tangential component of the electric field is much

smaller than the normal component of the electric field, that is, $E_t \ll E_n$. Consequently, the force due to polarisation stress can, at most, match the order of magnitude of the force due to electrostatic suction. Based on these considerations, we identified the electrostatic suction force to be the primary driving force in the current transfer region and derived the corresponding scaling laws.

To verify this assumption, we now compare the various driving forces in the current transfer region. Using the derived scaling laws for d and L , given by (4.22a) and (4.22b), we can write

$$\left(\frac{F_{et}}{F_{es}}\right)_{ct} \sim \frac{\varepsilon_0 E_n E_t L}{d \varepsilon_0 E_n^2} \sim \frac{E_t L}{d E_n} \sim \frac{E_t L d}{d E_t L} \sim 1 \quad (5.7a)$$

and

$$\left(\frac{F_{pol}}{F_{es}}\right)_{ct} \sim \frac{\varepsilon_0 \beta E_t^2}{\varepsilon_0 E_n^2} \sim \frac{\beta E_t^2}{E_t^2 \frac{L^2}{d^2}} \sim \frac{\beta d^2}{L^2} \sim \frac{\beta Re_0}{\left(\frac{Q Re_0}{Q_0}\right)^{1/4}}. \quad (5.7b)$$

From (5.7a), we note that tangential electric stress and electrostatic suction are of the same order of magnitude irrespective of the scaling relation of d and L . Moreover, from (5.1) and (5.7b), we note that the final scalings for τ_e/τ_h and $(F_{pol}/F_{es})_{ct}$ are identical. Therefore, from table 3, we infer that at the minimum flow rate condition, $(F_{pol}/F_{es})_{ct} \lesssim 1$ for all four liquids. This justifies our earlier assumption of electrostatic suction being the dominant driving force in the current transfer region. Furthermore, we check the validity of these assumptions for the experimental data of Scheideler & Chen (2014) and tabulate the findings in table 5. Similar to our experimental data, all the discussed assumptions are also satisfied by the liquids used by Scheideler & Chen (2014).

Therefore, our assumption that for highly viscous liquids, at the minimum flow rate condition, the viscous force acts as the dominant resisting force, while electrostatic suction serves as the dominant driving force in the current transfer region, is verified. Using (5.3a), (5.3b) and (5.7b), and considering $\beta - 1 \approx \beta$, we can now collectively represent these criteria as follows:

$$\left(\frac{Q_{min} Re_0}{Q_0}\right)^{3/4} \lesssim 1, \quad \frac{Re_0^{1/2}}{\left(\frac{Q_{min} Re_0}{Q_0}\right)^{1/8}} \lesssim 1 \quad \text{and} \quad \frac{\beta Re_0}{\left(\frac{Q_{min} Re_0}{Q_0}\right)^{1/4}} \lesssim 1, \quad (5.8)$$

which on further simplification yields

$$\frac{Q_{min} Re_0}{Q_0} \lesssim 1, \quad \frac{Q_{min}}{Q_0 Re_0^3} \gtrsim 1 \quad \text{and} \quad \frac{\left(\frac{Q_{min}}{Q_0 Re_0^3}\right)^{1/4}}{\beta - 1} \gtrsim 1. \quad (5.9)$$

Using the notation defined by Gañán-Calvo (2004), we can express (5.9) as

$$\alpha_\rho \lesssim \alpha_\mu^{1/4}, \quad \alpha_\mu \gtrsim 1 \quad \text{and} \quad \frac{\alpha_\mu^{1/4}}{\beta - 1} \gtrsim 1. \quad (5.10)$$

Here, $\alpha_\rho = Q_{min}/Q_0$ and $\alpha_\mu = \alpha_\rho/Re_0^3$ (Gañán-Calvo 2004). We note that (5.10) is the criteria for the viscous electrostatic suction regime (VE-regime) for steady cone-jetting of a viscous leaky dielectric liquid. Thus, the liquids considered in the present study and that of Scheideler & Chen (2014) with $\beta Re_0 < 1$, follow the VE scaling at the minimum flow

rate condition. However, liquids considered by Gañán-Calvo *et al.* (2013) and Ponce-Torres *et al.* (2018) having $\beta Re_0 < 1$, follow the IE scaling.

It is important to note that the term $\beta Re_0(Q_{min}Re_0/Q_0)^{-1/4}$ determines whether viscous or polarisation forces limit steady cone-jetting at Q_{min} for both highly and less viscous liquids. Specifically, when $\beta Re_0(Q_{min}Re_0/Q_0)^{-1/4} < 1$, the viscous force restricts jet emission at Q_{min} . Conversely, when $\beta Re_0(Q_{min}Re_0/Q_0)^{-1/4} > 1$, the polarisation force opposes the driving force at Q_{min} and stalls the jet beyond the current transfer region. Furthermore, by substituting $Q_{min} = Q_0Re_0^{-1}$ and $Q_{min} = \beta Q_0$, the criteria $\beta Re_0(Q_{min}Re_0/Q_0)^{-1/4} < 1$ and $\beta Re_0(Q_{min}Re_0/Q_0)^{-1/4} > 1$ simplify to $\beta Re_0 < 1$ and $\beta Re_0 > 1$, respectively, which is the criteria defined by Gañán-Calvo *et al.* (2013) for less viscous liquids.

5.1.4. Discussion on the assumption of Scheideler and Chen

From figure 9, we also note that for the present work and that of Scheideler & Chen (2014), $Q_{min}(\beta Q_0)^{-1} \lesssim 1$, which is in contrast to the observations of Gañán-Calvo *et al.* (2013) for which $Q_{min}(\beta Q_0)^{-1} > 1$. To explain this, we refer to the assumption of Scheideler & Chen (2014). They assumed that the viscous diffusion time scale across a length scale of d_n , $\tau_\mu \sim \rho d_n^2/\mu$, should become comparable to or smaller than the charge relaxation time $\tau_e \sim \beta \epsilon_0/\kappa$ and subsequently derived the scaling law for the minimum flow rate as $Q_{min} \sim \gamma d_n^2/\mu$. However, the assumption is not satisfied by most of the liquids considered by Scheideler & Chen (2014) and the present study. Therefore, we evaluate the viscous diffusion time across the length of the current transfer region L rather than d_n and compare it with the charge relaxation time. Using (4.22b), we get

$$\frac{\tau_\mu}{\tau_e} \sim \frac{\rho L^2 \kappa}{\mu \beta \epsilon_0} \sim \frac{Q}{\beta Q_0} \lesssim 1. \quad (5.11)$$

Therefore, at the minimum flow rate condition, for highly viscous liquids, the viscous diffusion time scale across the length of the current transfer region is comparable to or smaller than the charge relaxation time scale. As Gañán-Calvo *et al.* (2013) considered less viscous liquids in their cone-jetting experiments, the viscous diffusion time across the current transfer region was much greater than the charge relaxation time, $\tau_\mu/\tau_e > 1$, resulting in $Q_{min}(\beta Q_0)^{-1} > 1$.

5.2. Validity limits of the derived scaling laws

For deriving the theoretical scaling relations, we made several assumptions, such as fast charge relaxation, viscous force being the dominant resisting force and electrostatic suction being the dominant driving force in the current transfer region. In § 5.1, we discussed how the derived scaling laws for the minimum flow rate and the associated quantities are consistent with the assumptions made in §§ 4.1 and 4.3. Based on these verified assumptions, we also derived a set of inequalities in § 5.1, which define the constraints for the minimum flow rate. These inequalities are $Q_{min} \gtrsim \beta^4 Q_0 Re_0^3$ (from $\tau_e \lesssim \tau_h$), $Q_{min} \lesssim Q_0 Re_0^{-1}$ (from $F_{visc} > F_{int}$ in the current transfer region) and $Q_{min} \gtrsim Q_0 Re_0^3$ (from $F_{visc} > F_{st}$ in the current transfer region).

Figure 12(a) shows a log–log plot, where the abscissa is Re_0^{-1} and the ordinate is Q_{min}/Q_0 . In figure 12(a), we plot the derived constraints along with the minimum flow rate values of the present study, Scheideler & Chen (2014) and that of Gañán-Calvo *et al.* (2013). The shaded trapezoidal area in figure 12(a) satisfies all the constraints. Moreover, the minimum flow rate data of the present study and Scheideler & Chen (2014) fall in this shaded area. In addition, we note that the minimum flow rate data of Gañán-Calvo *et al.*

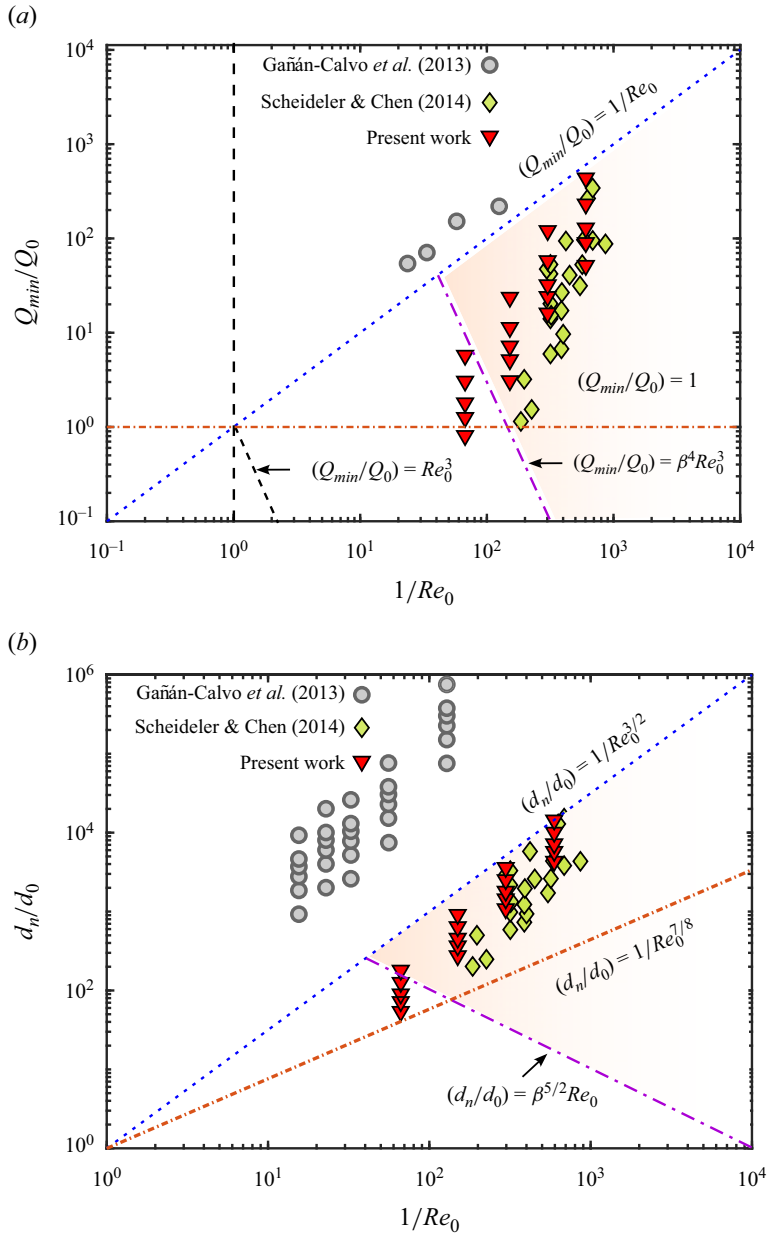


Figure 12. Regime plot showing the validity limits for the derived scaling laws for highly viscous liquids. (a) Validity limits of the scaling law for minimum flow rate expressed in terms of Q_0 and Re_0 . The experimental data from Scheideler & Chen (2014) and present work clearly fall within the region defined by the $Q_{min}/Q_0 = Re_0^{-1}$ line (upper limit), $Q_{min}/Q_0 = 1$ (lower limit) line and $Q_{min}/Q_0 = \beta^4 Re_0^3$ line (see tables 4 and 6). The experimental data from Gañán-Calvo *et al.* (2013) are located in the region where $Q_{min}/Q_0 > Re_0^{-1}$, but close to the $Q_{min}/Q_0 = Re_0^{-1}$ line because in these cases, inertia is the dominant resisting force. (b) Region in the 2-D regime plot between d_n/d_0 and $1/Re_0$, where Q_{min} depends on the needle diameter. This region is bounded by the $d_n/d_0 = Re_0^{-3/2}$ line (upper limit), $d_n/d_0 = Re_0^{-7/8}$ line (lower limit) and $d_n/d_0 = \beta^{5/2} Re_0$ line (see tables 4 and 6). For the experimental data from Gañán-Calvo *et al.* (2013), d_n/d_0 values are plotted for d_n ranging from 100 to 1000 μm . We note that even for small needle diameters ($d_n = 100 \mu\text{m}$), the minimum flow rate of steady cone-jetting remains independent of the needle diameter, as for those experiments, jetting occurs in an inertia-dominated regime.

d_n [μm]	κ [$\mu\text{S m}^{-1}$]	Q_0 [$\times 10^{-2} \text{ nl s}^{-1}$]	Re_0 [$\times 10^{-3}$]	$\gamma d_n^2/\mu$ [nl s^{-1}]	Q_{min}^{exp} [nl s^{-1}]	Q_{min}^{theory} [nl s^{-1}]	(Q_0/Re_0) [nl s^{-1}]	$d_0/Re_0^{7/8}$ [μm]	$d_0/Re_0^{3/2}$ [μm]
300.0	4.8	9281.4	15.0	4075.7	75.0	76.3	6169.7	218.4	3009.4
	55.0	810.0	6.7		25.0	28.8	1213.9	87.5	2004.3
	430.0	103.6	3.4		16.7	12.6	308.2	40.5	1422.7
	3430.0	12.9	1.7		6.7	5.5	77.2	18.6	1006.5
400.0	4.8	9281.4	15.0	7245.7	116.7	120.9	6169.7	218.4	3009.4
	55.0	810.0	6.7		41.7	45.6	1213.9	87.5	2004.3
	430.0	103.6	3.4		25.0	20.0	308.2	40.5	1422.7
	3430.0	12.9	1.7		11.7	8.7	77.2	18.6	1006.5
500.0	4.8	9281.4	15.0	11 321.4	166.7	172.8	6169.7	218.4	3009.4
	55.0	810.0	6.7		58.3	65.2	1213.9	87.5	2004.3
	430.0	103.6	3.4		33.3	28.6	308.2	40.5	1422.7
	3430.0	12.9	1.7		16.7	12.5	77.2	18.6	1006.5
700.0	4.8	9281.4	15.0	22 190.0	283.3	296.1	6169.7	218.4	3009.4
	55.0	810.0	6.7		100.0	111.6	1213.9	87.5	2004.3
	430.0	103.6	3.4		58.3	49.0	308.2	40.5	1422.7
	3430.0	12.9	1.7		30.0	21.4	77.2	18.6	1006.5
1000.0	4.8	9281.4	15.0	45 285.7	533.0	523.9	6169.7	218.4	3009.4
	55.0	810.0	6.7		191.7	197.5	1213.9	87.5	2004.3
	430.0	103.6	3.4		125.0	86.8	308.2	40.5	1422.7
	3430.0	12.9	1.7		56.7	37.8	77.2	18.6	1006.5

Table 4. Details of the needle diameter and the conductivity of the liquids used in the present work. Also tabulated are the experimental data for the minimum flow rate of steady Taylor cone-jets, alongside the corresponding values of minimum flow rate calculated based on the scaling laws provided by Scheideler & Chen (2014, see (4.15)), the present work ((4.26)) and Gañán-Calvo *et al.* (2013, see (4.8a)). The lower and upper limits of the needle diameter ((5.12a) and (5.12b)) for which the minimum flow rate depends on the needle diameter are also mentioned. For all the minimum flow rate data, the $\beta Re_0 < 1$ and $(Q_{min}/Q_0)Re_0 < 1$ criteria are satisfied.

(2013) are separated from the shaded area by the upper limit of the minimum flow rate $Q_{min} = Q_0 Re_0^{-1}$, as inertia is the dominant resisting force for the liquids considered in their study.

Next, we consider the limit $Q_{min} \gtrsim \beta^4 Re_0^3 Q_0$. From figure 12(a), we note that all the data points corresponding to the present study and that of Scheideler & Chen (2014) fall on the right side of $Q_{min} = \beta^4 Re_0^3 Q_0$, except solution I. For solution I, we have already shown that $\tau_e \sim \tau_h$ and that is why the minimum flow rate values lay close to the line $Q_{min} = \beta^4 Re_0^3 Q_0$, but slightly towards its left. We note that the assumptions based on which the scaling law for the minimum flow rate of steady cone-jetting is derived do not provide a lower bound for Q_{min} . Thus, the theoretical analysis could not rule out the possibility of Q_{min} falling below the corresponding Q_0 values. However, previous experimental work (Gañán-Calvo *et al.* 2013; Scheideler & Chen 2014; Ponce-Torres *et al.* 2018) and the experimental data of the present study on the minimum flow rate of steady cone-jetting suggest that $Q_{min} \gtrsim Q_0$. Therefore, we define $Q_{min} = Q_0$ as the lower limit for the minimum flow rate of steady cone-jetting for highly viscous liquids. In a recent study on the formation of an electrified jet using highly viscous soy lecithin oil ($\mu = 20 \text{ Pa s}$) (Tian *et al.* 2019), it was observed that for flow rates much below Q_0 ($Q \ll Q_0$), jetting was possible. However, in this case, the viscous liquid did not show a sharp cone-to-jet transition. Rather, a gradual transition was observed. In particular, for this case, the length

of the current transfer region was greater or comparable to the outer diameter of the needle, $L \gtrsim d_n$, and the jet diameter was comparable to the needle diameter, $d \sim d_n$. Therefore, we discard the area of the trapezoid which falls below the $Q_{min} = Q_0$ line, as liquids falling in this region do not show a sharp cone-to-jet transition.

We now substitute the derived scaling law for the minimum flow rate $Q_{min}/Q_0 \sim (\gamma d_n d_0 / \mu Q_0)^{8/5} Re_0^{-1/5}$ (4.25) into the derived constraints for the minimum flow rate to get the corresponding limits for the needle diameter, d_n .

- (i) For $Q_{min} = Q_0 Re_0^{-1}$, we get $d_n = d_0 Re_0^{-3/2}$.
- (ii) For $Q_{min} = \beta^4 Q_0 Re_0^3$, we get $d_n = \beta^{5/2} d_0 Re_0$.
- (iii) For $Q_{min} = Q_0$, we get $d_n = d_0 Re_0^{-7/8}$.

Figure 12(b) shows a log–log plot, where the abscissa denotes Re_0^{-1} and the ordinate denotes d_n/d_0 . In figure 12(b), following a similar procedure, we plot the d_n/d_0 values along with their constraints. We note that all the data points lie within the shaded trapezoidal area. Within these defined limits, the minimum flow rate of highly viscous liquids that satisfy the $\beta Re_0 < 1$ criterion depends on the needle diameter.

Therefore, we can represent the limits for Q_{min} and d as

$$1 \lesssim \frac{Q_{min}}{Q_0} \lesssim Re_0^{-1} \quad \text{and} \quad \frac{Q_{min}}{Q_0} \gtrsim \beta^4 Re_0^3, \quad (5.12a)$$

and

$$Re_0^{-7/8} \lesssim \frac{d_n}{d_0} \lesssim Re_0^{-3/2} \quad \text{and} \quad \frac{d_n}{d_0} \gtrsim \beta^{5/2} Re_0. \quad (5.12b)$$

Next, we examine the experimental data obtained from Gañán-Calvo *et al.* (2013) for liquids such as 3-ethylene glycol, 4-ethylene glycol and EMI-Im of varying electrical conductivities, satisfying the $\beta Re_0 < 1$ criterion (see table 2). We note that these data points lie in the region $(Q_{min}/Q_0) > 1/Re_0$, but remain close to the $(Q_{min}/Q_0)Re_0 = 1$ line, as illustrated in figure 12(a). Therefore, the minimum flow rate data exceed the limit defined by (5.12a). This is attributed to the fact that, for these liquids, inertia is the dominant resisting force in the current transfer region. Subsequently, we plot the d_n/d_0 values for the same data set in figure 12(b). We note that the d_n/d_0 value exceeds the upper limit $d_n = d_0/Re_0^{3/2}$, even for small values of needle diameter such as $d_n = 100 \mu\text{m}$ for the liquids considered by Gañán-Calvo *et al.* (2013) and Ponce-Torres *et al.* (2018). This is because the viscous effects are not prominent in the cone region of these liquids and are localised to the current transfer region. That is why the minimum flow rate of steady cone-jetting did not depend on the needle diameter for the cases considered by Gañán-Calvo *et al.* (2013) and Ponce-Torres *et al.* (2018).

In § 4.1, we assumed that at the minimum flow rate condition, the steady cone-jet mode persists. A previous work (Singh *et al.* 2022, 2023) on the regimes of steady cone-jetting suggests that the diameter of the jet and the length of the current transfer region with respect to the characteristic length of the needle are the important parameters to distinguish a cone-jet (Taylor 1964) from a moderately stretched and a thick cone-jet. In some studies in the literature, it was observed that the length of the current transfer region exceeded the outer diameter of the needle used in the experiments (Arscott *et al.* 2005; Carlier *et al.* 2005; Tian *et al.* 2019). Therefore, we try to find out the limits for the d/d_n and L/d_n ratios for our case and for Scheideler & Chen (2014). To find the lower and upper limits for d/d_n , we use (4.27a) and (5.12b) to get

$$Re_0 \lesssim \frac{d}{d_n} \lesssim Re_0^{3/4} \quad \text{and} \quad \frac{d}{d_n} \lesssim \beta^{-1}. \quad (5.13)$$

Similarly, using (4.27b) and (5.12b), we get

$$Re_0^{1/2} \lesssim \frac{L}{d_n} \lesssim Re_0^{3/8} \quad \text{and} \quad \frac{L}{d_n} \lesssim \beta^{-1/2}. \quad (5.14)$$

As for highly viscous liquids $Re_0 \ll 1$, from (5.13) and (5.14), we can conclude that at the minimum flow rate condition, the jet diameter and the length of the current transfer region are significantly smaller than the outer diameter of the needle. This observation is consistent with the experimental data of Scheideler & Chen (2014) and that of the present study. Therefore, the present discussion focuses on the scenarios where $d \ll d_n$ and $L \ll d_n$, and excludes the cases where $d \lesssim d_n$ and $L > d_n$.

6. Conclusion

In this work, we have presented a systematic experimental and theoretical investigation of the minimum flow rate of steady electrohydrodynamic jetting of highly viscous liquids. Through this study, we were able to define the conditions for which the minimum flow rate of a highly viscous liquid having $\beta Re_0 < 1$ depends on the needle diameter. To this end, we performed experiments for Taylor cone-jetting using five different needles with outer diameters varying from 300 μm to 1000 μm and four salt-doped glycerol solutions with electrical conductivity spanning over four orders of magnitude. Our experiments suggest that the minimum flow rate of steady cone-jetting of a viscous liquid depends on the needle diameter and the electrical conductivity of the liquid. In particular, experiments show that $Q_{min} \sim d_n^{1.7}$ and $Q_{min} \sim \kappa^{-0.34}$. Subsequently, we performed a dimensional analysis, which suggests that the parameters that influence the minimum flow rate can be effectively grouped into two dimensionless groups, $\tilde{d}_n = d_n/d_0$ and Re_0 , for liquids with low to moderate electrical permittivity. The minimum flow rate can therefore be represented as $\tilde{Q}_{min} = f(\tilde{d}_n, Re_0)$.

To elucidate the physical mechanism underlying the observed dependence of Q_{min} on the needle diameter d_n and electrical conductivity κ , we performed an order-of-magnitude analysis. Through an order-of-magnitude analysis, we showed that for our experiments with highly viscous liquids, the viscous force is the dominant resisting force, whereas electrostatic suction is the dominant supporting force at the current transfer region. Moreover, the viscous and the interfacial tension forces are equally important in the cone region, as the viscous diffusion time scale is small enough to allow viscous effects to propagate to the cone region. Subsequently, from the balance of the dominant forces, we get the scaling laws for the minimum flow rate and the jet diameter at the minimum flow rate condition as $\tilde{Q}_{min} = 0.5\tilde{d}_n^{8/5} Re_0^{7/5}$ and $\tilde{d} = 0.75\tilde{d}_n^{3/5} Re_0^{2/5}$, respectively. The dependence of minimum flow rate and the corresponding jet diameter, on the needle diameter, is attributed to the dominant viscous force in the cone and in the current transfer region. The derived scaling laws are in agreement with the experimental observations of the present study and that of the literature (Scheideler & Chen 2014).

Finally, we define the limits of the regime within which the minimum flow rate depends on the needle diameter. The experimentally validated regime, where the minimum flow rate depends on the needle diameter, can be useful in reducing the number of trials to find the proper operating conditions for applications such as EHD-jet printing, electrospinning and electrospraying, where cone-jetting near the minimum flow rate condition is desired. In short, the derived scaling laws for the minimum flow rate and the jet diameter will be useful for the design and control of EHD-jet printing, electrospinning and electrospraying.

d_n [μm]	κ [$\mu\text{S m}^{-1}$]	Q_0 [$\times 10^{-2} \text{ nl s}^{-1}$]	Re_0 [$\times 10^{-3}$]	Q_{min}^{exp} [nl s^{-1}]	τ_e/τ_h , (F_{pol}/F_{es}) _{ct}	(F_{int}/F_{visc}) ct cone		(F_{st}/F_{visc}) ct cone	
125.0	180.0	247.5	4.5	3.8	0.62	0.02	0.01	0.12	1.7
	910.0	48.9	2.6	3.3	0.29	0.05	0.02	0.08	1.2
150.0	100.0	445.5	5.5	5.1	0.78	0.02	0.01	0.14	2.0
	500.0	89.1	3.2	5.3	0.34	0.05	0.02	0.09	1.3
	1000.0	44.5	2.5	4.3	0.25	0.06	0.02	0.08	1.2
	2500.0	17.8	1.9	5.6	0.15	0.12	0.04	0.06	0.8
	10000.0	4.4	1.2	3.9	0.08	0.19	0.08	0.05	0.7
	910.0	48.9	2.6	8.4	0.23	0.10	0.04	0.08	1.2
210.0	2830.0	15.7	1.8	8.3	0.13	0.17	0.07	0.06	0.9
	4990.0	8.9	1.5	8.5	0.10	0.23	0.10	0.05	0.8
250.0	500.0	89.1	3.2	12.5	0.28	0.10	0.04	0.08	1.3
	120.0	371.3	5.1	11.9	0.57	0.05	0.02	0.12	2.4
	500.0	89.1	3.2	18.1	0.25	0.13	0.06	0.08	1.3
	520.0	85.7	3.1	13.1	0.27	0.10	0.05	0.08	1.6
330.0	930.0	47.9	2.6	12.8	0.20	0.14	0.07	0.07	1.4
	1420.0	31.3	2.2	12.8	0.16	0.16	0.08	0.06	1.2
	3100.0	14.4	1.7	13.9	0.11	0.26	0.13	0.05	1.0
	500.0	89.1	3.2	37.5	0.21	0.22	0.11	0.07	1.4
840.0	410.0	108.7	3.4	51.4	0.21	0.25	0.17	0.07	1.8
	500.0	89.1	3.2	47.2	0.20	0.26	0.18	0.07	1.8
	1160.0	38.4	2.4	36.1	0.14	0.33	0.26	0.06	1.8
	3870.0	11.5	1.6	30.6	0.08	0.53	0.45	0.04	1.4
	4990.0	8.9	1.5	30.6	0.07	0.61	0.52	0.04	1.4
	500.0	89.1	3.2	37.5	0.21	0.22	0.11	0.07	1.4

Table 5. Experimental data of minimum flow rate along with the corresponding needle diameter and electrical conductivity of the liquids used by Scheideler & Chen (2014). Also tabulated is the corresponding values of τ_e/τ_h , (F_{int}/F_{visc}) , (F_{st}/F_{visc}) and $(F_{pol}/F_{es})_{ct}$. From these values, we infer that the viscous force is the dominant resisting force and the electrostatic suction is the dominant driving force in the current transfer region. Moreover, viscous and interfacial tension forces are of the same order of magnitude at the cone-tip.

As discussed, the present work primarily focused on the steady cone-jetting of highly viscous liquids near the minimum flow rate condition. Here, the dominant resisting and supporting forces considered were the viscous and electrostatic suction forces, respectively. A natural extension of the present study will be to consider highly viscous liquids where the polarisation force dominates the electrostatic suction force.

Acknowledgements. We thank the anonymous reviewers for their constructive and insightful suggestions. These suggestions have significantly contributed to improving the overall quality of the manuscript.

Funding. S.S.B. acknowledges the financial support received from the Science and Engineering Research Board (SERB), Government of India, under the Core Research Grant scheme (Grant No. CRG/2022/002148).

Declaration of interests. The authors report no conflict of interest.

Data availability statement. All data that support the findings of this article are included within the article.

Appendix A. Verification of the assumptions based on literature

In this section, we verify whether the assumptions, made in §§ 4.1 and 4.3 for deriving the scaling laws, hold true for the experimental data of Scheideler & Chen (2014). To verify,

d_n [μm]	κ [$\mu\text{S m}^{-1}$]	Q_0 [$\times 10^{-2} \text{ nl s}^{-1}$]	Re_0 [$\times 10^{-3}$]	$\gamma d_n^2/\mu$ [nl s^{-1}]	Q_{min}^{exp} [nl s^{-1}]	Q_{min}^{theory} [nl s^{-1}]	Q_0/Re_0 [nl s^{-1}]	$d_0/Re_0^{7/8}$ [μm]	$d_0/Re_0^{3/2}$ [μm]
125.0	180.0	247.5	4.5	707.6	3.8	4.4	550.7	56.1	1644.9
	910.0	48.9	2.6		3.3	2.3	186.9	30.6	1255.6
	100.0	445.5	5.5		5.1	7.5	814.8	70.0	1814.2
150.0	500.0	89.1	3.2	1018.9	5.3	3.9	278.7	38.3	1387.3
	1000.0	44.5	2.5		4.3	3.0	175.6	29.5	1235.9
	2500.0	17.8	1.9		5.6	2.1	95.3	20.9	1060.9
	10000.0	4.4	1.2		3.9	1.2	37.8	12.4	842.1
210.0	910.0	48.9	2.6	1997.1	8.4	5.3	186.9	30.6	1255.6
	2830.0	15.7	1.8		8.3	3.4	87.7	20.0	1039.2
	4990.0	8.9	1.5		8.5	2.7	60.1	16.1	945.5
250.0	500.0	89.1	3.2	2830.3	12.5	8.9	278.7	38.3	1387.3
	120.0	371.3	5.1		11.9	24.5	721.6	65.3	1759.9
	500.0	89.1	3.2		18.1	13.9	278.7	38.3	1387.3
	520.0	85.7	3.1		13.1	13.6	271.5	37.7	1378.3
330.0	930.0	47.9	2.6	4931.6	13.3	10.8	184.2	30.3	1251.0
	1420.0	31.3	2.2		12.8	9.1	138.9	25.9	1165.8
	3100.0	14.4	1.7		13.9	6.7	82.6	19.3	1023.6
550.0	500.0	89.1	3.2	13 698.9	37.5	31.4	278.7	38.3	1387.3
	410.0	108.7	3.4		51.4	66.9	318.1	41.2	1434.0
	500.0	89.1	3.2		47.2	61.8	278.7	38.3	1387.3
840.0	1160.0	38.4	2.4	31 953.6	36.1	44.1	159.0	27.9	1205.8
	3870.0	11.5	1.6		30.6	27.3	71.2	17.8	986.4
	4990.0	8.9	1.5		30.6	24.6	60.1	16.1	945.5

Table 6. Experimental data of minimum flow rate along with the corresponding needle diameter and electrical conductivity of the liquids used by Scheideler & Chen (2014). Also tabulated is the experimental data for the minimum flow rate of steady Taylor cone-jets, alongside the corresponding values of minimum flow rate calculated based on the scaling laws provided by Scheideler & Chen (2014, see (4.15)), the present work (4.26) and Gañán-Calvo *et al.* (2013, see (4.8a)). The lower and upper limits of the needle diameter ((5.12a) and (5.12b)) for which the minimum flow rate depends on the needle diameter are also mentioned. For all the minimum flow rate data, the $\beta Re_0 < 1$ and $(Q_{min}/Q_0)Re_0 < 1$ criteria are satisfied.

we follow a similar procedure as discussed in § 5.1 and include the results in table 5. We observe that, similar to the liquids used in the present work, the liquids employed by Scheideler & Chen (2014) satisfy all assumptions based on which the scaling laws for Q_{min} and associated parameters were derived.

Appendix B. Validation of the scaling laws with the literature

In this section, we compare the predicted minimum flow rate values based on the derived scaling laws with the experimental conditions of Scheideler & Chen (2014). The experimental data provided by Scheideler & Chen (2014) and the theoretical predictions are summarised in table 6. From table 6, we infer that the theoretical scaling law for the minimum flow rate, given by (4.26), predicts well the experimental data of Scheideler & Chen (2014). Furthermore, we note that the scaling laws for Q_{min} proposed by Gañán-Calvo *et al.* (2013) and Scheideler & Chen (2014) overpredict the experimental results of Scheideler & Chen (2014), highlighting the importance of the scaling laws proposed in the present work.

REFERENCES

- ARSCOTT, S., LE GAC, S. & ROLANDO, C. 2005 A polysilicon nanoelectrospray–mass spectrometry source based on a microfluidic capillary slot. *Sensors Actuators B* **106** (2), 741–749.
- BAHGA, S.S. 2021 *Experimental Uncertainty Analysis: A Textbook for Science and Engineering Students*. White Falcon.
- BASARAN, O.A. & SCRIVEN, L.E. 1990 Axisymmetric shapes and stability of pendant and sessile drops in an electric field. *J. Colloid Interface Sci.* **140** (1), 10–30.
- BOBER, D.B. & CHEN, C.-H. 2011 Pulsating electrohydrodynamic cone-jets: from choked jet to oscillating cone. *J. Fluid Mech.* **689**, 552–563.
- CARLIER, J., ARSCOTT, S., THOMY, V., CAMART, J.-C., CREN-OLIVÉ, C. & LE GAC, S. 2005 Integrated microfabricated systems including a purification module and an on-chip nano electrospray ionization interface for biological analysis. *J. Chromatogr. A* **1071** (1–2), 213–222.
- CASTRO, S. & FERNÁNDEZ DE LA MORA, J. 2009 Effect of tip curvature on ionic emissions from Taylor cones of ionic liquids from externally wetted tungsten tips. *J. Appl. Phys.* **105** (3), 034903.
- CHEN, C.H. 2011 Electrohydrodynamic stability. In *Electrokinetics and Electrohydrodynamics in Microsystems*, pp. 177–220. Springer.
- CHO, T.H., FARJAM, N., BARTON, K. & DASGUPTA, N.P. 2024 Subtractive patterning of nanoscale thin films using acid-based electrohydrodynamic-jet printing. *Small Meth.* **8** (5), 2301407.
- CHOUHEY, A. & BAHGA, S.S. 2024 Electrohydrodynamic jet printed templates for hot embossing of microfluidic devices. *J. Micromech. Microengng* **34** (10), 105004.
- CHOUHEY, A., DUBEY, K. & BAHGA, S.S. 2023 Rapid prototyping of polydimethylsiloxane (PDMS) microchips using electrohydrodynamic jet printing: application to electrokinetic assays. *Electrophoresis* **44** (7–8), 725–732.
- CLOUPEAU, M. & PRUNET-FOCH, B. 1989 Electrostatic spraying of liquids in cone-jet mode. *J. Electrostat.* **22** (2), 135–159.
- DE LA MORA, J.F. 1996 On the outcome of the coulombic fission of a charged isolated drop. *J. Colloid Interface Sci.* **178** (1), 209–218.
- DE LA MORA, J.F. & LOSCERTALES, I.G. 1994 The current emitted by highly conducting Taylor cones. *J. Fluid Mech.* **260**, 155–184.
- EVANS, J.H. 1972 Dimensional analysis and the Buckingham pi theorem. *Am. J. Phys.* **40** (12), 1815–1822.
- FENG, J.J. 2002 The stretching of an electrified non-Newtonian jet: a model for electrospinning. *Phys. Fluids* **14** (11), 3912–3926.
- FERNANDEZ DE LA MORA, J., NAVASCUES, J., FERNANDEZ, F. & ROSELL-LLOMPART, J. 1990 Generation of submicron monodisperse aerosols in electrosprays. *J. Aerosol Sci.* **21**, S673–S676.
- FERNÁNDEZ-MARTÍNEZ, D., VEGA, E.J., GAÑÁN-CALVO, A.M. & MONTANERO, J.M. 2025 A new emitter for electrospray and electrohydrodynamic jet printing. *Phys. Fluids* **37** (4), 042010.
- GAMERO-CASTAÑO, M. & MAGNANI, M. 2019 The minimum flow rate of electrosprays in the cone-jet mode. *J. Fluid Mech.* **876**, 553–572.
- GAÑÁN-CALVO, A.M., LASHERAS, J.C., DÁVILA, J. & BARRERO, A. 1994 The electrostatic spray emitted from an electrified conical meniscus. *J. Aerosol Sci.* **25** (6), 1121–1142.
- GAÑÁN-CALVO, A.M. 1994 The size and charge of droplets in the electrospraying of polar liquids in cone-jet mode, and the minimum droplet size. *J. Aerosol Sci.* **25**, 309–310.
- GAÑÁN-CALVO, A.M. 1997a On the theory of electrohydrodynamically driven capillary jets. *J. Fluid Mech.* **335**, 165–188.
- GAÑÁN-CALVO, A.M. 1997b Cone-jet analytical extension of Taylor’s electrostatic solution and the asymptotic universal scaling laws in electrospraying. *Phys. Rev. Lett.* **79** (2), 217.
- GAÑÁN-CALVO, A.M. 1999 The surface charge in electrospraying: its nature and its universal scaling laws. *J. Aerosol Sci.* **30** (7), 863–872.
- GAÑÁN-CALVO, A.M. & LÓPEZ-GARCÍA, P. 2025 Taylor cone film-jet electrospray. *Phys. Fluids* **37** (10), 102016.
- GAÑÁN-CALVO, A.M., LÓPEZ-HERRERA, J.M., HERRADA, M.A., RAMOS, A. & MONTANERO, J.M. 2018 Review on the physics of electrospray: from electrokinetics to the operating conditions of single and coaxial Taylor cone-jets, and AC electrospray. *J. Aerosol Sci.* **125**, 32–56.
- GAÑÁN-CALVO, A.M. 2004 On the general scaling theory for electrospraying. *J. Fluid Mech.* **507**, 203–212a.
- GAÑÁN-CALVO, A.M. 2025 The spatial structure of jets from steady Taylor cone-jets. *J. Fluid Mech.* **1023**, A49.
- GAÑÁN-CALVO, A.M., REBOLLO-MUÑOZ, N. & MONTANERO, J.M. 2013 The minimum or natural rate of flow and droplet size ejected by Taylor cone–jets: physical symmetries and scaling laws. *New J. Phys.* **15** (3), 033035.

- HIGUERA, F.J. 2003 Flow rate and electric current emitted by a Taylor cone. *J. Fluid Mech.* **484**, 303–327.
- KU, B.K. & KIM, S.S. 2003 Electrohydrodynamic spraying characteristics of glycerol solutions in vacuum. *J. Electrostat.* **57** (2), 109–128.
- KU, B.K., KIM, S.S., KIM, Y.D. & LEE, S.Y. 2001 Direct measurement of electrospray droplets in submicron diameter using a freezing method and a TEM image processing technique. *J. Aerosol Sci.* **32** (12), 1459–1477.
- FERNÁNDEZ DE LA MORA, J. 2007 The fluid dynamics of Taylor cones. *Annu. Rev. Fluid Mech.* **39** (1), 217–243.
- LEE, A., JIN, H., DANG, H.-W., CHOI, K.-H. & AHN, K.H. 2013 Optimization of experimental parameters to determine the jetting regimes in electrohydrodynamic printing. *Langmuir* **29** (44), 13630–13639.
- LOEB, L.B., KIP, A.F., HUDSON, G.G. & BENNETT, W.H. 1941 Pulses in negative point-to-plane corona. *Phys. Rev.* **60** (10), 714.
- LÓPEZ-HERRERA, J.M., HERRADA, M.A. & GAÑÁN-CALVO, A.M. 2023 Electrokinetic modelling of cone-jet electrosprays. *J. Fluid Mech.* **964**, A19.
- LOZANO, P. & MARTÍNEZ-SÁNCHEZ, M. 2005 On the dynamic response of externally wetted ionic liquid ion sources. *J. Phys. D* **38** (14), 2371.
- MELCHER, J.R. & WARREN, E.P. 1971 Electrohydrodynamics of a current-carrying semi-insulating jet. *J. Fluid Mech.* **47** (1), 127–143.
- NGANGUIA, H., YOUNG, Y.-N., LAYTON, A.T., LAI, M.-C. & HU, W.-F. 2016 Electrohydrodynamics of a viscous drop with inertia. *Phys. Rev. E* **93** (5), 053114.
- ONSES, M.S., SUTANTO, E., FERREIRA, P.M., ALLEYNE, A.G. & ROGERS, J.A. 2015 Mechanisms, capabilities, and applications of high-resolution electrohydrodynamic jet printing. *Small* **11** (34), 4237–4266.
- PANTANO, C., GAÑÁN-CALVO, A.M. & BARRERO, A. 1994 Zeroth-order, electrohydrostatic solution for electrospraying in cone-jet mode. *J. Aerosol Sci.* **25** (6), 1065–1077.
- PONCE-TORRES, A., REBOLLO-MUÑOZ, N., HERRADA, M.A., GAÑÁN-CALVO, A.M. & MONTANERO, J.M. 2018 The steady cone-jet mode of electrospraying close to the minimum volume stability limit. *J. Fluid Mech.* **857**, 142–172.
- QIAN, L., LAN, H. & ZHANG, G. 2018 A theoretical model for predicting the feature size printed by electrohydrodynamic jet printing. *Appl. Phys. Lett.* **112** (20), 203505.
- ROSELL-LLOMPART, J., GRIFOLL, J. & LOSCERTALES, I.G. 2018 Electrosprays in the cone-jet mode: from Taylor cone formation to spray development. *J. Aerosol Sci.* **125**, 2–31.
- SAVILLE, D.A. 1997 Electrohydrodynamics: the Taylor–Melcher leaky dielectric model. *Annu. Rev. Fluid Mech.* **29** (1), 27–64.
- SCHEIDELER, W.J. & CHEN, C.H. 2014 The minimum flow rate scaling of Taylor cone-jets issued from a nozzle. *Appl. Phys. Lett.* **104** (2), 024103.
- SINGH, A.K., CHOUBEY, A., SRIVASTAVA, R.K. & BAHGA, S.S. 2023 Physics of moderately stretched electrified jets in electrohydrodynamic jet printing. *Phys. Rev. E* **107** (4), 045103.
- SINGH, A.K., SRIVASTAVA, R.K. & BAHGA, S.S. 2022 Regimes of steady jetting in electrohydrodynamic jet printing. *Phys. Rev. Fluids* **7** (6), 063701.
- SMITH, D.P.H. 1986 The electrohydrodynamic atomization of liquids. *IEEE Trans. Ind. Applics* **3**, 527–535.
- TAYLOR, G.I. 1964 Disintegration of water drops in an electric field. *Proc. R. Soc. Lond. A* **280** (1382), 383–397.
- TIAN, J., LI, J., SAURET, A., KONG, T., WU, X., LU, Y. & SHUM, H.C. 2019 Facile control of liquid-rope coiling with tunable electric field configuration. *Phys. Rev. Appl.* **12** (1), 014034.
- TURNBULL, R.J. 2002 Self-acceleration of a charged jet. *IEEE Trans. Ind. Applics* **25** (4), 699–704.
- YARIN, A.L. 2003 *Electrospinning of Nanofibers From Polymer Solutions and Melts*. Institute of Fundamental Technological Research.
- YARIN, A.L., KOOMBHONGSE, S. & RENEKER, D.H. 2001 Taylor cone and jetting from liquid droplets in electrospinning of nanofibers. *J. Appl. Phys.* **90** (9), 4836–4846.
- ZELENY, J. 1914 The electrical discharge from liquid points, and a hydrostatic method of measuring the electric intensity at their surfaces. *Phys. Rev.* **3**, 69–91.
- ZELENY, J. 1935 The role of surface instability in electrical discharges from drops of alcohol and water in air at atmospheric pressure. *J. Franklin Inst.* **219** (6), 659–675.