

Countability of Sets

- Sets
- relations
- functions
- One-to-one correspondence
- Countable
- Uncountable
- Infinite number of infinite sets of different sizes
- Continuum hypothesis

Set: is a collection of well defined objects

Example: $\{1, 3, 4, 6, 8\}$

Example: $\{1, 2, 3, \dots, 66\}$ or $\{2, 4, 6, 8, \dots\}$

Example: $\{x : x \text{ is an even positive integer}\}$ which we read as:

the set of x such that x is an even positive integer

Example: $\{x : x \text{ is a prime number less than a million}\}$ which we read as:

The set of x such that x is a prime number less than a million

Cartesian Product

- **Definition:** Let A and B be two sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid (a \in A) \wedge (b \in B) \}$$

- The Cartesian product is also known as the **cross product**
- **Definition:** A subset of a Cartesian product, $R \subseteq A \times B$ is called a **relation**.
- Note: $A \times B \neq B \times A$ unless $A = \emptyset$ or $B = \emptyset$ or $A = B$. Find a counter example to prove this.

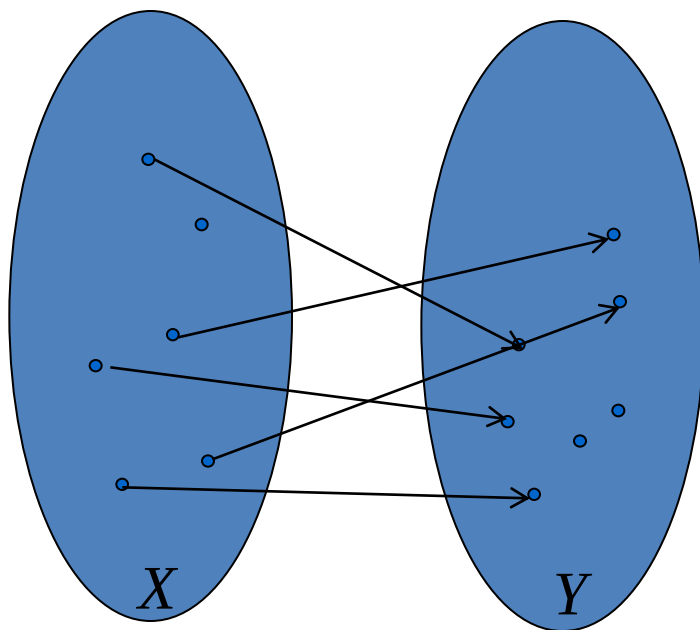
Cartesian Product

- Cartesian Products can be generalized for any n-tuple
- **Definition:** The Cartesian product of n sets, $A_1, A_2, \dots, A_n$, denoted $A_1 \times A_2 \times \dots \times A_n$, is
$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \dots, n \}$$

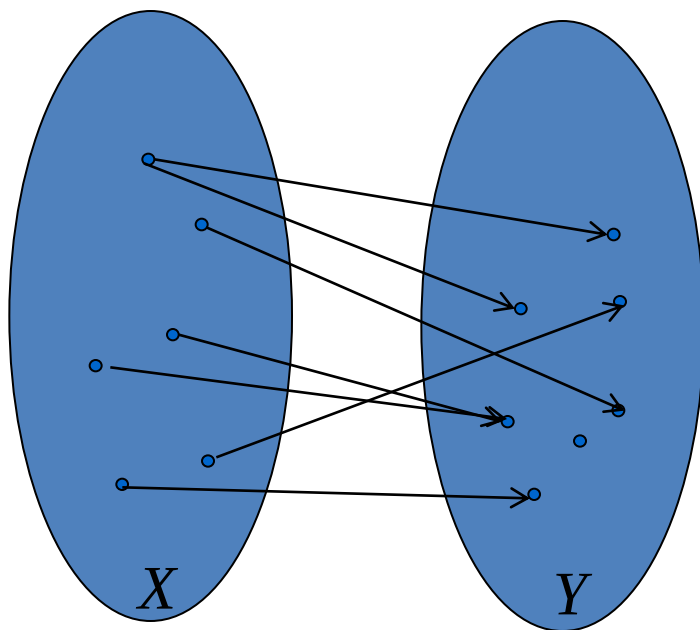
Relations and Functions

- A relation R from A to B is a subset of $A \times B$
- A function from A to B is a relation that associates every elements of A to one and only one element of B .

Is this a Function? (I)

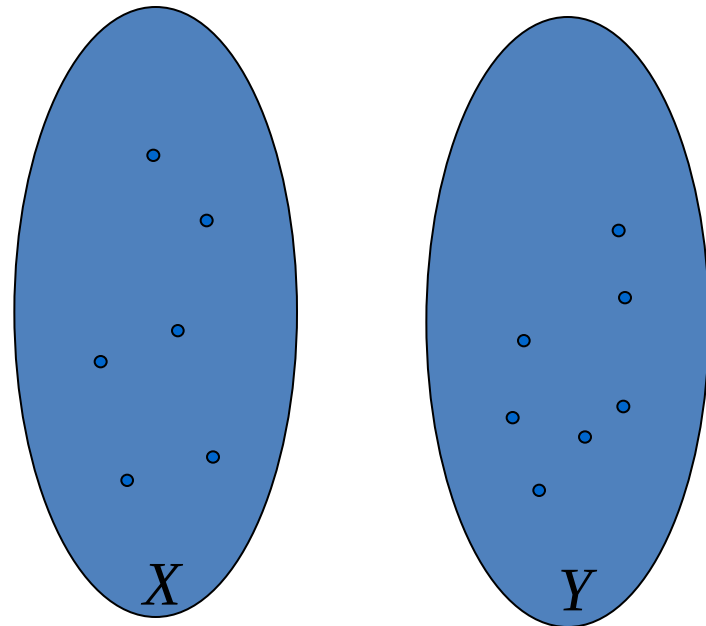


Is this a Function? (II)



One-to-One Functions

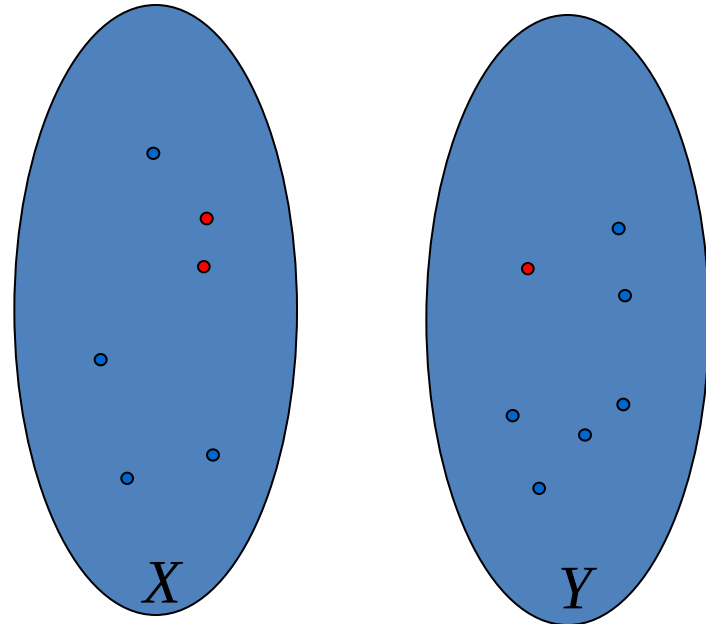
$f : X \rightarrow Y$ is **1-1 (injective)** if
for each $y \in Y$, there is at most one $x \in X$
such that $f(x) = y$



Equivalent Criteria

For $x_1, x_2 \in X$,

if $f(x_1) = f(x_2)$ then $x_1 = x_2$



Example 1

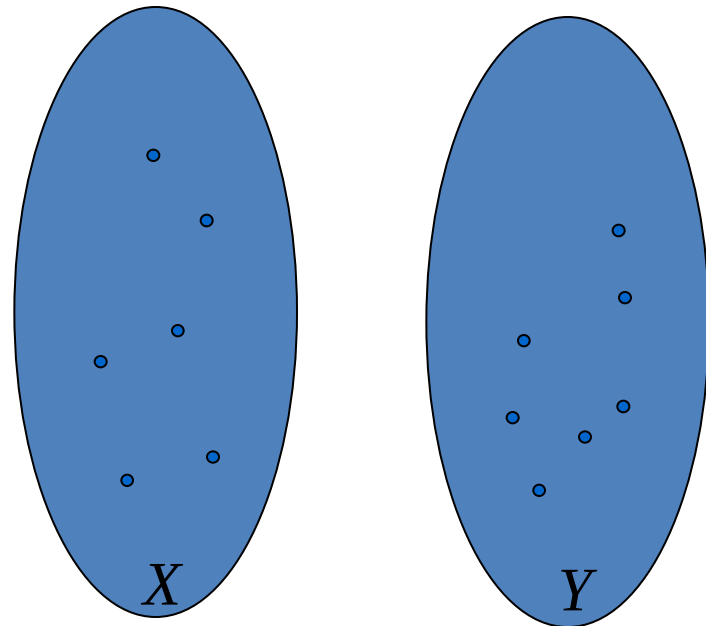
Determine if the given function is injective. Prove your answer.

$$f : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$f(n) = 3n - 1$$

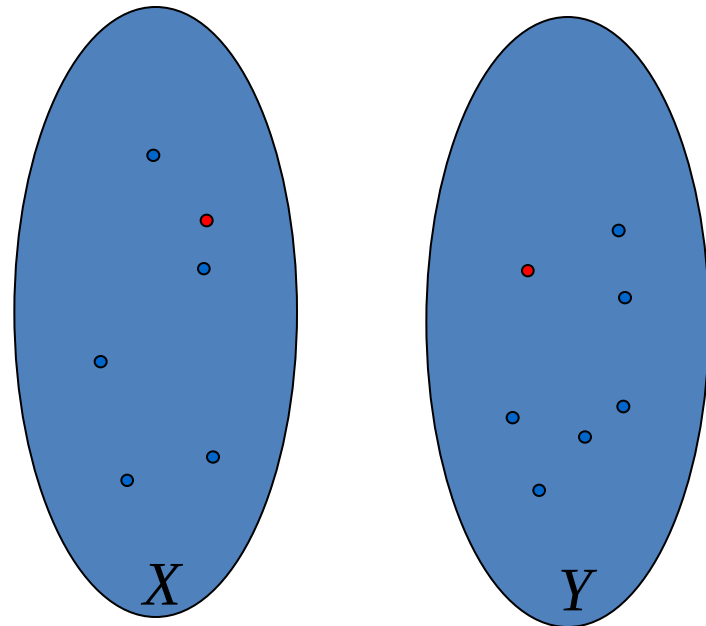
Onto Functions

$f : X \rightarrow Y$ is **onto (surjective)** if
the range of f is Y .



Equivalent Criteria

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y$$



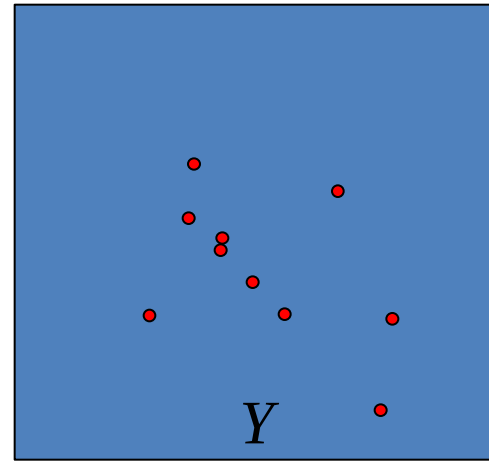
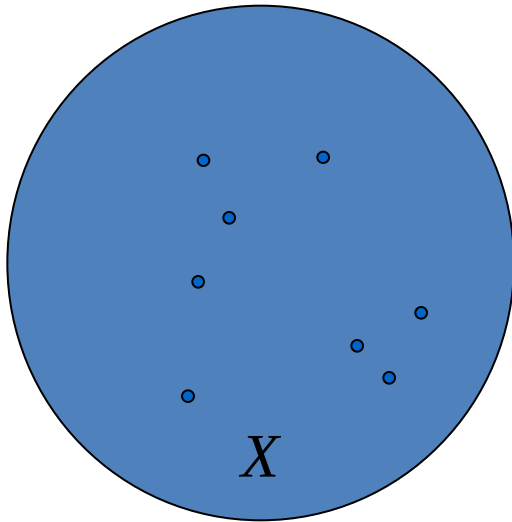
Example 2

Determine if the given function is surjective. Prove your answer.

$$f : \mathbb{Z} \rightarrow \mathbb{Z}$$

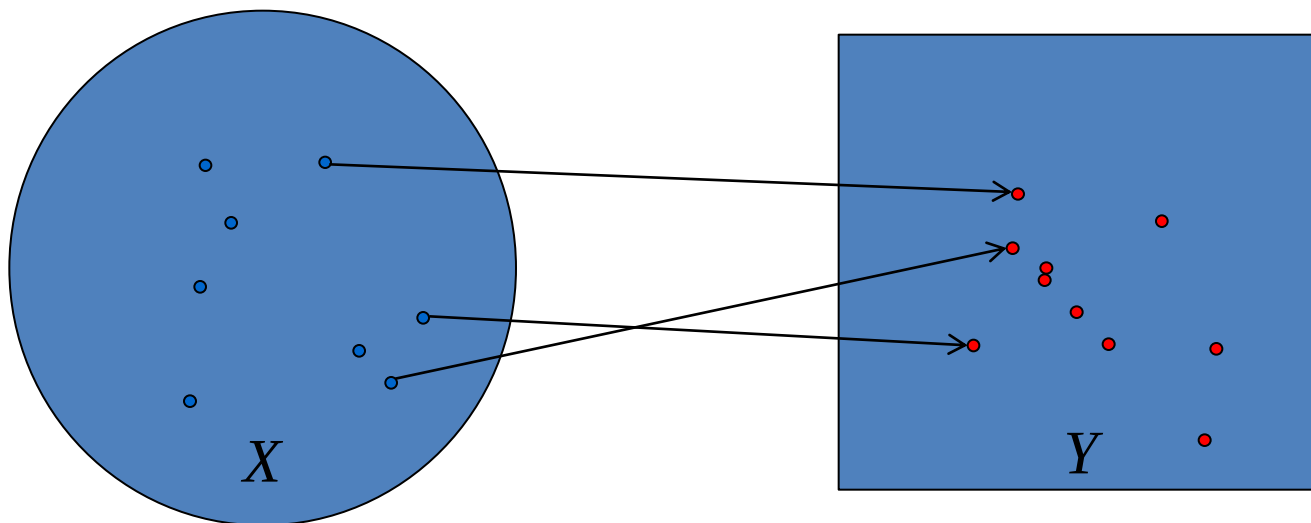
$$f(n) = 3n - 1$$

Counting Problems...



$$|X| \stackrel{?}{=} |Y|$$

Counting Problems...



$$|X| \stackrel{?}{=} |Y|$$

Bijections

$f : X \rightarrow Y$ is **bijective** if it is both 1-1 and onto.

Inverse Functions

If $f : X \rightarrow Y$ is **bijective**

then its inverse function $f^{-1} : Y \rightarrow X$ exists

and is also bijective.

Equivalent Sets

Two sets A and B are *equivalent* if there is a 1-1 function from A onto B .

Example 3

The set of odd integers (O) and even integers (E) are equivalent.

Plan:

1. Define a function from O to E .
2. Show that the function is well defined.
3. Show that the function is bijective.

Countable Sets

A set A is *countable* if it is either finite or $\mathbb{N}$ is equivalent to A .

One-to-one correspondence

Two sets $\mathbf{M}$ and $\mathbf{N}$ are equivalent ... if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.

If $\mathbf{M}$ and $\mathbf{N}$ are *equivalent* we often say that they have the same cardinality

If $\mathbf{M}$ and $\mathbf{N}$ are finite this means they have the same number of elements

But what about the case when $\mathbf{M}$ and $\mathbf{N}$ are infinite?

Countable

$\mathbf{N} = \{1, 2, 3, \dots\}$ the set of natural numbers

$\mathbf{E} = \{2, 4, 6, \dots\}$ the set of even natural numbers

N:	1	2	3	4	5	...	n	...
								
E:	2	4	6	8	10	...	2n	...

Countable

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N:	1	2	3	4	5	...	n	...
								
E:	2	4	6	8	10	...	2n	...

A set is *infinite* if it can be put into one-to-one correspondence with a proper subset of itself.
A *proper* subset does not contain all the elements of the set.

Countable

$\mathbf{N} = \{1, 2, 3, \dots\}$ the set of natural numbers

$\mathbf{E} = \{2, 4, 6, \dots\}$ the set of even natural numbers

N:	1	2	3	4	5	...	n	...
								
E:	2	4	6	8	10	...	2n	...

$\mathbf{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$ the set of all integers

N:	1	2	3	4	5	6	7	8	9	...
										
Z:	0	1	-1	2	-2	3	-3	4	-4	...

Any set that could be put into one-to-one correspondence with $\mathbf{N}$ is called *countably infinite* or *denumerable*

The symbol he chose to denote the size of a countable set was $\aleph_0$ which is read as aleph-nought or aleph-null.

It is named after the first letter of the Hebrew alphabet.

Cardinality of $\mathbf{E} = \text{cardinality of } \mathbf{Z} = \text{cardinality of } \mathbf{N} = \aleph_0$

The positive rationals are countable

the first row lists the integers,
the second row lists the ‘halves’,
the third row the thirds
the fourth row the quarters
and so on.

We then ‘snake around’ the
diagonals of this array of numbers,
deleting any numbers that we have
seen before: this gives the list

$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, 1\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, 2\frac{1}{2}, \dots$

This list contains all the positive fractions, so the positive fractions are
countable.

1	2	3	4	5	6	...
$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	...
$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	...
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	...
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$1\frac{1}{5}$	...

The Reals

We will prove that the set of real numbers in the interval from 0 up to 1 is not countable. We use *proof by contradiction*

Suppose they are *countable* then we can create a list like

1	$\leftrightarrow$	$x_1 = 0.256173\dots$
2	$\leftrightarrow$	$x_2 = 0.654321\dots$
3	$\leftrightarrow$	$x_3 = 0.876241\dots$
4	$\leftrightarrow$	$x_4 = 0.60000\dots$
5	$\leftrightarrow$	$x_5 = 0.67678\dots$
6	$\leftrightarrow$	$x_6 = 0.38751\dots$
.	.	.
.	.	.
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5 \dots a_n \dots$
.	.	.
.	.	.

1	$\leftrightarrow$	$x_1 = 0.256173\dots$
2	$\leftrightarrow$	$x_2 = 0.654321\dots$
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n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5\dots a_n\dots$
.	.	.
.	.	.

Construct the number

$$b = 0.b_1b_2b_3b_4b_5\dots$$

Choose

b_1 not equal to 2 say 4

b_2 not equal to 5 say 7

b_3 not equal to 6 say 8

b_4 not equal to 0 say 3

b_5 not equal to 8 say 7

b_n not equal to a_n

Then $b = 0.b_1b_2b_3b_4b_5 \dots = 0.47837\dots$ is **NOT** in the list
The reals are uncountable!

1	$\leftrightarrow$	$x_1 = 0.256173\dots$
2	$\leftrightarrow$	$x_2 = 0.654321\dots$
3	$\leftrightarrow$	$x_3 = 0.876241\dots$
4	$\leftrightarrow$	$x_4 = 0.60000\dots$
5	$\leftrightarrow$	$x_5 = 0.67678\dots$
6	$\leftrightarrow$	$x_6 = 0.38751\dots$
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5 \dots a_n \dots$
.	.	.
.	.	.

Construct the number

$$b = 0.b_1b_2b_3b_4b_5 \dots$$

Choose

b_1 not equal to 2 say is 4

b_2 not equal to 2 say is 7

b_3 not equal to 2 say is 8

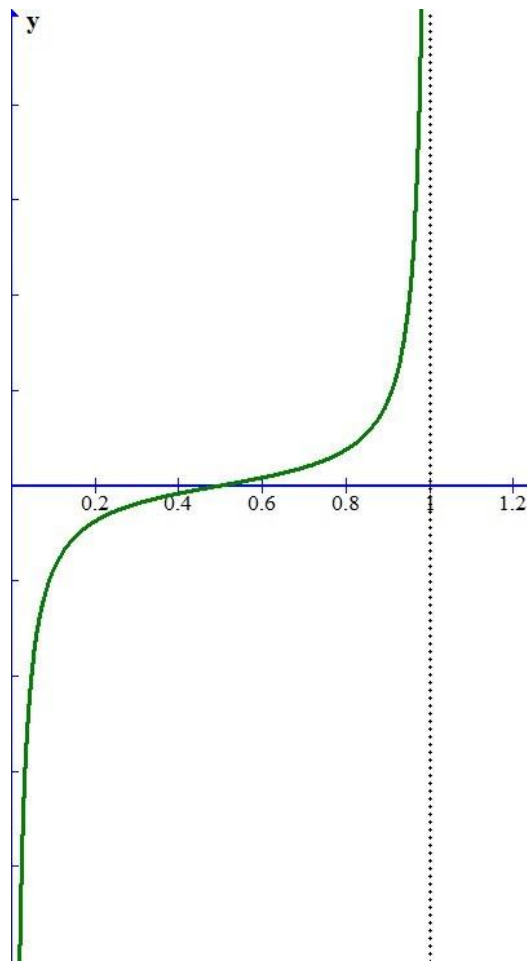
b_4 not equal to 2 say is 3

b_5 not equal to 2 say is 7

b_n not equal to a_n

The cardinality of the reals is the same as that of the interval of the reals between 0 and 1

The cardinality of the reals is often denoted by $\mathfrak{c}$ for the continuum of real numbers.



$$y = \frac{2x-1}{x-x^2}$$

The rationals can be thought of as precisely the collection of decimals which terminate or repeat e.g.

$$5/4 = 1.25000000 \dots$$

$$17/7 = 2.428571428571428571 \dots$$

$$-133/990 = -0.134343434 \dots$$

The decimal expansion of a fraction must terminate or repeat because when you divide the bottom integer into the top one there are only a limited number of remainders you can get.

1/7 starts with

0.1	remainder 3 then
0.14	remainder 2 then
0.142	remainder 6 then
0.1428	remainder 4 then
0.14285	remainder 5 then
0.142857	remainder 1 which we have

had **before** at the start so process **repeats**

A repeating decimal is a fraction e.g.
Consider $x = 0.123123123123 \dots$

This has a repeating block of length 3

Multiply by 10^3 to get

$$1000x = 123.123123123 \dots$$

Subtract x

$$x = 0.123123123 \dots$$

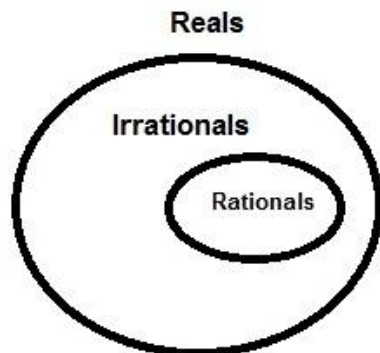
$$999x = 123$$

$$x = 123/999 = 41/333$$

The *irrationals* are those real numbers which are *not* rational
So their decimal expansions do not terminate or repeat

Cardinality of some sets

Set	Description	Cardinality
Natural numbers	1, 2, 3, 4, 5, ...	$\aleph_0$
Integers	..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...	$\aleph_0$
Rational numbers or fractions	All the decimals which terminate or repeat	$\aleph_0$
Irrational numbers	All the decimals which do not terminate or repeat	$\mathfrak{c}$
Real numbers	All decimals	$\mathfrak{c}$



Some Results

- Theorem 1:
 - Countable Union of Countable sets is countable
 - The set of all C programs is countable
 - The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.
 - There are functions which cannot be computed by a C program

Power set of a set

Given a set $\mathbf{A}$, the *power set* of $\mathbf{A}$, denoted by $P[\mathbf{A}]$, is the set of all subsets of $\mathbf{A}$.

$$\mathbf{A} = \{a, b, c\}$$

Then $\mathbf{A}$ has eight $= 2^3$ subsets and the power set of $\mathbf{A}$ is the set containing these eight subsets.

$$P[\mathbf{A}] = \{ \{ \}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$$

$\{ \}$ is the empty set and if a set has n elements it has 2^n subsets.

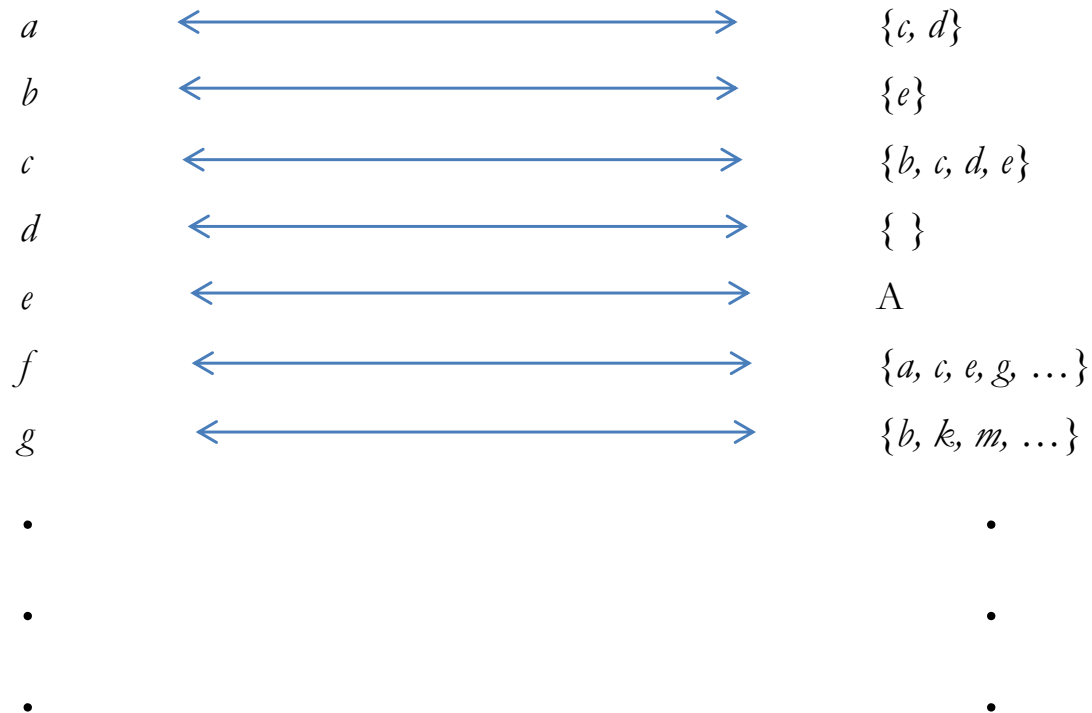
The power set is itself a set

No set can be placed in one-to-one correspondence with its power set

Elements of A

Elements of $P[A]$

(i.e. subsets of A)

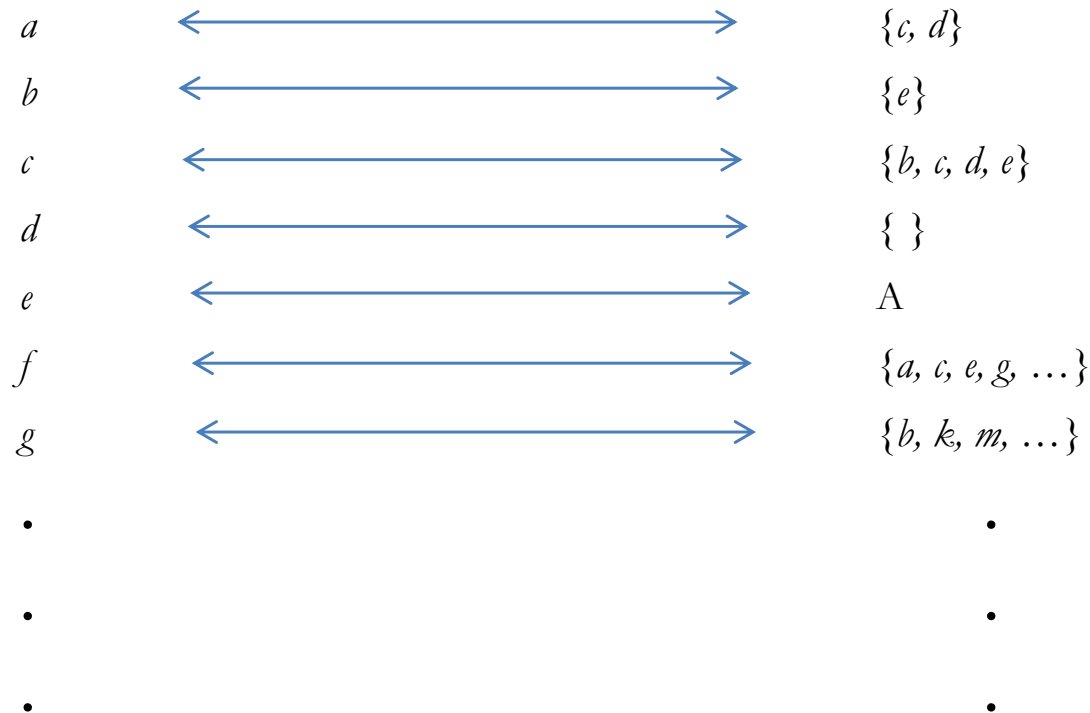


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Elements of A

Elements of $P[A]$

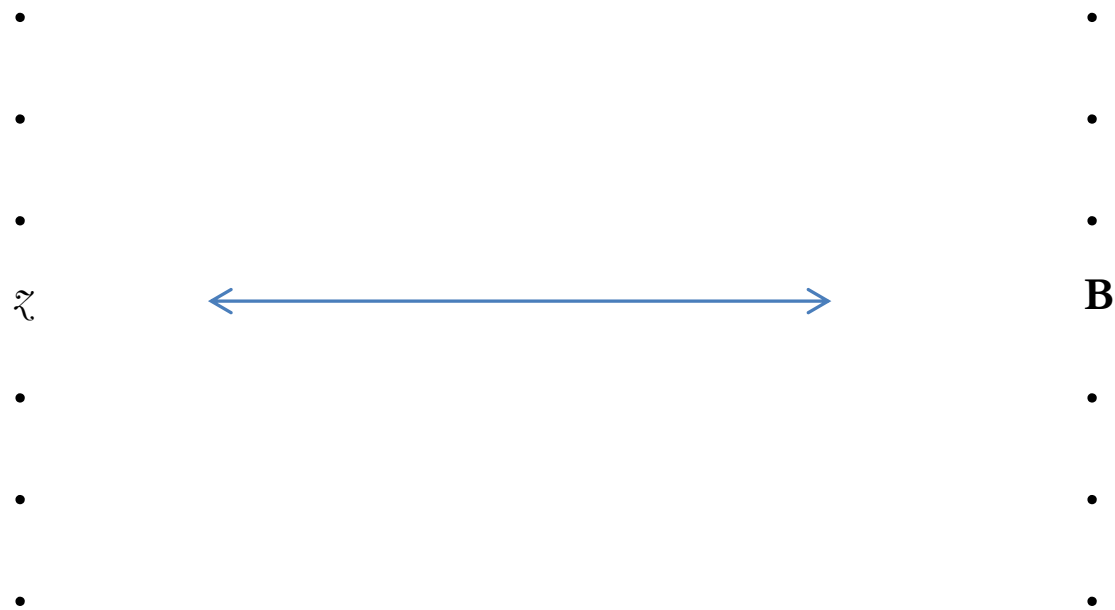
(i.e. subsets of A)



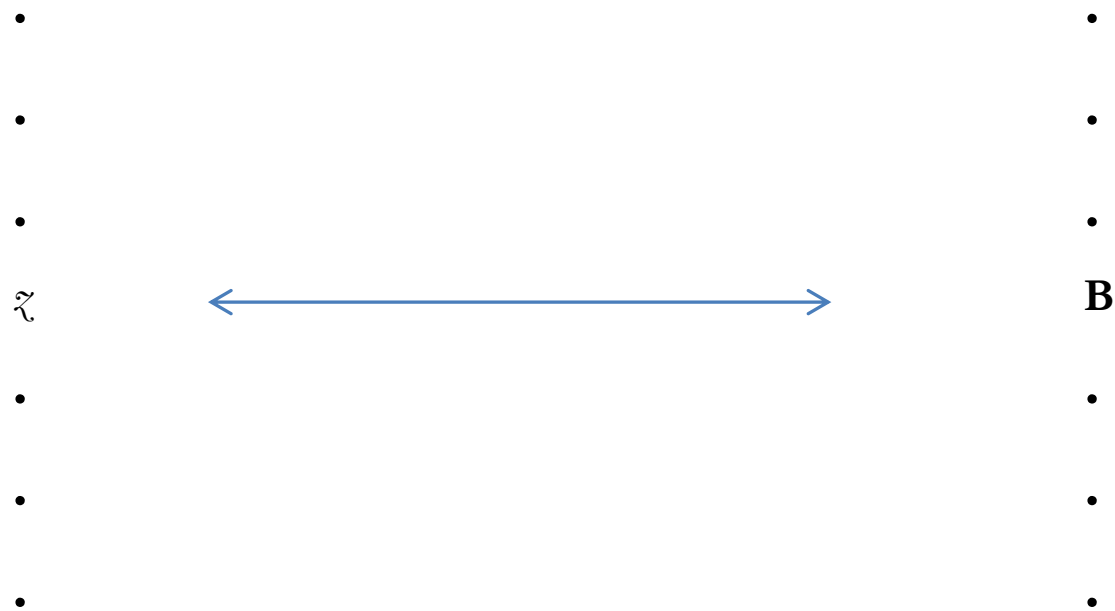
B is the set of each and every element of the original set A that is not a member of the subset with which it is matched.

$$\mathbf{B} = \{a, b, d, f, g, \dots\}$$

Now **B** is just a subset of **A** so must appear somewhere in the right-hand column and so is matched with some element of **A** say z



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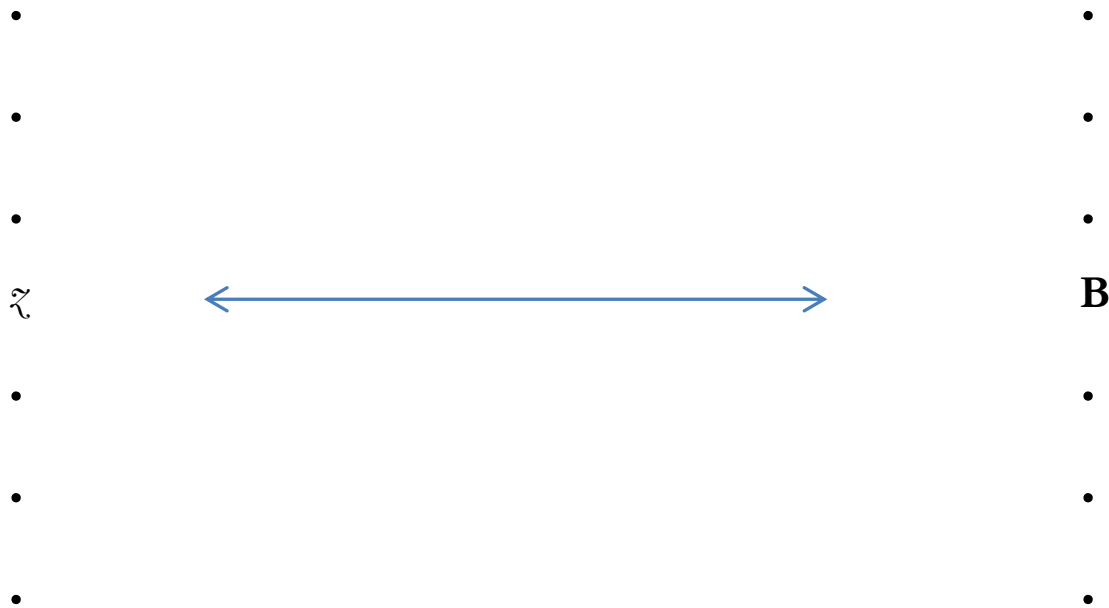


Is z an element of **B**?

Case 1: Suppose z is an element of **B**

Then z satisfies the defining property of **B** which is that it consists of elements which do not belong to their matching subset so z does not belong to **B**!

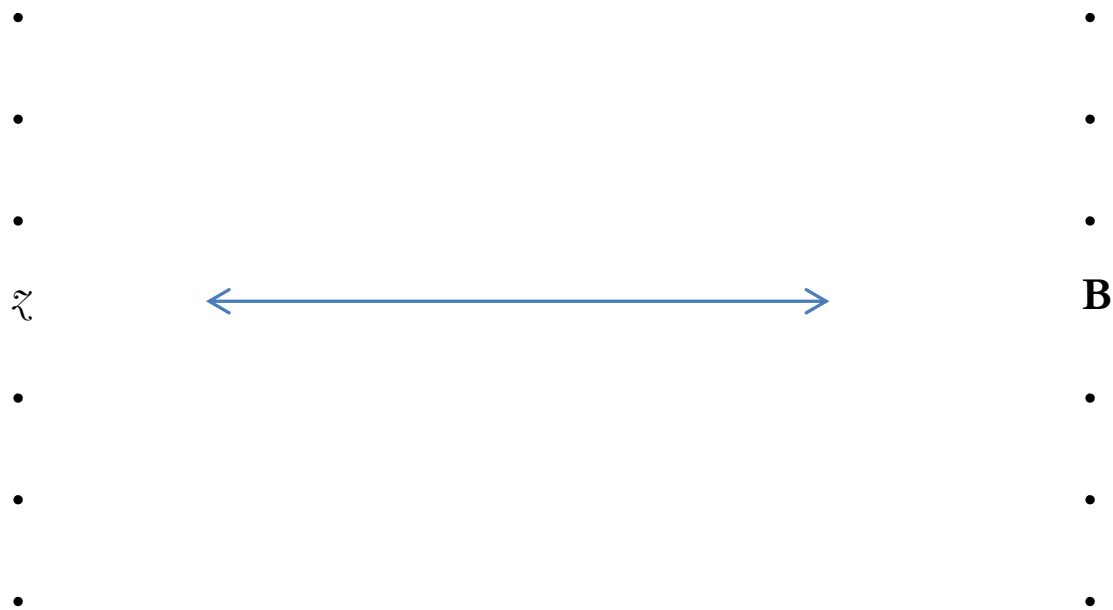
Contradiction



Case 2: Suppose z *is not* an element of **B**

Then z satisfies the defining property of **B** which is that it consists of elements which do not belong to their matching subset so z does belong to **B**!

Contradiction!



Infinity of infinities

Reals have smaller cardinality than the power set of the reals.

Which is smaller than the power set of the power set of the reals

Which is smaller than the power set of the power set of the power set of the reals

Which is smaller than the power set of the power set of the power set of the power set of the reals

etc!

$$\mathfrak{c} = 2^{\aleph_0}$$

Indeed we can show that the reals have the cardinality of the power set of the natural numbers which is often written as above

Continuum hypothesis

The Continuum hypothesis states:

there is no transfinite cardinal falling *strictly*
between $\aleph_0$ and c

Work of Gödel (1940) and of Cohen (1963)
together implied that the continuum hypothesis
was independent of the other axioms of set
theory