Cut vertices, Cut Edges and Biconnected components

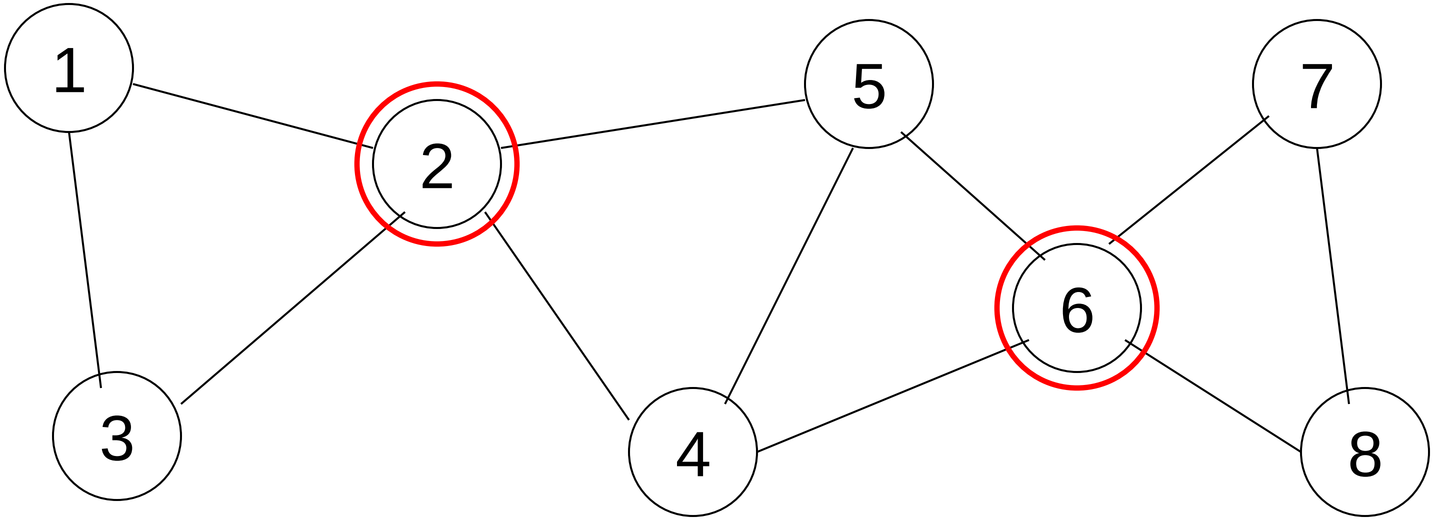
MTL776 Graph algorithms

Articulation points, Bridges, Biconnected Components

- Let G = (V;E) be a connected, undirected graph.
- An ***articulation point*** of G is a vertex whose removal disconnects G.
- A ***bridge*** of G is an edge whose removal disconnects G.
- A ***biconnected component*** of G is a maximal set of edges such that any two edges in the set lie on a common simple cycle

- These concepts are important because they can be used to identify vulnerabilities of networks

Articulation points – Example



How to find all articulation points ?

- **Brute-force approach**: one by one remove all vertices and see if removal of a vertex causes the graph to disconnect:

```
For every vertex v, do :
    Remove v from graph
    See if the graph remains connected (use BFS or DFS)
    If graph is disconnected, add v to AP list
    Add v back to the graph
```

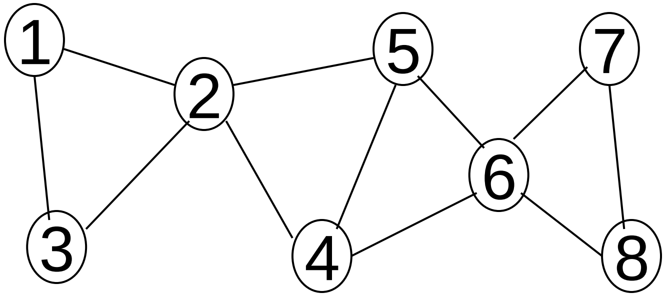
- Time complexity of above method is O(n*(n+m)) for a graph represented using adjacency list.
- Can we do better?

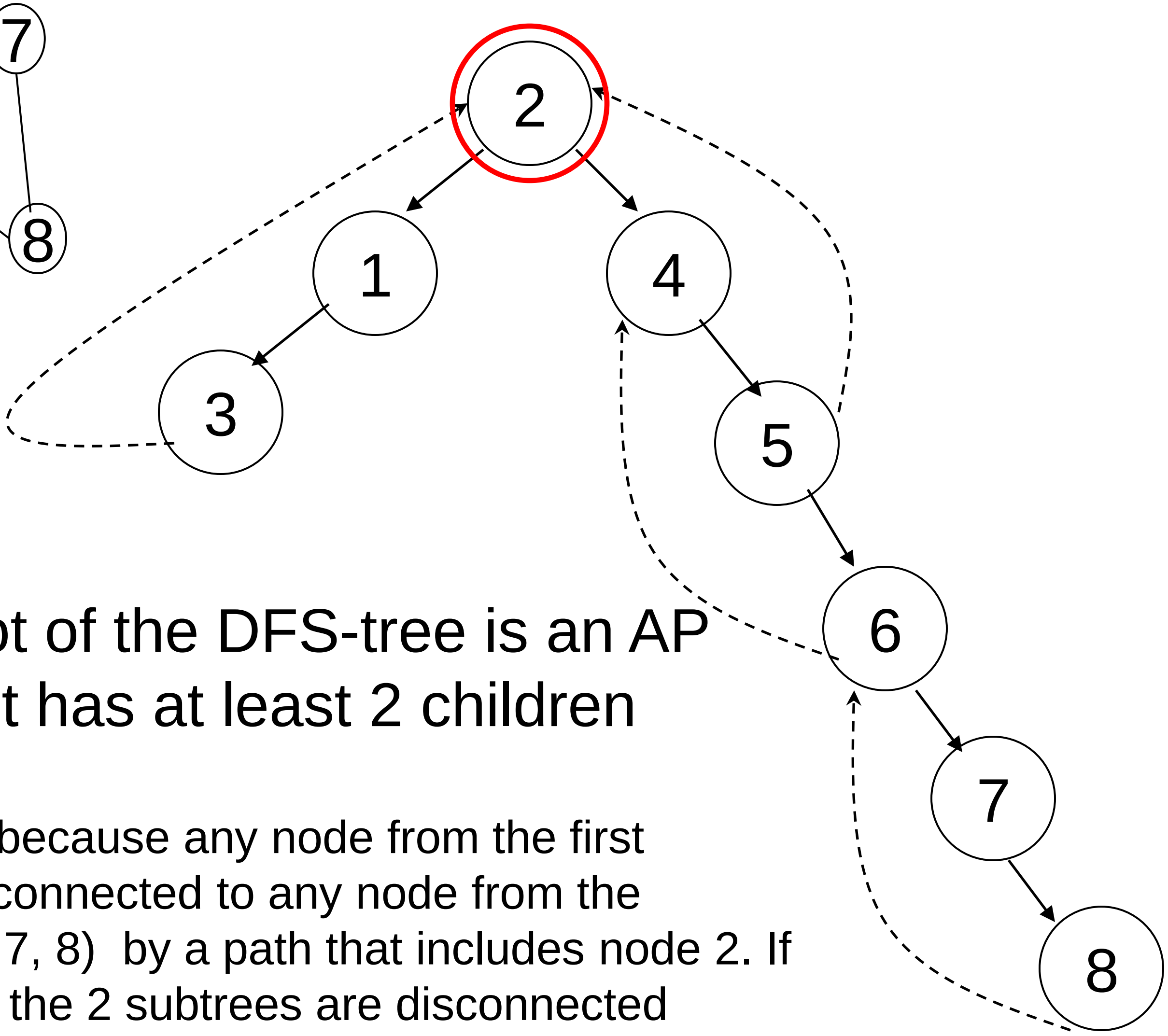
How to find all articulation points ?

- **DFS- based-approach**:
- We can prove following properties:

1. The root of a DFS-tree is an articulation point if and only if it has at least two children.
2. A nonroot vertex v of a DFS-tree is an articulation point of G if and only if has a child s such that there is no back edge from s or any descendant of s to a proper ancestor of v.
3. Leafs of a DFS-tree are never articulation points

Finding articulation points by DFS

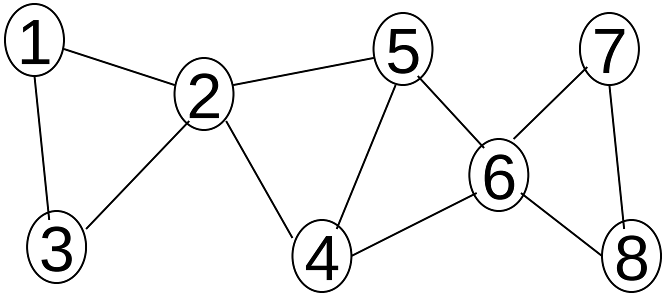


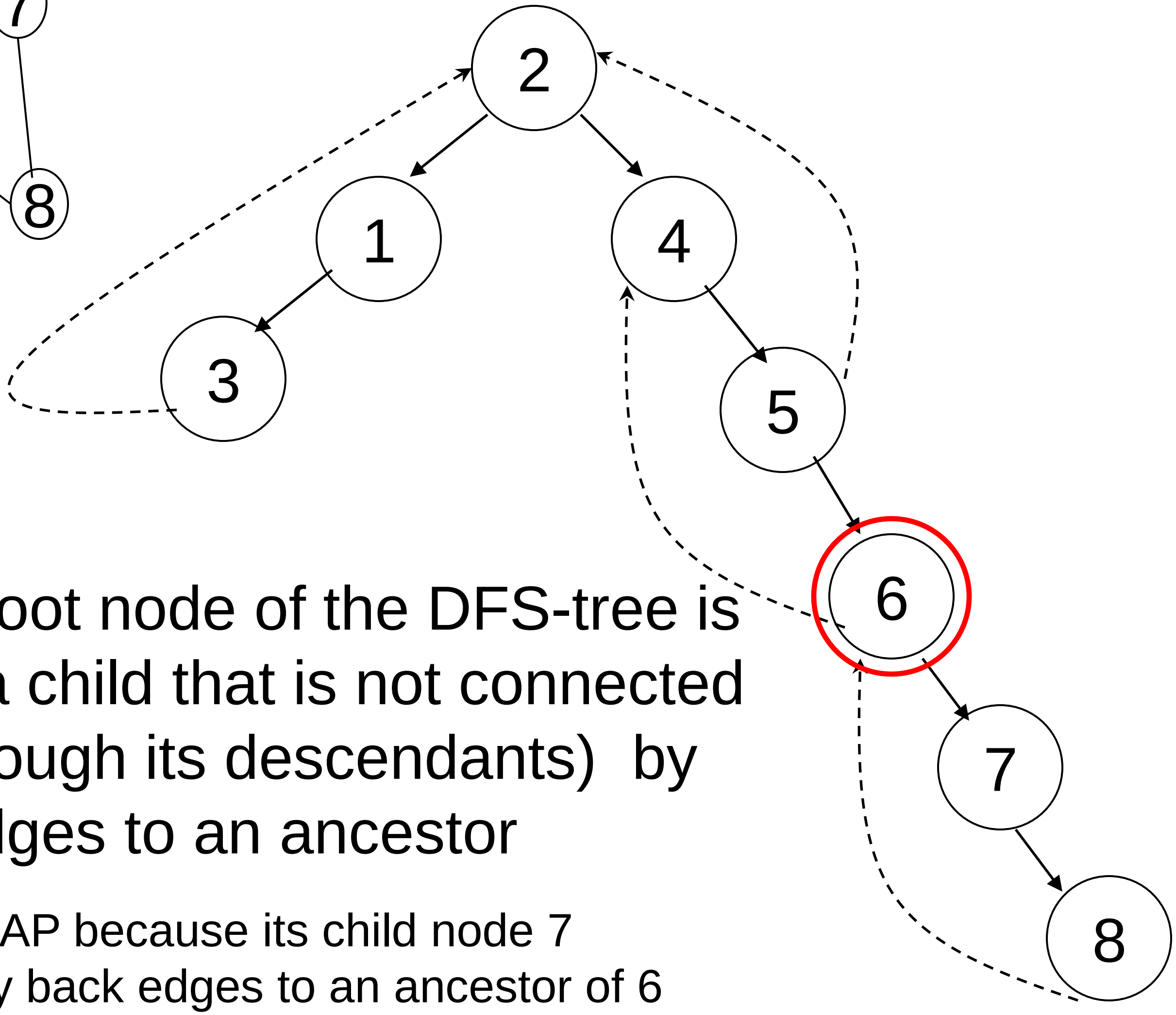


Case 1: The root of the DFS-tree is an AP
if and only if it has at least 2 children

Node 2 is an AP because any node from the first subtree (1, 2) is connected to any node from the second subtree (4, 5, 6, 7, 8) by a path that includes node 2. If node 2 is removed, the 2 subtrees are disconnected

Finding articulation points by DFS

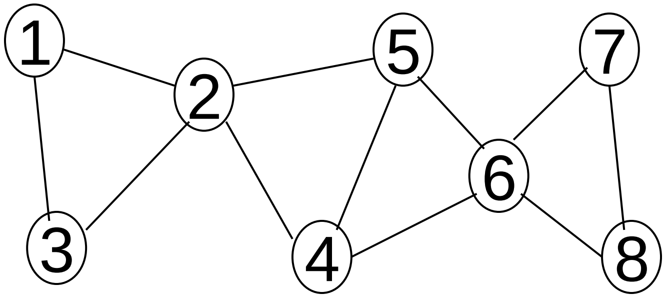


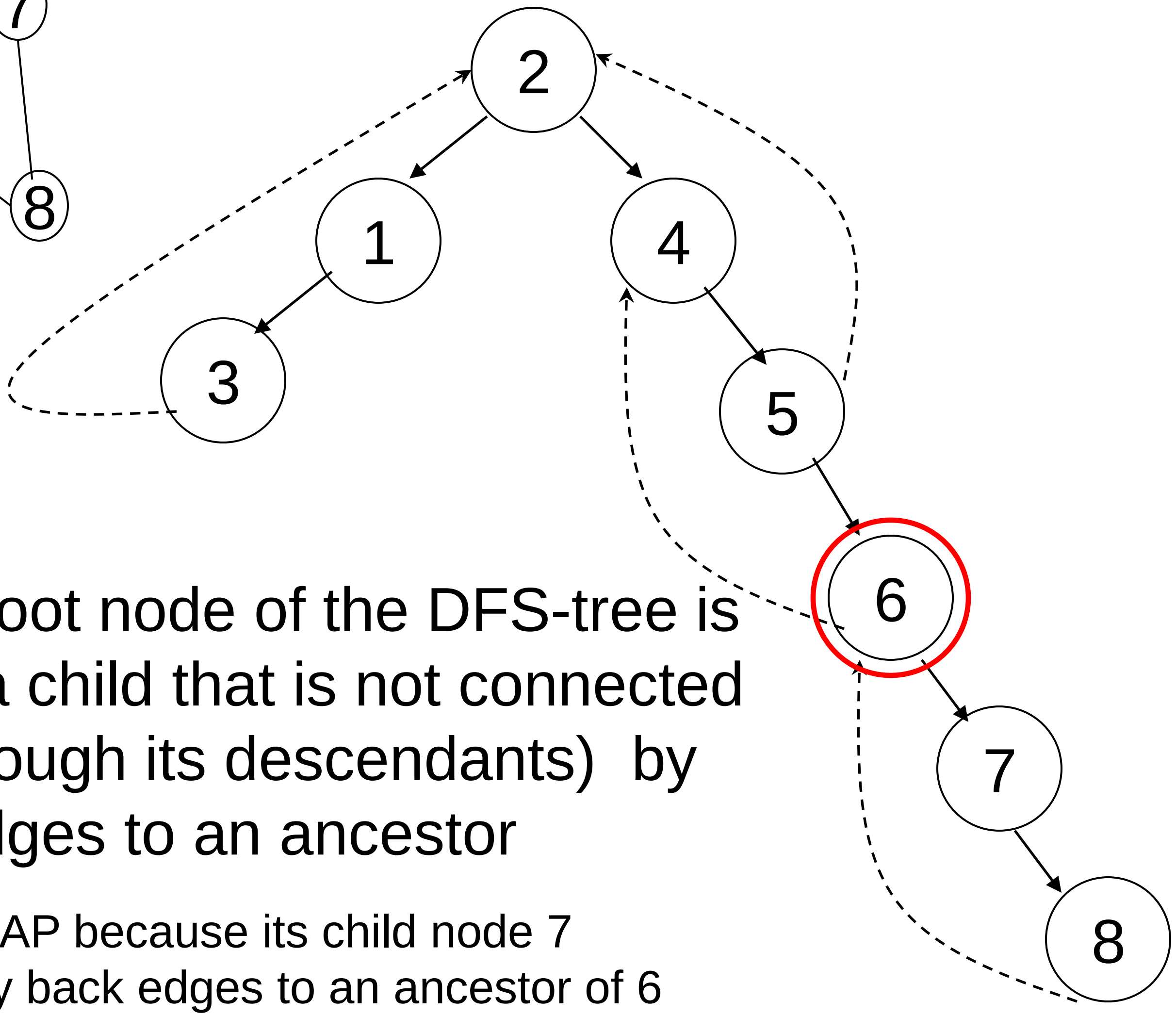


Case 2: A non-root node of the DFS-tree is an AP if it has a child that is not connected (directly or through its descendants) by back edges to an ancestor

Node 6 is an AP because its child node 7
is not connected by back edges to an ancestor of 6

Finding articulation points by DFS





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Node 6 is an AP because its child node 7
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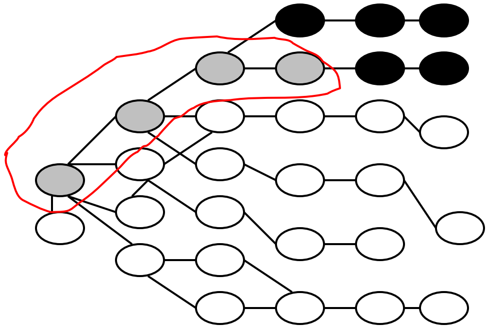
Depth-First Search: Revisited

- *Depth-first search* is another strategy for exploring a graph
  - Explore “deeper” in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*’s edges have been explored, backtrack to the vertex from which *v* was discovered

Depth-First Search

- Vertices initially colored white
- Then colored gray when discovered
- Then black when their exploration is finished



Depth-First Search

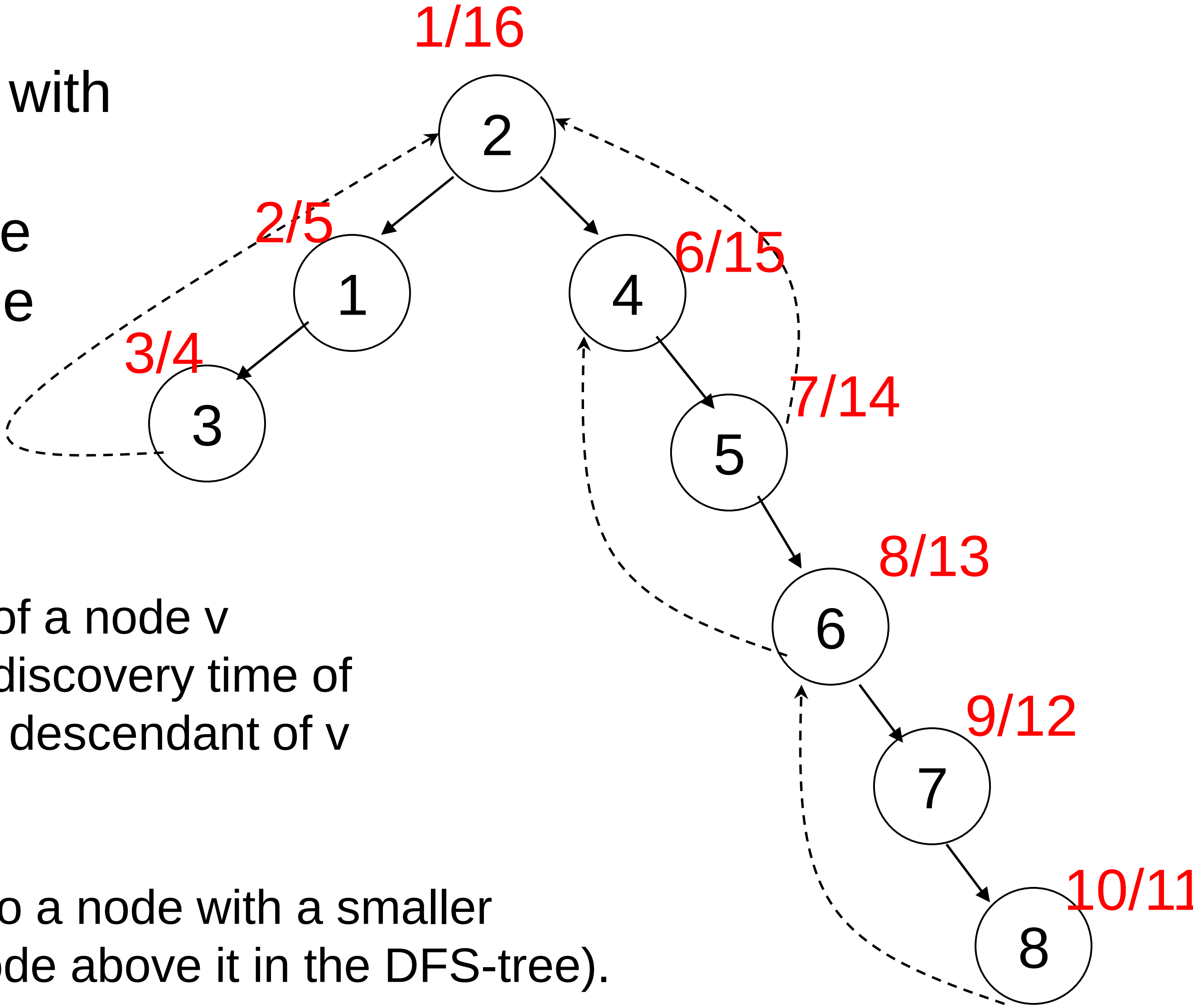
- Every vertex v will get following attributes:
  - v.color: (white, grey, black) – represents its exploration status
  - v.pi represents the “parent” node of v (v has ben reached as a result of exploring adjacencies of pi)
  - v.d represents the time when the node is discovered
  - v.f represents the time when the exploration is finished

```
DFS($G$)
for each vertex $u \in G.V$
    $u.color$ = WHITE
    $u.\pi$ = NIL
$time = 0$
for each vertex $u \in G.V$
    if $u.color$ == WHITE
        DFS-VISIT($G, u$)
```

```
DFS-VISIT($G, u$)
$time = time + 1$
$u.d = time$
$u.color$ = GRAY
for each $v \in G.Adj[u]$
    if $v.color$ == WHITE
        $v.\pi = u$
        DFS-VISIT($G, v$)
$u.color$ = BLACK
$time = time + 1$
$u.f = time$
```

Reminder: DFS – v.d and v.f

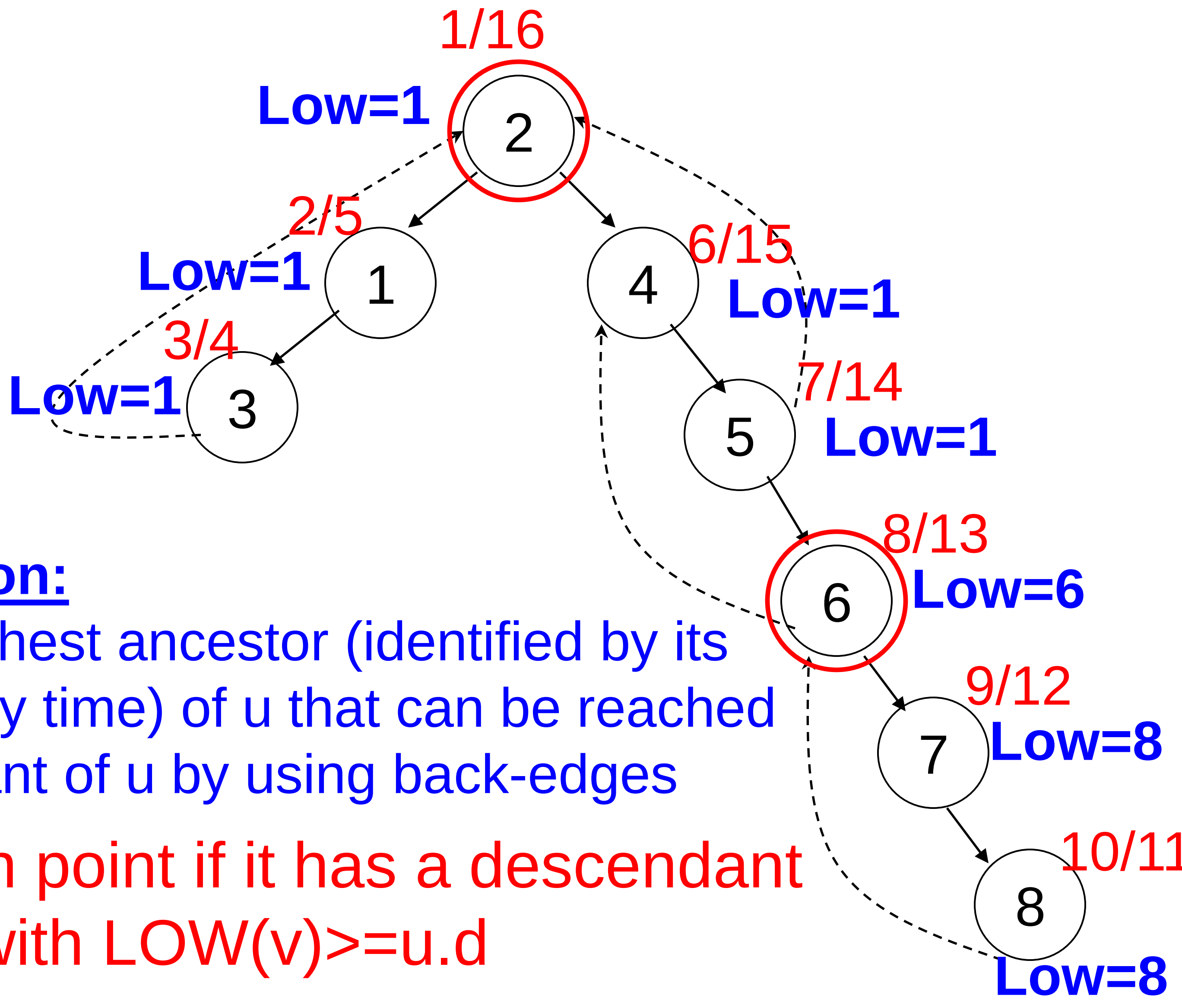
DFS associates with
every vertex v
its discovery time
and its finish time
v.d /v.f



The discovery time of a node v is smaller than the discovery time of any node which is a descendant of v in the DFS-tree.

A back-edge leads to a node with a smaller discovery time (a node above it in the DFS-tree).

The LOW function



The LOW function:

LOW(u) = the highest ancestor (identified by its smallest discovery time) of u that can be reached from a descendant of u by using back-edges

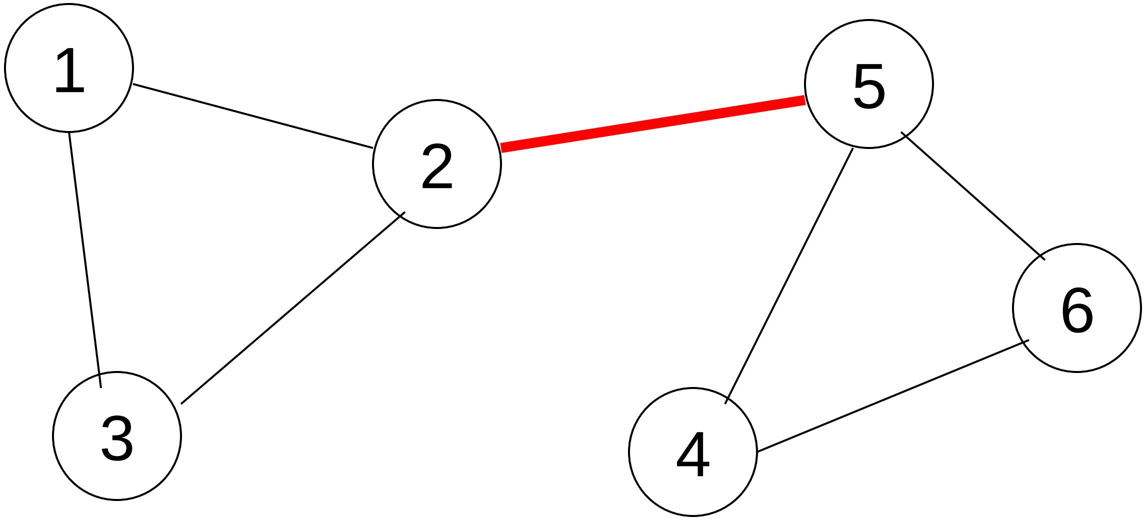
u is articulation point if it has a descendant v with LOW(v)>=u.d

Finding Articulation Points

- *Algorithm principle:*
  - *During DFS, calculate also the values of the LOW function for every vertex*
  - *After we finish the recursive search from a child v of a vertex u, we update u.low with the value of v.low. Vertex u is an articulation point, disconnecting v, if v.low >=u.d*
  - *If vertex u is the root of the DFS tree, check whether v is its second child*
  - *When encountering a back-edge (u,v) update u.low with the value of v.d*

```
DFS_VISIT_AP(G, u)
      time=time+1
      u.d=time
      u.color=GRAY
      u.low=u.d
      for each v in G.Adj[u]
            if v.color==WHITE
                  v.pi=u
                  DFS_VISIT_AP(G,v)
                  if (u.pi==NIL)
                        if (v is second son of u)
                              "u is AP"    // Case 1
                  else
                        u.low=min(u.low, v.low)
                        if (v.low>=u.d)
                               "u is AP"    // Case 2
            else  if ((v<>u.pi) and (v.d <u.d))
                        u.low=min(u.low, v.d)
      u.color=BLACK
      time=time+1
      u.f=time
```

Bridge edges – Example



How to find all bridges ?

- **Brute-force approach**: one by one remove all edges and see if removal of an edge causes the graph to disconnect:

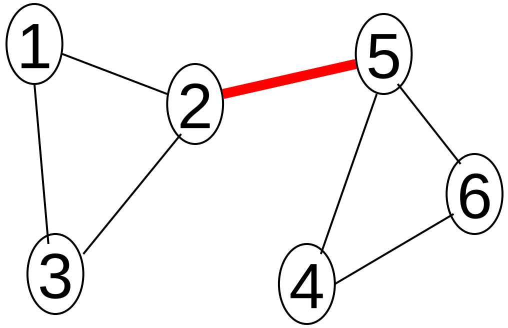
```
For every edge e, do :
    Remove e from graph
    See if the graph remains connected (use BFS or DFS)
    If graph is disconnected, add e to B list
    Add e back to the graph
```

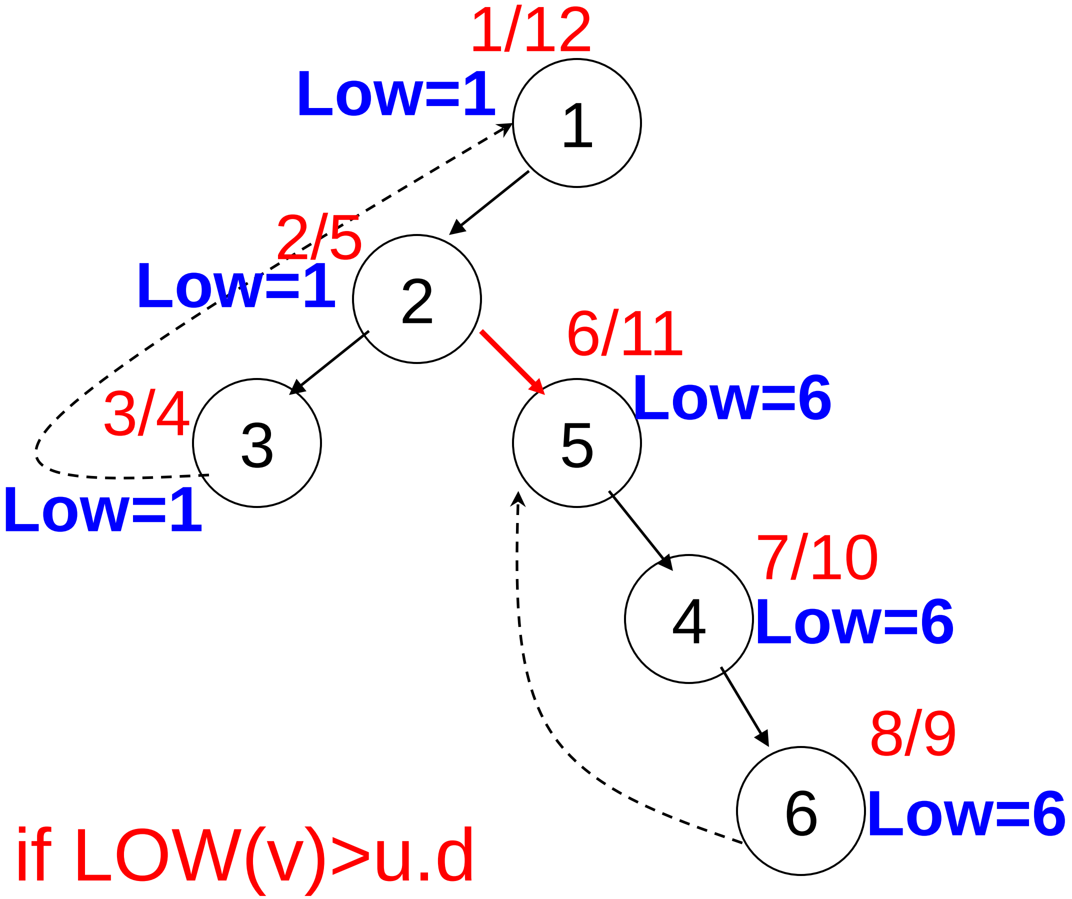
- Time complexity of above method is O(m*(n+m)) for a graph represented using adjacency list.
- Can we do better?

How to find all bridges ?

- **DFS- approach**:
- An edge of G is a bridge if and only if it does not lie on any simple cycle of G.
- if some vertex u has a back edge pointing to it, then no edge below u in the DFS tree can be a bridge. The reason is that each back edge gives us a cycle, and no edge that is a member of a cycle can be a bridge.
- if we have a vertex v whose parent in the DFS tree is u, and no ancestor of v has a back edge pointing to it, then (u, v) is a bridge.

Finding bridges by DFS

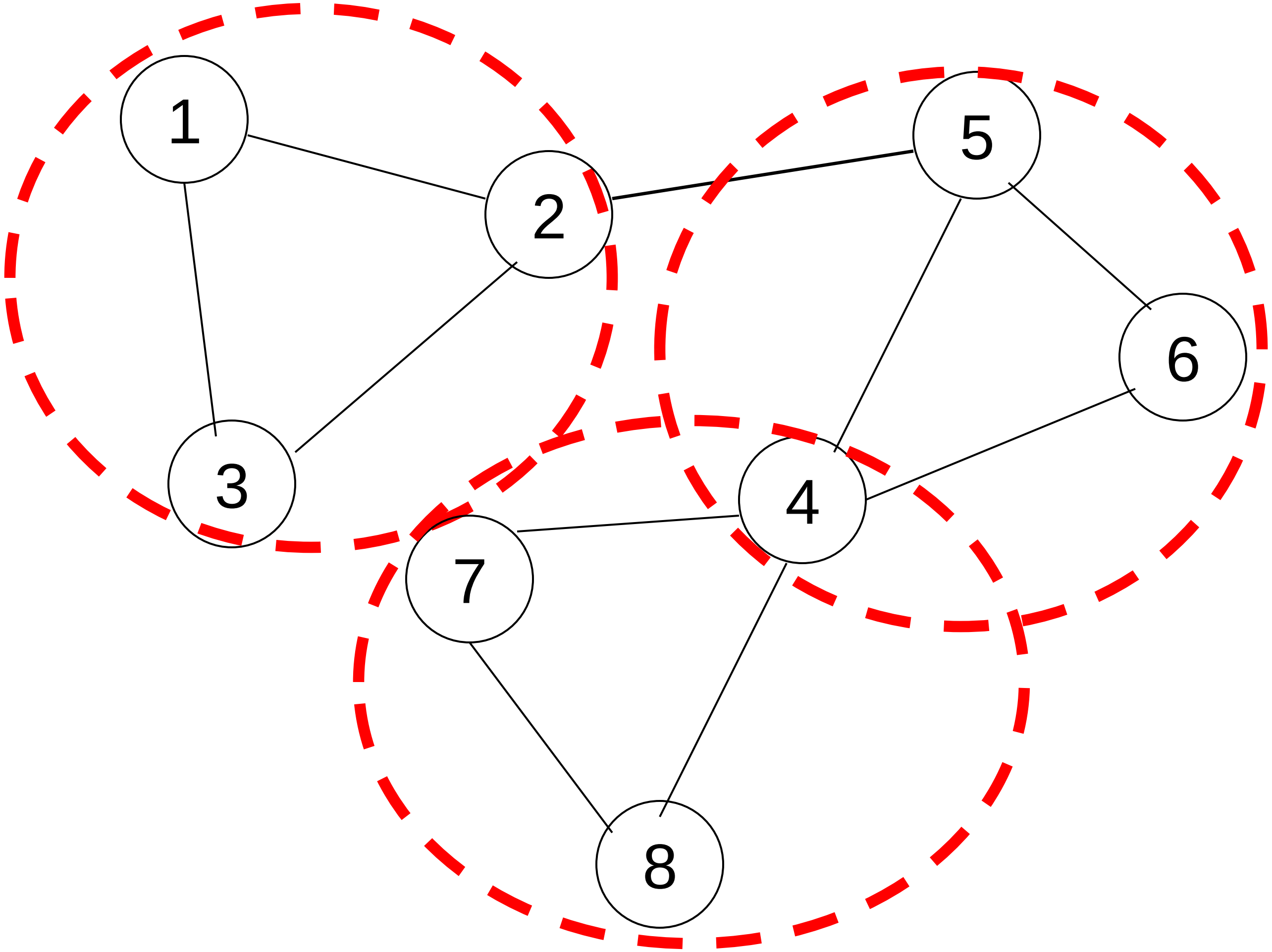




(u,v) is a bridge if LOW(v)>u.d

```
DFS_VISIT_Bridges(G, u)
      time=time+1
      u.d=time
      u.color=GRAY
      u.low=u.d
      for each v in G.Adj[u]
            if v.color==WHITE
                  v.pi=u
                  DFS_VISIT_AP(G,v)

                  u.low=min(u.low, v.low)
                  if (v.low>u.d)
                               "(u,v) is Bridge"
            else  if ((v<>u.pi) and  (v.d <u.d))
                        u.low=min(u.low, v.d)
      u.color=BLACK
      time=time+1
      u.f=time
```

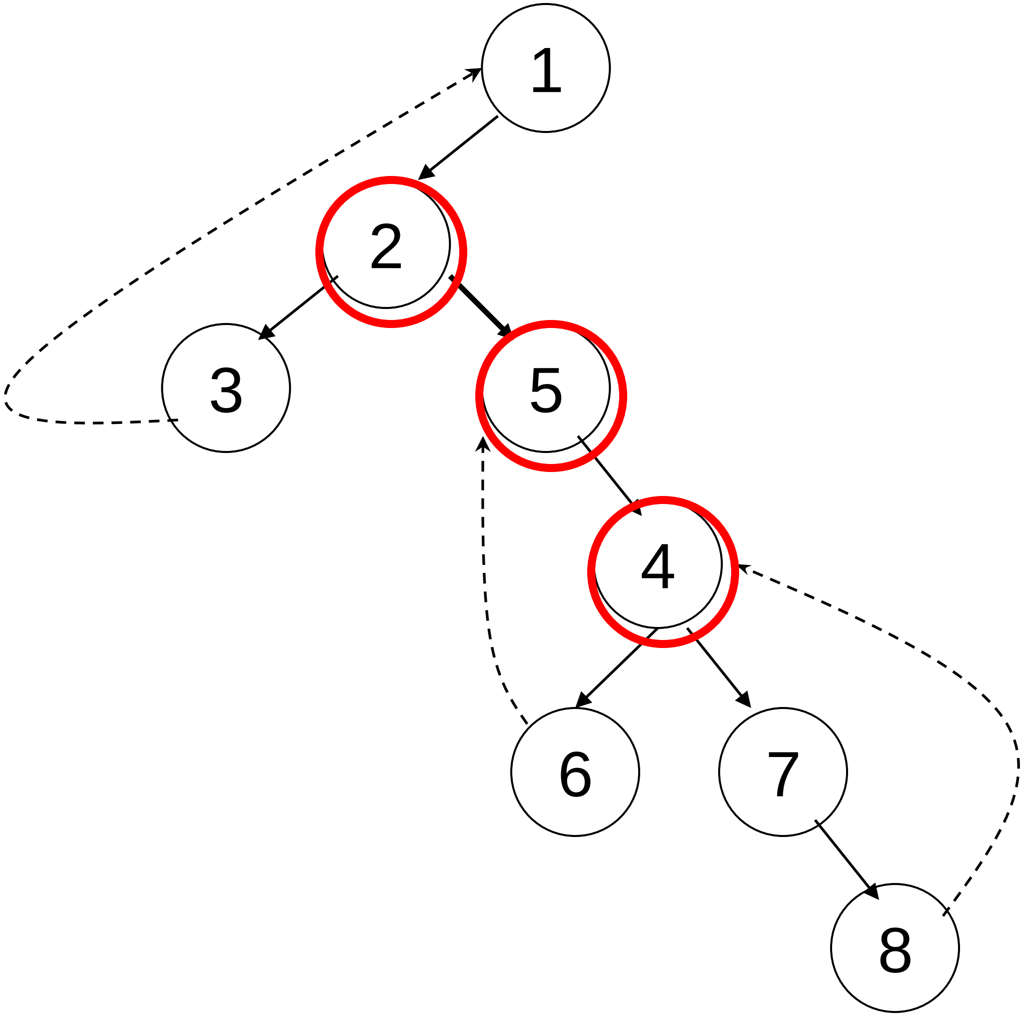
Biconnected components – Example



Finding biconnected components

- Two biconnected components cannot have a common edge, but they can have a common vertex
  - -> We will mark the edges with an id of their biconnected component
- The common vertex of several biconnected components is an articulation point
- The articulation points separate the biconnected components of a graph. If the graph has no articulation points, it is biconned
  - -> We will try to identify the biconnected components while searching for articulation points

Finding biconnected components



Finding biconnected components

- *Algorithm principle:*
  - *During DFS, use a stack to store visited edges (tree edges or back edges)*
  - *After we finish the recursive search from a child v of a vertex u, we check if u is an articulation point for v. If it is, we output all edges from the stack until (u,v). These edges form a biconnected component*
  - *When we return to the root of the DFS-tree, we have to output the edges even if the root is no articulation point (graph may be biconnex) – we will not test the case of the root being an articulation point*

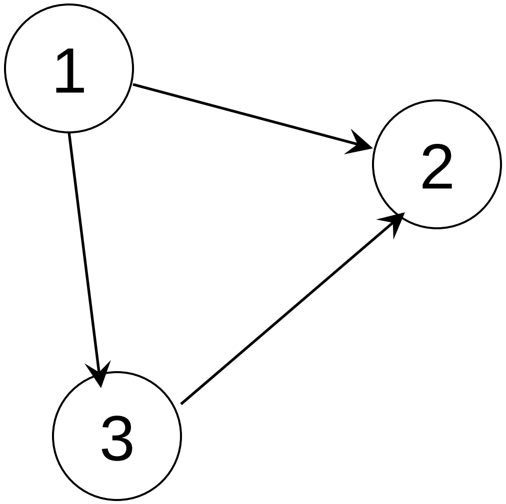
```
DFS_VISIT_BiconnectedComp(G, u)
    time=time+1
    u.d=time
    u.color=GRAY
    u.low=u.d
    u.AP=false
    for each v in G.Adj[u]
        if v.color==WHITE
            v.pi=u
            EdgeStack.push(u,v)
            DFS_VISIT_AP(G,v)
            u.low=min(u.low, v.low)
            if (v.low>=u.d)
                pop all edges from EdgeStack until (u,v)
                these are the edges of a Biconn Comp
        else   if ((v<>u.pi) and (v.d <u.d))
                EdgeStack.push(u,v)
                u.low=min(u.low, v.d)

    u.color=BLACK
    time=time+1
    u.f=time
```

Applications of DFS

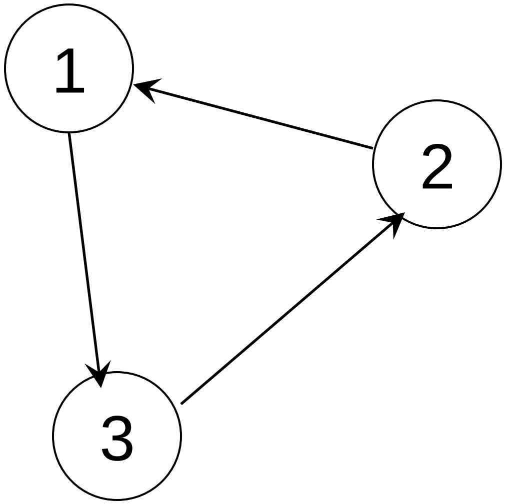
- DFS has many applications
- For undirected graphs:
  - Connected components
  - Connectivity properties
- For directed graphs:
  - Finding cycles
  - Topological sorting
  - Connectivity properties: Strongly connected components

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles



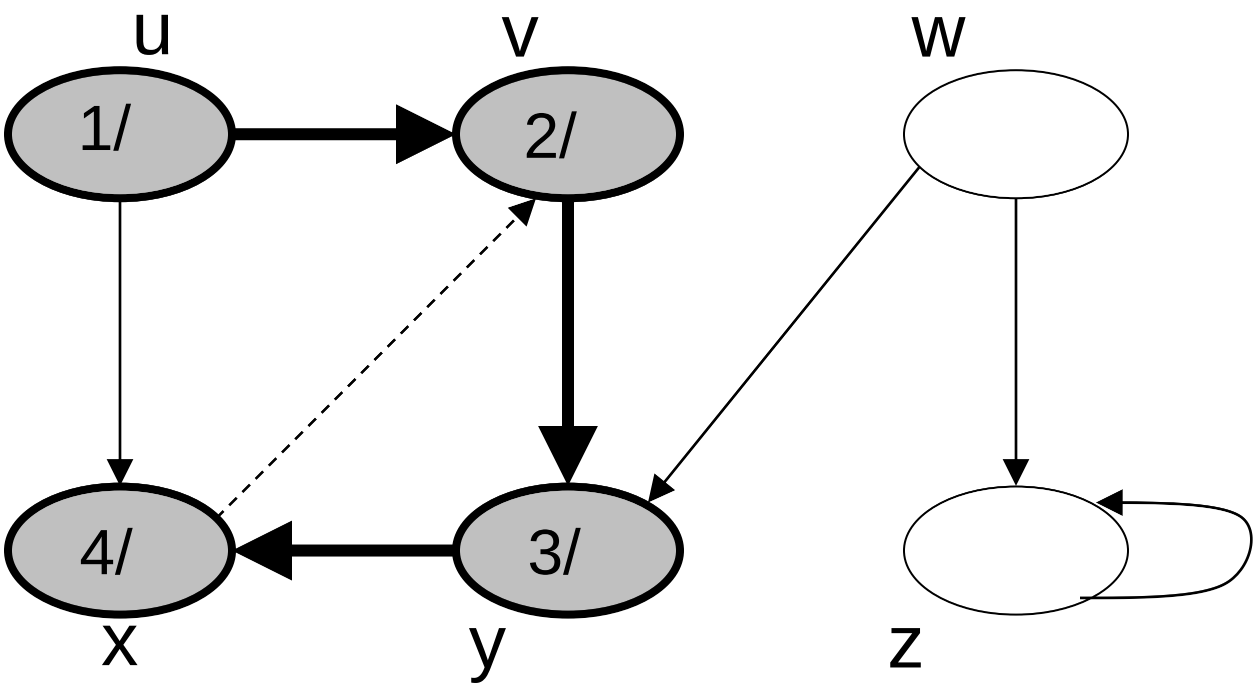
acyclic



cyclic

DFS and cycles in graph

- A graph G is acyclic if a DFS of G results in no back edges

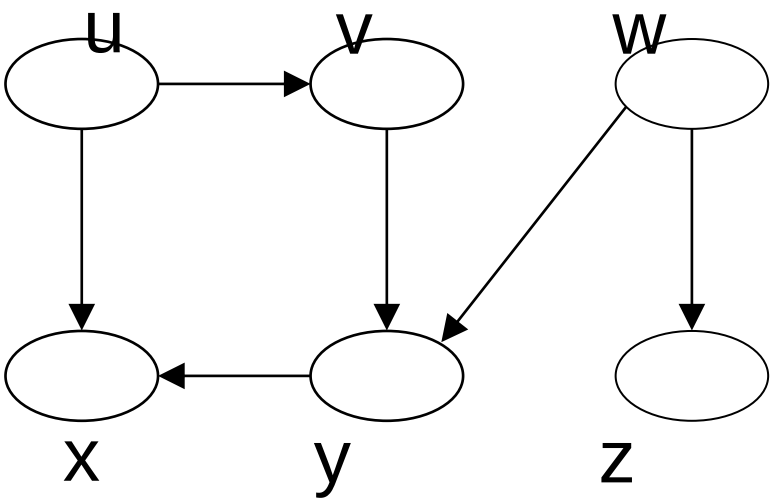


Topological Sort

- *Topological sort* of a DAG (Directed Acyclic Graph):
  - Linear ordering of all vertices in a DAG G such that vertex *u* comes before vertex *v* if there is an edge (*u*, *v*) $\in$ G

  - This property is important for a class of *scheduling* problems

Example – Topological Sorting



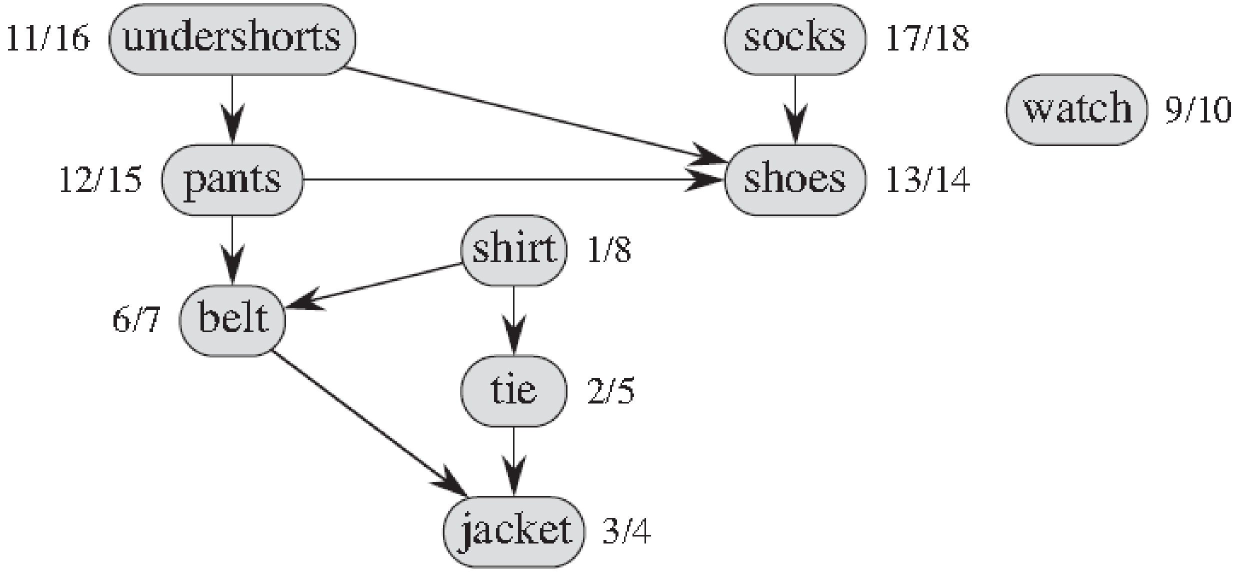
- There can be several orderings of the vertices that fulfill the topological sorting condition:
  - u, v, w, y, x, z
  - w, z, u, v, y, x
  - w, u, v, y, x, z
  - …

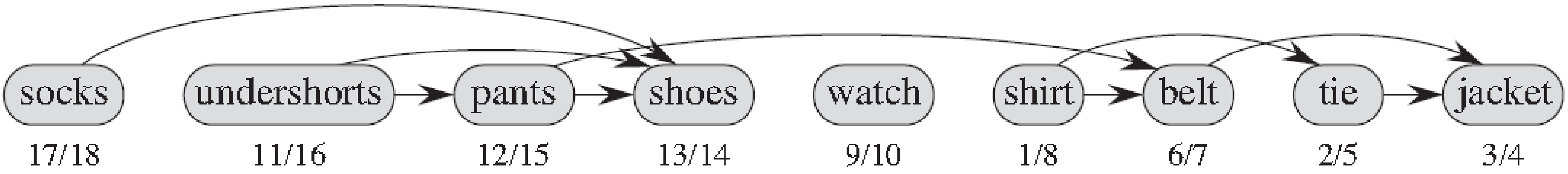
Topological Sorting

- *Algorithm principle:*
  1. *Call DFS to compute finishing time v.f for every vertex*
  2. *As every vertex is finished (BLACK) insert it onto the front of a linked list*
  3. *Return the list as the linear ordering of vertices*

- Time: O(V+E)

Using DFS for Topological Sorting





Correctness of Topological Sort

- Claim: $(u,v) \in G \Rightarrow u.\mathrm{f} > v.\mathrm{f}$
  - When (u,v) is explored, *u* is grey
    - *v* = grey $\Rightarrow$ (u,v) is back edge. Contradiction, since G is DAG and contains no back edges
    - *v* = white $\Rightarrow$ *v* becomes descendent of *u* $\Rightarrow$ $v.\mathrm{f} < u.\mathrm{f}$ (since it must finish *v* before backtracking and finishing *u*)
    - *v* = black $\Rightarrow$ *v* already finished $\Rightarrow$ $v.\mathrm{f} < u.\mathrm{f}$

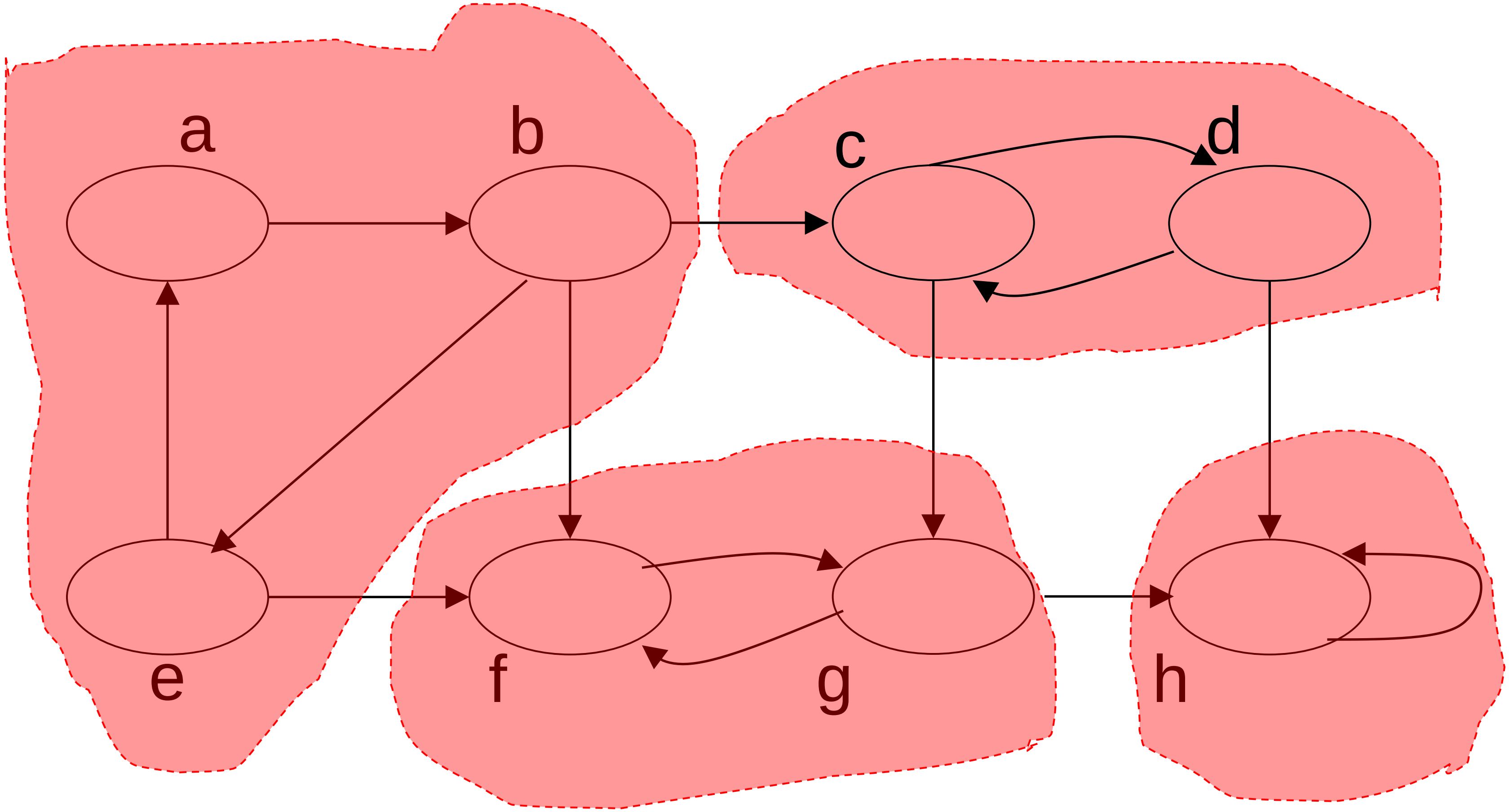
Applications of DFS

- DFS has many applications
- For undirected graphs:
  - Connected components
  - Connectivity properties
- For directed graphs:
  - Finding cycles
  - Topological sorting
  - Connectivity properties: Strongly connected components

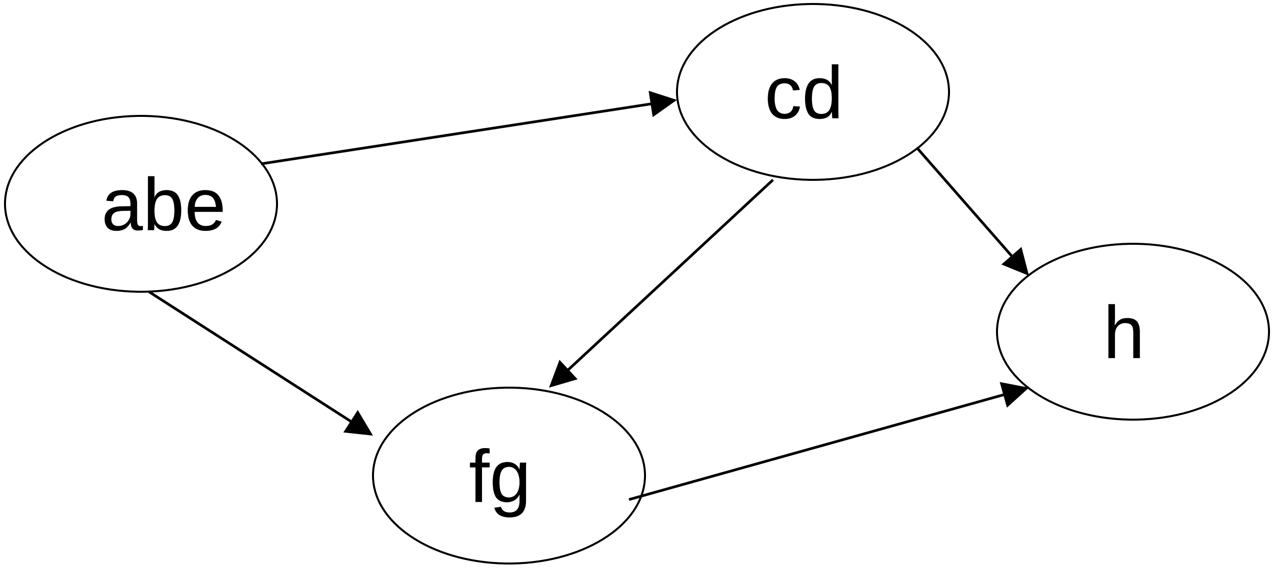
Strongly Connected Components

- A strongly connected component of a directed graph G=(V,E) is a maximal set of vertices C such that for every pair of vertices u and v in C, both vertices u and v are reachable from each other.

Strongly connected components - Example



Strongly connected components – Example – The Component Graph



The Component Graph results by collapsing each strong component into a single vertex

The Component Graph

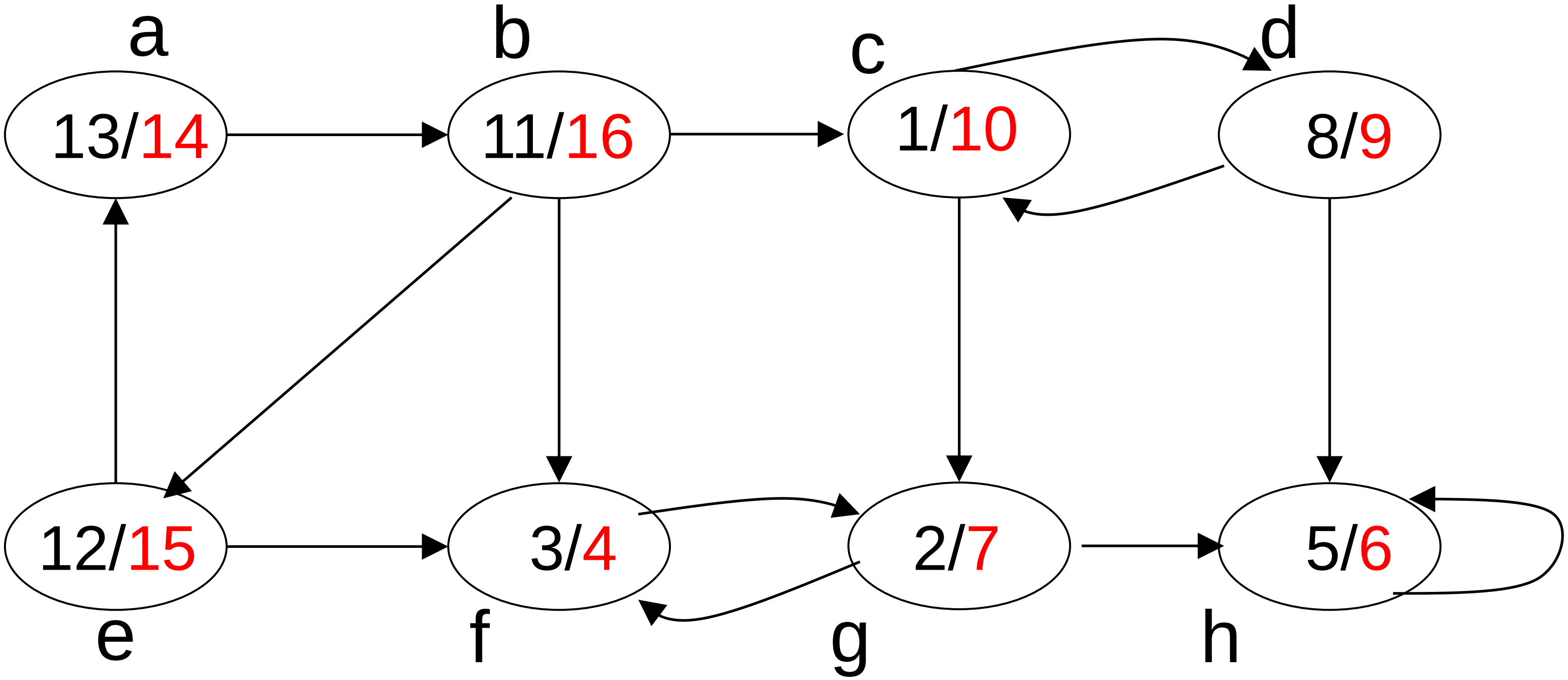
- Property: The Component Graph is a DAG (directed acyclic graph)

Strongly connected components

- *Strongly connected components of a directed graph G*
- *Algorithm principle:*
  1. *Call DFS(G) to compute finishing times u.f for every vertex u*
  2. *Compute GT*
  3. *Call DFS(GT), but in the main loop of DFS, consider the vertices in order of decreasing u.f as computed in step 1*
  4. *Output the vertices of each DFS-tree formed in step 3 as the vertices of a strongly connected component*

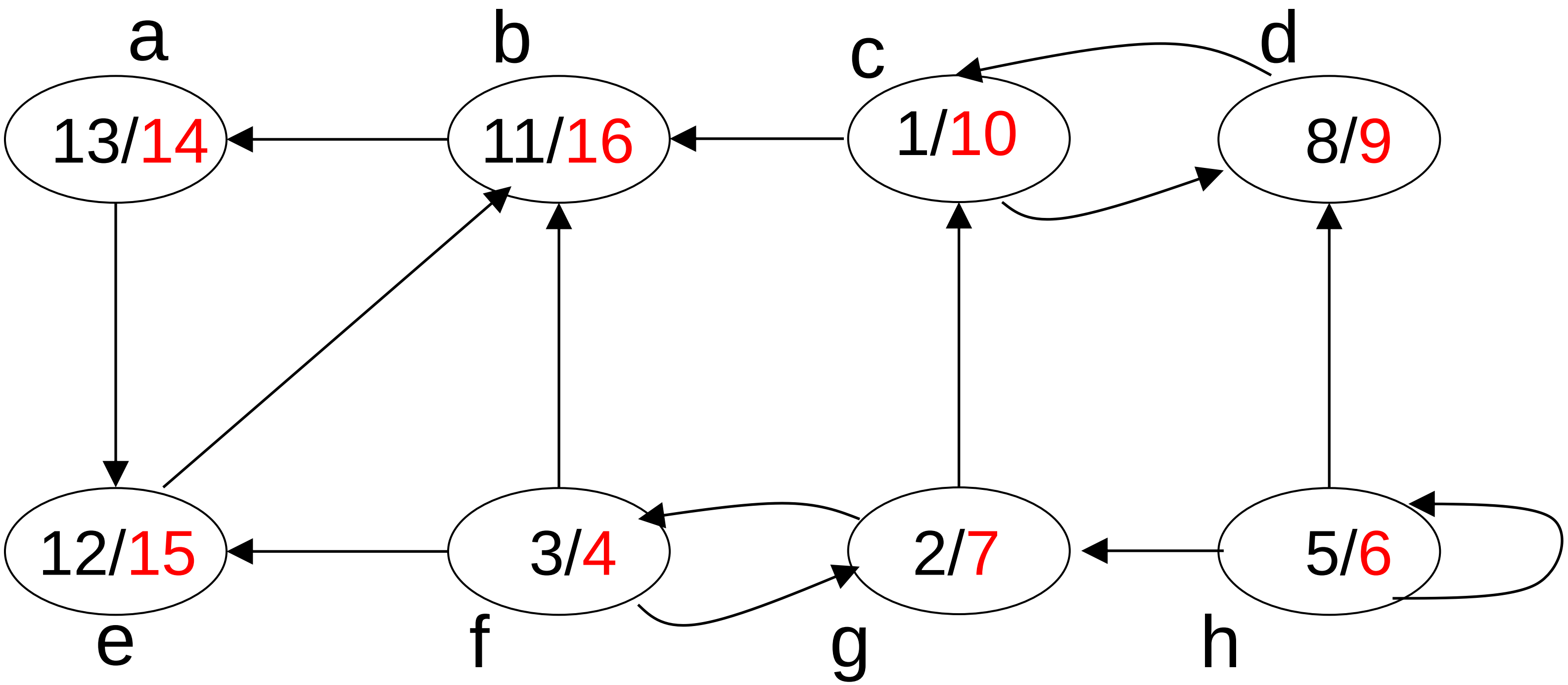
Strongly connected components - Example

Step1: call DFS(G), compute u.f for all u



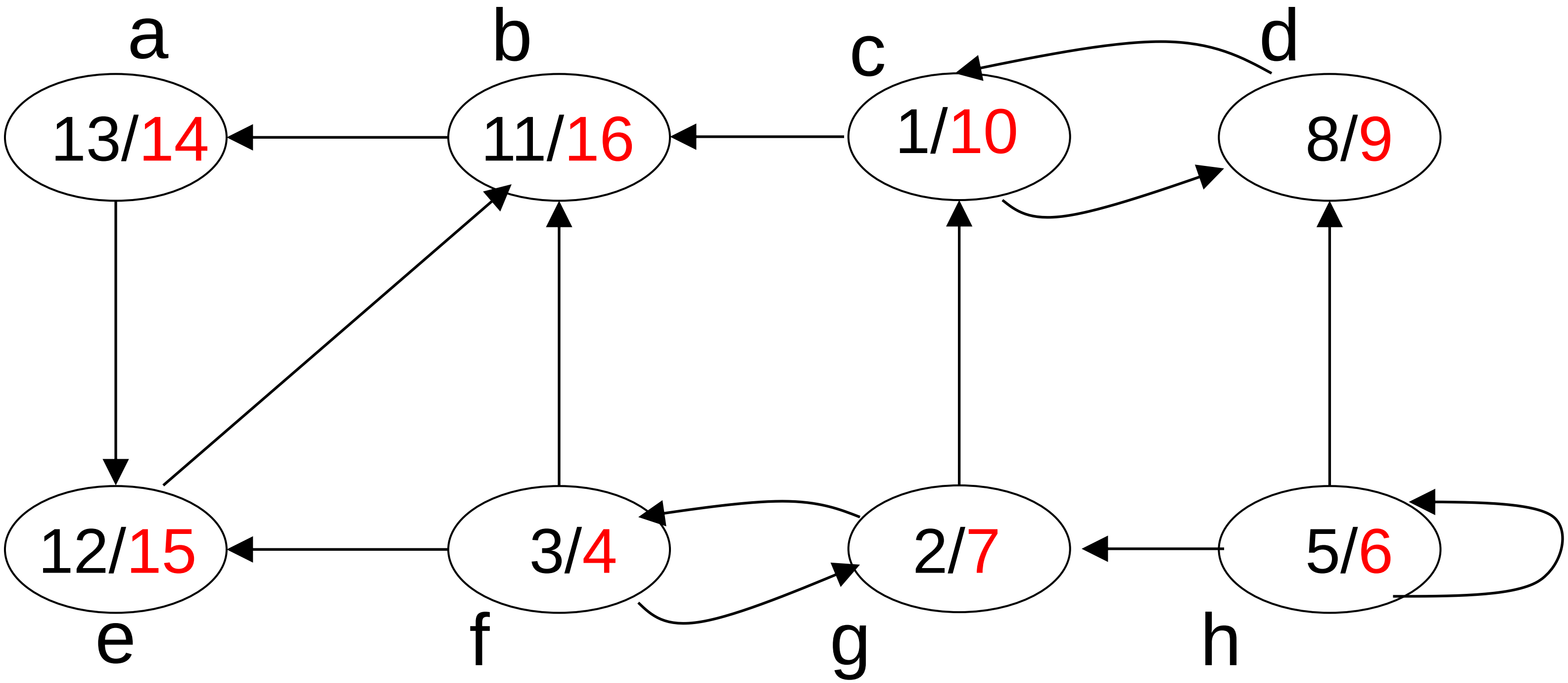
Strongly connected components - Example

Step2: compute GT



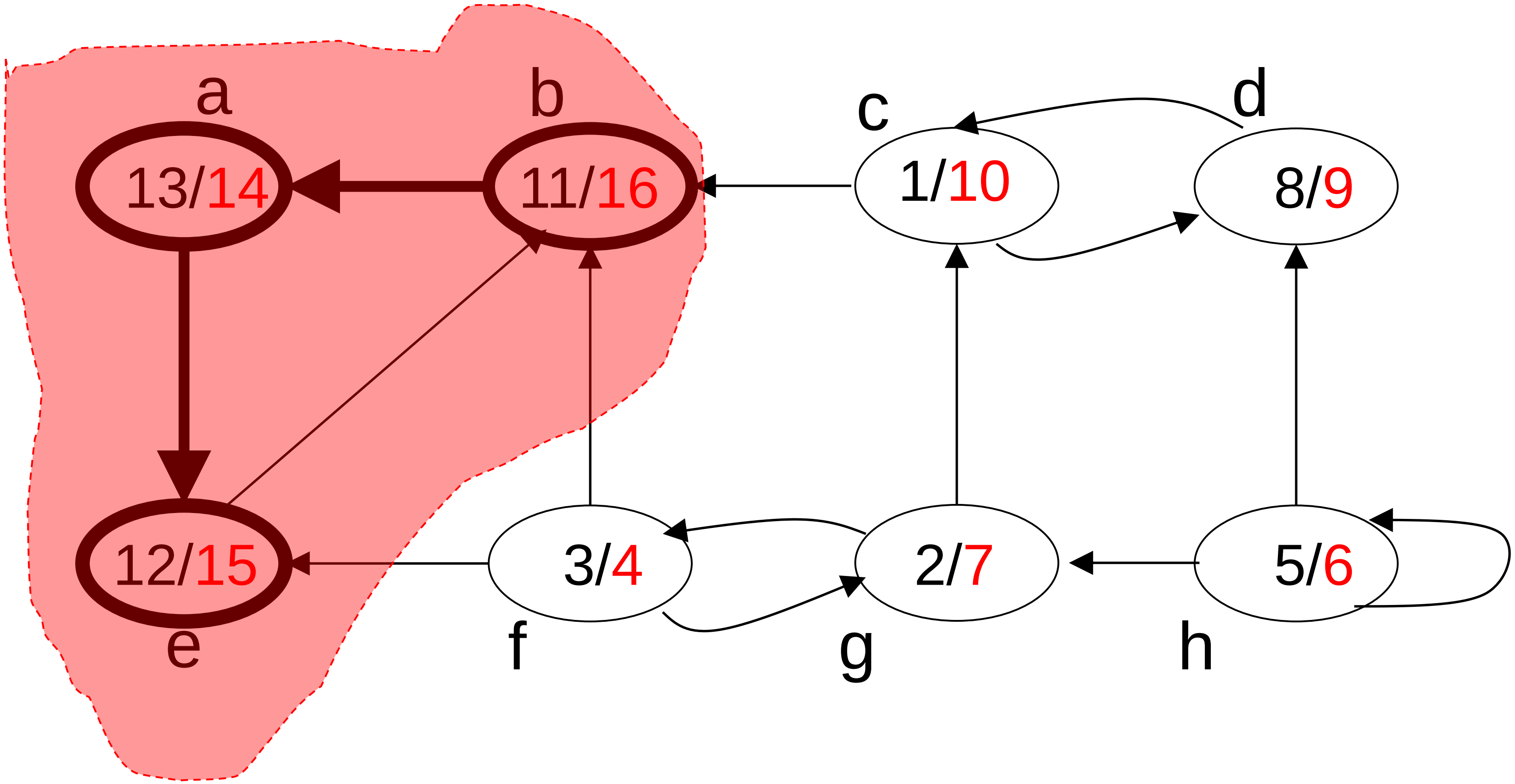
Strongly connected components - Example

Step3: call DFS(GT), consider vertices in order of decreasing u.f



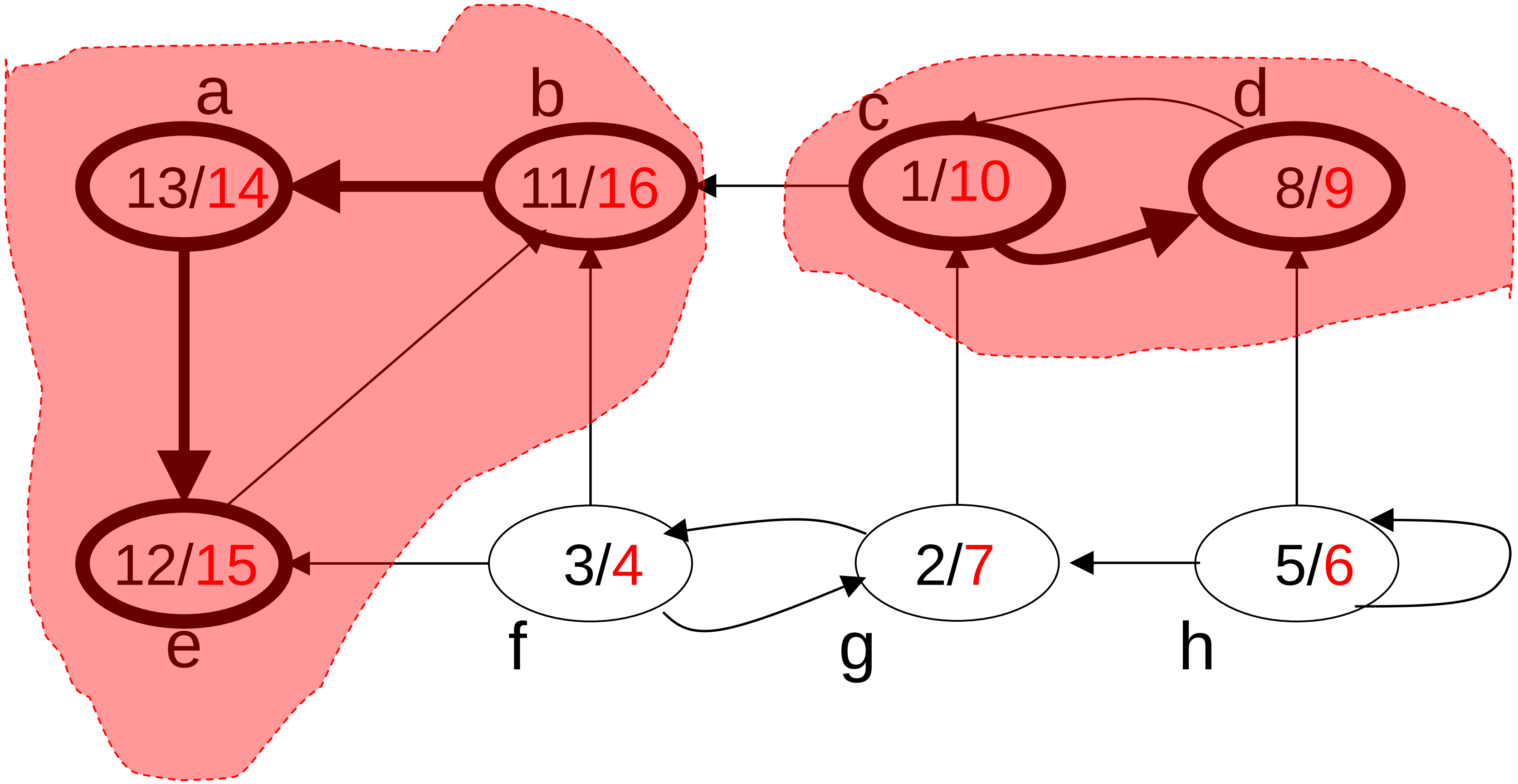
Strongly connected components - Example

Step3: call DFS(GT), consider vertices in order of decreasing u.f



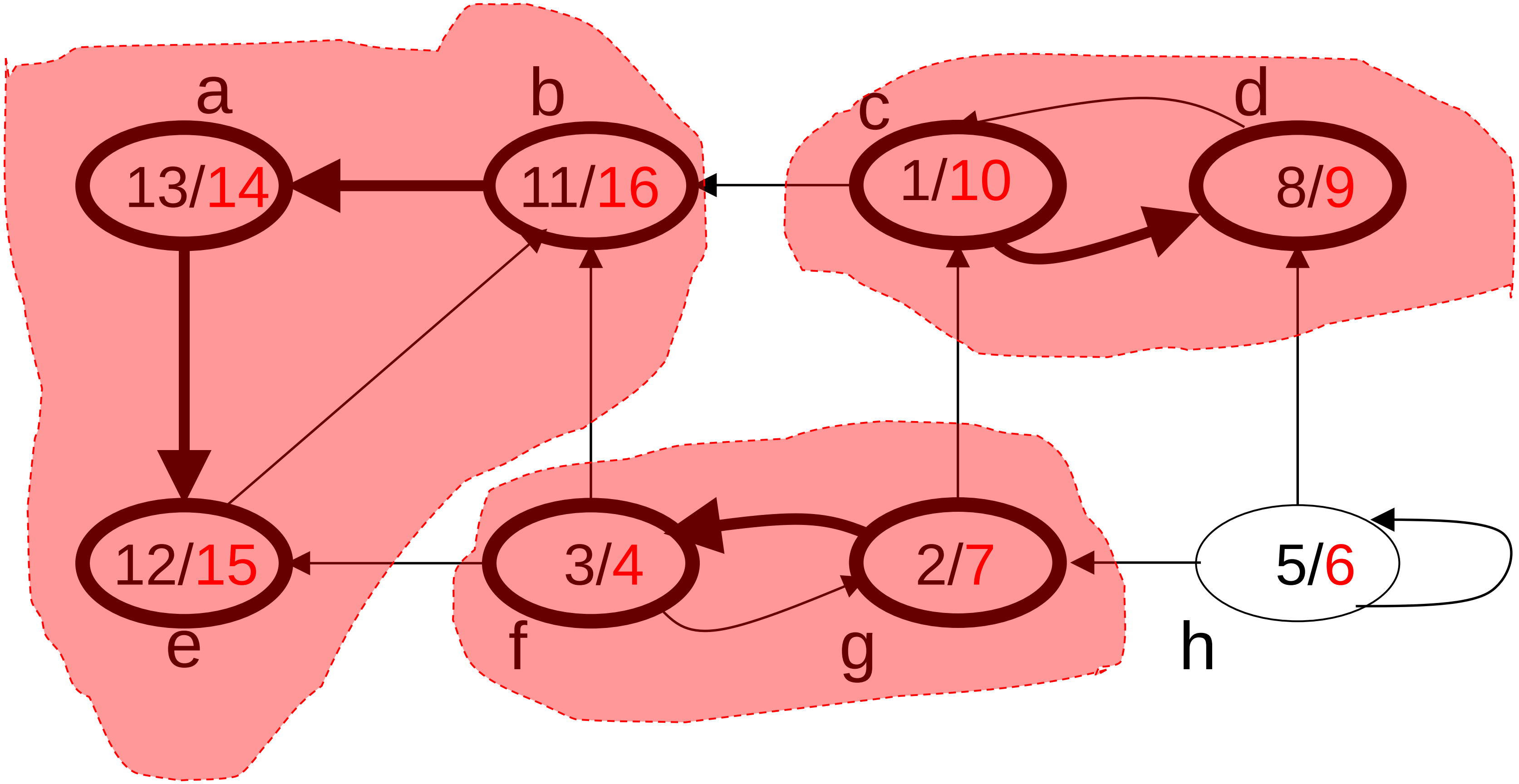
Strongly connected components - Example

Step3: call DFS(GT), consider vertices in order of decreasing u.f



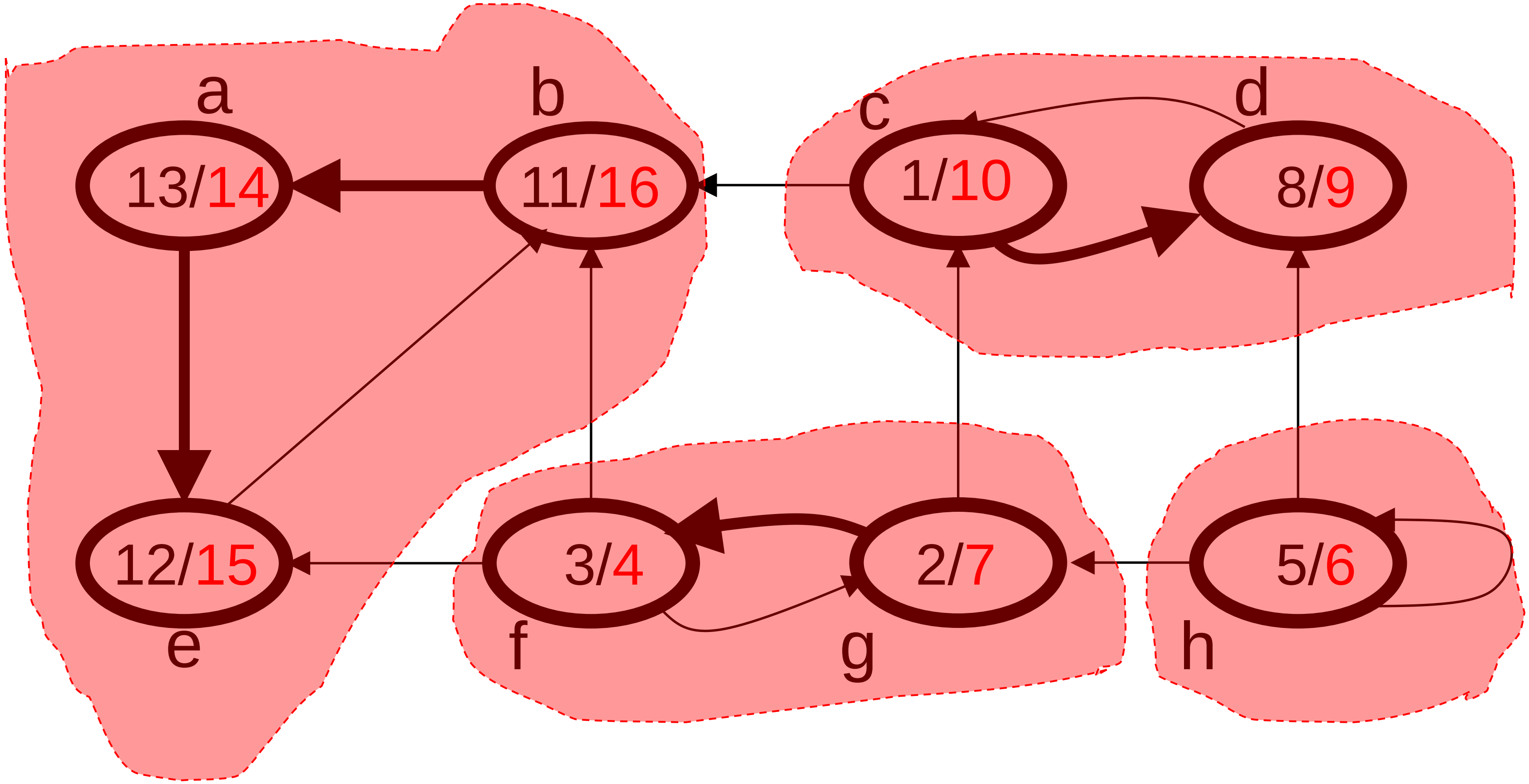
Strongly connected components - Example

Step3: call DFS(GT), consider vertices in order of decreasing u.f



Strongly connected components - Example

Step3: call DFS(GT), consider vertices in order of decreasing u.f



Proof of
Strong Components Algorithm

- *Theorem*

  The Algorithm presented before finds the strong components of G.

- *Proof*

  We must show that a set of vertices forms a strong components *if and only if* they are vertices of a tree in the DFS-forest of G^T

Proof (cont)

- *Suppose that* v,w are vertices in the same strong component
- There is a DFS search in GT which starts at a vertex r and reaches v.
- Since v, w are in the same strong component, there is a path from v to w and from w to v. => then w will also be reached in this DFS search
- => Vertices v,w belong to the same spanning tree of GT.

Proof (cont)

- *Suppose* v,w are two vertices in the same DFS-spanning tree of GT.
- Let r be the root of that spanning tree.
- Then there exists paths in GT *from r* to each of v and w.
- So there exists paths in G *to r* from each of v and w.

Proof (cont)

- We will prove that there are paths in G from r to v and w as well
- We know that r.f > v.f (when r was selected as a root of its DFS-tree)
- If there is no path from r to v, then it is also no path from v to r, which is a contradiction
- Hence, there exists path in G from r to v, and similar argument gives path from r to w.
- So v and w are in a cycle of G and must be in the same strong component.

Summary

- Applications of Depth-First Search
  - Undirected graphs:
    - Connected components, articulation points, bridges, biconnected components
  - Directed graphs:
    - Cyclic/acyclic graphs
    - Topological sort
    - Strongly connected components