Optimal Design of Parabolic Canal Section

Bhagu R. Chahar[1]

Abstract: Optimal design equations for a parabolic canal section are presented in this paper. The design equations for a minimum earthwork cost section and a minimum cost lined section are in explicit form and result in optimal dimensions of a canal in single step computations. These have been obtained after applying the Fibbonaci search method on a nonlinear unconstrained optimization problem. The study also addresses the bounds on the canal dimensions and the velocity of flow. A nondimensional parameter approach has been used to simplify the analysis, and a set of graphs for nondimensional parameters are presented as an alternative for design. Design procedures for different cases have been presented to demonstrate the simplicity of the method.

DOI: 10.1061/(ASCE)0733-9437(2005)131:6(546)

CE Database subject headings: Canals; Cost control; Optimization; Design; Linings.

Introduction

Networks of canals are used to convey, distribute, and apply water to the land. A canal in a network may be either unlined or lined. River beds, unlined canals, and irrigation furrows all tend to approximate a stable parabolic shape (Mironenko et al. 1984). Therefore, unlined canals can be made more hydraulically stable by initially constructing them in a parabolic shape. Because the channel side slopes along the cross section are always less than the maximum allowable side slope at the water surface, parabolic channels are physically more stable (Mironenko et al. 1984; Babaeyan-Koopaei et al. 2000; Babaeyan-Koopaei 2001). A lined parabolic channel has no sharp angles of stress concentration where cracks may occur and can be prefabricated in molded sections. Small parabolic ditches can be constructed by bulldozers and other types of earth moving equipment (Mironenko et al. 1984).

Irrigation canals are lined for several purposes (Swamee et al. 2000a). Lined canals are designed for uniform flow considering hydraulic efficiency, practicability, and economy (Streeter 1945). Factors to be considered in the design include: (1) the material forming the channel surface, which determines the roughness coefficient; (2) the minimum permissible velocity, to avoid deposition of silt or debris; (3) the limiting velocity, to avoid erosion of the channel surface; (4) the topography of the channel route, which fixes the channel bed slope; and (5) the efficiency of the channel section, which indicates how much the section is hydraulically and/or economically efficient (Chow 1973). A maximum hydraulic radius results in a section of minimum excavation area and the best hydraulic design. Monadjemi (1994) and others (Froehlich 1994; Swamee 1995) presented a fundamental approach for determining the best hydraulic section based on Lagrange's method of undetermined multipliers. Loganathan (1991) proposed optimality for a parabolic canal design accounting for freeboard together with limitations on velocity and canal dimensions and presented results in a tabular form, which are inconvenient to use in designing canals.

When an open channel is constructed, the excavation and lining constitute a major cost. Obviously it is desirable to keep this cost at a minimum by adopting the most economical canal cross section. Explicit equations are not available for designing the minimum earthwork cost section and minimum cost lined section of a parabolic shape. Also, explicit equations are not available for setting bounds to the parabolic canal dimensions and the canal flow velocity. The present study is an attempt to address these design problems in parabolic canals.

Starting from the geometric properties of a parabolic section and the governing uniform flow equation, the optimization method for minimum area section has been described. The resultant optimal parameters have been obtained in a nondimensional form. The next section proposes design equations for the minimum cost parabolic section, which have been obtained by formulating a cost function involving earthwork cost and lining cost. The optimization problem, involving a nonlinear cost function with the nonlinear equality constraint, has been converted into an unconstrained optimization problem in nondimensional form and minimized using the Fibbonacci search. The bounds on the canal dimensions and the velocity of flow have also been treated. The proposed design equations have been obtained in explicit form through minimization of errors or regression analysis. Finally, a step-by-step design procedure with design examples and limitations of the method is presented.

Geometric Properties of Parabolic Section

A parabolic canal (Fig. 1) is described by

$$Y = aX^2 \tag{1}$$

in which Y=ordinate; X=abscissa; and a=parameter.

[1]Assistant Professor, Dept. of Civil Engineering, Indian Institute of Technology, New Delhi–110 016, India. Email: chahar@civil.iitd.ac.in

Note. Discussion open until May 1, 2006. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on November 11, 2003; approved on December 30, 2004. This paper is part of the ***Journal of Irrigation and Drainage Engineering***, Vol. 131, No. 6, December 1, 2005. ©ASCE, ISSN 0733-9437/2005/6-546–554/$25.00.

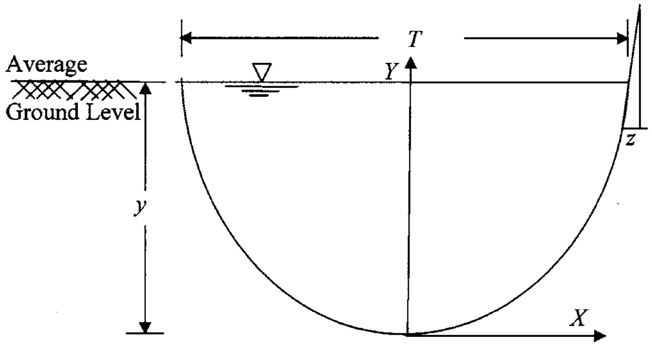

Fig. 1. Cross section of parabolic channel

The flow area $A(\mathrm{m}^2)$ of a parabolic canal section is computed as

$$A = 2\left[\frac{yT}{2} - \int_0^{T/2} YdX\right] = \frac{2}{3}yT = \frac{8}{3}y^2 z \tag{2}$$

where y=flow depth (m); and T=top width of the canal at the water surface (m), given by

$$T = 4yz \tag{3}$$

in which z=side slope at $Y=y$.

The perimeter P (m) is obtained by integrating length ds of the parabola as

$$P = \int ds = \sqrt{(dX)^2 + (dY)^2}$$

$$= 2yz^2\left[\frac{1}{z}\sqrt{1+\frac{1}{z^2}} + \ln\left(\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right)\right] \tag{4}$$

Alternatively, Eq. (4) can be expressed as

$$P = yf_z \tag{5}$$

where f_z, which is a function of z only, is given by

$$f_z = 2z^2\left[\frac{1}{z}\sqrt{1+\frac{1}{z^2}} + \ln\left(\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right)\right] \tag{6}$$

Optimal Canal Section

A lined canal section is designed for uniform flow. The most commonly used uniform flow formula is the Manning equation (Chow 1973). The uniform flow rate or discharge Q $(\mathrm{m}^3/\mathrm{s})$ in a canal by Manning's equation is

$$Q = AV = \frac{1}{n}AR^{2/3}S_f^{1/2} = \frac{1}{n}A\left(\frac{A}{P}\right)^{2/3} S_f^{1/2} = \frac{1}{n}A\left(\frac{A}{P}\right)^{2/3} S_o^{1/2} \tag{7}$$

where V=mean velocity of uniform flow (m/s); R=hydraulic radius (m), defined as the ratio of flow area to the flow perimeter; n=Manning's roughness coefficient; S_f=energy slope (dimensionless); and S_o=bed slope of the canal (dimensionless). For uniform flow, $S_f = S_o$. In the Manning's formula, all the terms except n can be directly measured. The roughness coefficient is a parameter representing the integrated effects of the channel cross-sectional resistance. The selection of a value of n is subjective, based on experience and engineering judgment. Chow (1973) lists values of n for different conditions of a canal.

Classical Optimal Section

The classical optimal section is the best hydraulic section which has the maximum flow velocity or the minimum flow area and wetted perimeter for a specified discharge and canal bed slope. Mathematically, it can be stated as

$$\text{Minimize} \quad A = A(y,z) \tag{8a}$$

$$\text{Subject to} \quad \phi = Q - \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S_o^{1/2} = \phi(A,P) = \phi(y,z) = 0 \tag{8b}$$

Because a parabolic canal is completely described by two independent variables y and z, applying Lagrange's method of undetermined multipliers (Kreyszig 2001):

$$\frac{\partial A}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \tag{9a}$$

$$\frac{\partial A}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \tag{9b}$$

$$\phi = Q - \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S_o^{1/2} = \phi(A,P) = \phi(y,z) = 0 \tag{9c}$$

where λ=undetermined multiplier. Eliminating λ from Eqs. (9a) and (9b):

$$\frac{\partial A}{\partial y}\frac{\partial \phi}{\partial z} = \frac{\partial A}{\partial z}\frac{\partial \phi}{\partial y} \tag{10}$$

From Eq. (9c):

$$\frac{\partial \phi}{\partial z} = \frac{S_o^{1/2}A^{2/3}}{3nP^{5/3}}\left(2A\frac{\partial P}{\partial z} - 5P\frac{\partial A}{\partial z}\right) \tag{11a}$$

$$\frac{\partial \phi}{\partial y} = \frac{S_o^{1/2}A^{2/3}}{3nP^{5/3}}\left(2A\frac{\partial P}{\partial y} - 5P\frac{\partial A}{\partial y}\right) \tag{11b}$$

Using Eqs. (11a) and (11b) in Eq. (10):

$$\frac{\partial A}{\partial y}\frac{\partial P}{\partial z} = \frac{\partial A}{\partial z}\frac{\partial P}{\partial y} \tag{12}$$

Substituting A from Eq. (2) and P from Eq. (4) in Eq. (12):

$$\left(\frac{16}{3}yz\right)\left[4yz\left[\ln\left(\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right)\right]\right]$$

$$= \left(\frac{8}{3}y^2\right)\left[2z^2\left[\frac{1}{z}\sqrt{1+\frac{1}{z^2}} + \ln\left(\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right)\right]\right] \tag{13}$$

and simplifying:

$$3\ \ln\left[\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right] = \frac{1}{z}\sqrt{1+\frac{1}{z^2}} \tag{14}$$

Solving by trial and error:

$$z^* = 0.514 \tag{15}$$

where superscript *=optimum value. The optimal values of the other parameters are found using Eq. (15) in Eqs. (2)–(4):

$$A^* = 1.37067y^2; \quad T^* = 2.056y; \quad P^* = 2.9985y \cong 3y \tag{16}$$

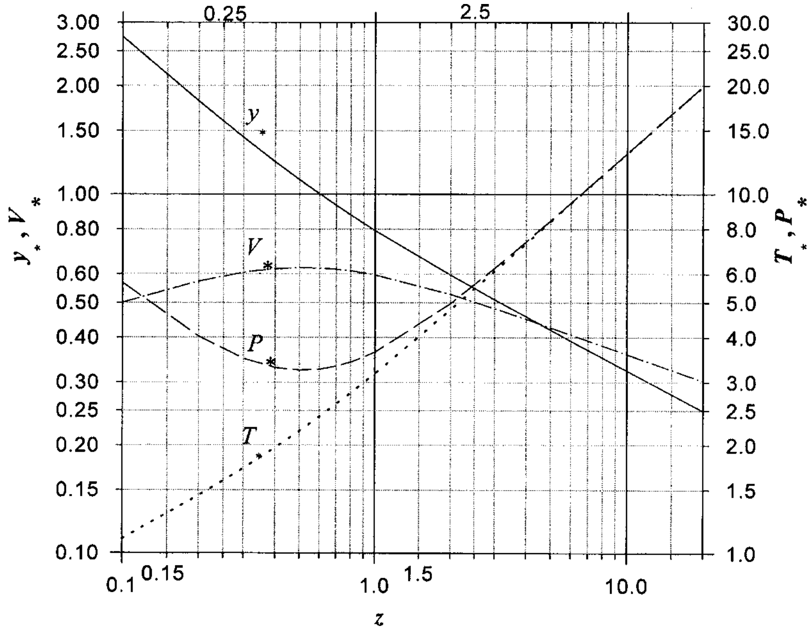

Fig. 2. Variations in nondimensional parameters with z (exact perimeter)

Nondimensionalization of Parameters

Assuming a length scale L (m):

$$L = (Qn/\sqrt{S_o})^{3/8} \tag{17}$$

the following nondimensional variables have been defined:

$$y_* = y/L; \quad P_* = P/L; \quad T_* = T/L; \quad A_* = A/L^2; \quad V_* = VL^2/Q \tag{18}$$

Using Eqs. (2), (4), (7), and (17):

$$L^{8/3} = \left(\frac{8}{3}y^2 z\right)^{5/3} \Big/ (yf_z)^{2/3} \tag{19}$$

and solving for y:

$$y = \left(\frac{3}{8z}\right)^{5/8} L f_z^{1/4}; \tag{20a}$$

or

$$y_* = \left(\frac{3}{8z}\right)^{5/8} f_z^{1/4} \tag{20b}$$

Substitution of y from Eq. (20a) in Eqs. (2), (3), (5), and (7) have yielded:

$$A_* = \left(\frac{3}{8z}\right)^{1/4} f_z^{1/2}; \tag{21a}$$

$$T_* = 4\left(\frac{3}{8}\right)^{5/8} z^{3/8} f_z^{1/4} \tag{21b}$$

$$P_* = \left(\frac{3}{8z}\right)^{5/8} f_z^{5/4}; \tag{21c}$$

$$V_* = \left(\frac{8z}{3}\right)^{1/4} \frac{1}{f_z^{1/2}} \tag{21d}$$

The nondimensional expressions for the depth of flow [Eq. (20b)], the flow area [Eq. (21a)], the top width [Eq. (21b)], the wetted perimeter [Eq. (21c)], and the uniform velocity [Eq. (21d)] are only a function of the side slope z. Graphical representation of the preceding equations can be used to obtain the values of these nondimensional parameters for a wide range of z. Using the equations, a set of graphs have been plotted for variations in y_*, T_*, P_*, and V_* with z, as shown in Fig. 2.

Nondimensional Optimal Parameters

Substituting the value of z^* from Eq. (15) in Eqs. (20b) and (21), the following nondimensional optimal values of other parameters have been obtained:

$$y_*^* = 1.08055; \quad A_*^* = 1.60036; \quad T_*^* = 2.22161; \quad P_*^* = 3.24003; \quad V_*^* = 0.62486 \tag{22}$$

Minimum Cost Canal Section

Although the best hydraulic section has the minimum flow area and the minimum wetted perimeter, it is not necessarily the most economical section. The design of a minimum cost canal section involves minimization of canal section cost subject to flow requirements.

Canal Section Cost

Canals in alluvium are generally lined (Swamee et al. 2000b). The major costs of a lined canal are earthwork costs and the cost of the lining.

Earthwork Cost. The cost of earthwork depends on the volume and depth of cut and fill. It also depends on the strata to be excavated and the distance of haulage if required in transporting the soil materials. For a canal section with the normal water surface at the average ground level as shown in Fig. 1, the earthwork cost C_e (monetary unit per unit length, e.g., \$/m) could be given (Chahar 2000; Swamee et al. 2000a, 2001) by

$$C_e = c_e A + c_r A\bar{y} \tag{23}$$

where c_e=cost per unit volume of earthwork at ground level (\$/m^3); c_r=additional cost per unit volume of excavation per unit depth ($/m^3/m of depth); and $\bar{y}$=depth (m) of the centroid of the area of excavation from the ground surface ($\bar{y}=2y/5$ for a parabolic section). It was assumed in Eq. (23) that the cost per unit volume of excavation is a linear function of the depth of excavation.

Lining Cost. Lining of an existing canal is found to be prohibitive because of the increased cost of lining material and construction and the restriction on closure of the canal. If the lining is envisaged in the planning stage, a smaller cross section can be adopted and the lining could be justified from an economic point of view. Assuming the cost per unit surface area of lining c_l ($/m^2) independent of the depth of placement, the cost of lining C_l ($/m) for the flow section would be

$$C_l = c_l P \tag{24}$$

Unit Length Canal Section Cost. Adding Eqs. (23) and (24), the cost of a parabolic canal per unit length C ($/m) has been obtained as

$$C = C_e + C_l = c_e A + c_r A \bar{y} + c_l P = \frac{8c_e}{3} y^2 z + \frac{16c_r}{15} y^3 z + c_l y f_z \tag{25}$$

Flow Requirements

This analysis assumes that a lined canal sustains a uniform flow through it. Manning's equation, Eq. (7), as an equality constraint function, may be used to provide uniform flow condition in the canal. Also, the design of a canal section must take into account the minimum permissible velocity, the limiting velocity, and the freeboard. The minimum permissible velocity or nonsilting velocity is the lowest velocity that will not initiate sedimentation and will not allow the growth of vegetation. Sedimentation and growth of vegetation decrease the carrying capacity and increase the maintenance cost of a canal. In general, an average velocity of 0.6–0.9 m/s will prevent sedimentation when the silt load of the flow is low, and a velocity of 0.75 m/s is usually sufficient to prevent the growth of vegetation (Chow 1973). Hence, the minimum permissible velocity $V_{\min}$ (m/s) can be assumed in the range from 0.75 to 0.9 m/s. Higher velocities are desired in lined (or rigid boundary) canals to reduce costs. However, high velocities may cause scour and erosion at the boundaries of the canal. In lined canals, the maximum permissible velocity or the limiting velocity V_L (m/s) that will not cause erosion depends on the lining material. Swamee et al. (2002) and Bureau of Indian Standards (1982) list limiting velocities for different types of linings.

Design Method

The cost of canal construction in an irrigation project is a major cost item, and maximum economy is usually achieved by constructing minimum cost canal sections assuming there are no other costs such as pumping or loss of small hydro. As stated earlier, the design of a minimum cost canal section involves minimization of canal section cost subject to the uniform flow condition. Essentially, it is a problem of minimization of a nonlinear objective function subject to a nonlinear equality constraint. The problem of determination of a minimum cost canal section has been formulated as

$$\text{Minimize} \quad C = \frac{8c_e}{3} y^2 z + \frac{16c_r}{15} y^3 z + c_l y f_z \tag{26a}$$

$$\text{Subject to} \quad \phi = Q - \frac{1}{n}\frac{A^{5/3}}{P^{2/3}} S_o^{1/2} = \phi(y,z) = 0 \tag{26b}$$

Using the length scale L, the following nondimensional cost variables have been defined:

$$C_* = \frac{C}{c_e}\frac{1}{L^2}; \quad c_{r^*} = \frac{c_r}{c_e}L; \quad c_{l^*} = \frac{c_l}{c_e}\frac{1}{L} \tag{27}$$

As c_e/c_r and c_l/c_e have length dimensions (m), they remain unaffected by the monetary unit chosen. These ratios can be obtained for various types of linings and soil strata using appropriate unit rates (Swamee et al. 2000b).

Using Eqs. (26) and (27), the determination of the optimal canal section shape in nondimensional form becomes

$$\text{Minimize} \quad C_* = \frac{8}{3} y_*^2 z + \frac{16c_{r^*}}{15} y_*^3 z + c_{l^*} y_* f_z \tag{28a}$$

$$\text{Subject to} \quad \phi = A_*^{5/3} - P_*^{2/3} = \left(\frac{8z}{3}\right)^{5/3} y_*^{10/3} - f_z^{2/3} y_*^{2/3}$$

$$= \left(\frac{8z}{3}\right)^{5/3} y_*^{8/3} - f_z^{2/3} = 0 \tag{28b}$$

Solving Eq. (28*b*) for y_* and then substituting its value in Eq. (28*a*):

$$\text{Minimize} \quad C_* = \left(\frac{3}{8z}\right)^{1/4} f_z^{1/2} + \frac{2c_{r^*}}{5}\left(\frac{3}{8z}\right)^{7/8} f_z^{3/4} + c_{l^*}\left(\frac{3}{8z}\right)^{5/8} f_z^{5/4} \tag{29}$$

This is a nonlinear unconstrained optimization problem in a single variable z. Using the Fibbonacci search (Bazaraa and Shetty 1979), the optimal values of z can be obtained for wide ranges of nondimensional cost variables. The cost and constraint functions as well as the optimization algorithm have been developed in dimensionless form. The methodology can be extended for different or more extensive sets of input and/or design variables.

Minimum Earthwork Cost Canal Section

A canal passing through hard/firm strata may be kept unlined, but it is designed as a rigid boundary channel. Sometimes canals are lined with low cost lining materials, in which case the cost of the earthwork is more significant than the cost of the lining. Neglecting the lining cost, the application of the optimization algorithm with input variables $0 \leqslant c_{r^*} \leqslant 20$ has yielded a large number of optimal sections. The variations in the optimal side slope, depth of flow, and cost with c_{r^*} have been plotted as symbols in Fig. 3. Fig. 3 also shows the regressed curves for each case, and it can be seen that the regressed curves almost represent the variation in these parameters (correlation coefficient=0.999). The equations of these regressed curves for the optimal parameters of a parabolic canal are as follows:

$$z^* = 0.514 + 0.3536(c_r L/c_e) - 0.0244(c_r L/c_e)^2 + 2.8 \times 10^{-3}(c_r L/c_e)^3 \tag{30a}$$

$$\frac{y^*}{L} = 0.7993 - 0.2121[\log(c_r L/c_e)] - 0.1231[\log(c_r L/c_e)]^2 - 0.0261[\log(c_r L/c_e)]^3 \tag{30b}$$

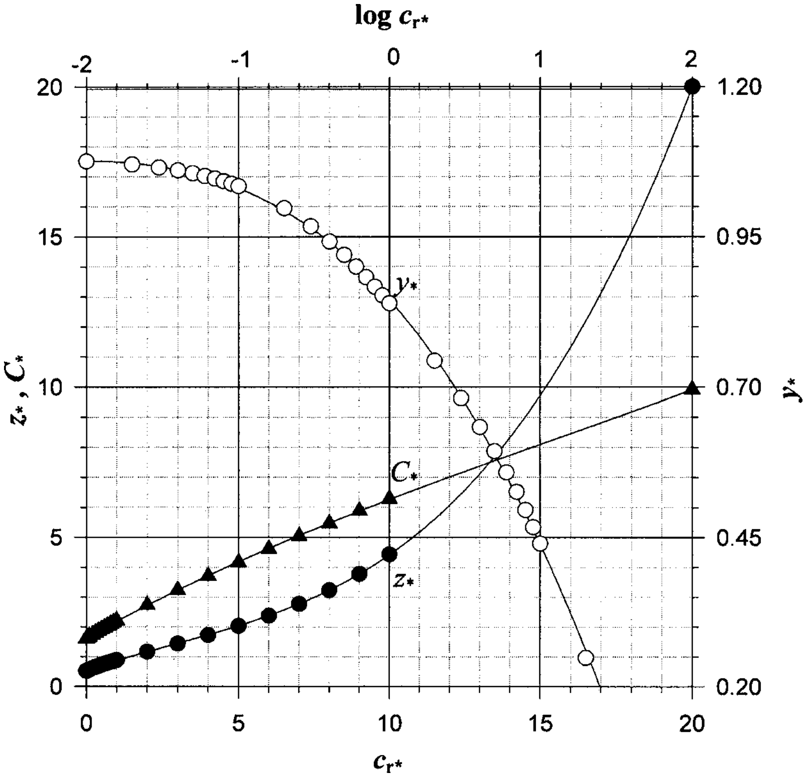

Fig. 3. Variations in z^*, y_*^*, and C_*^* with c_r^*

$$\frac{C^*}{c_e L^2} = 1.6001 + 0.5762(c_r L/c_e) - 0.0143(c_r L/c_e)^2 + 3.1 \times 10^{-4}(c_r L/c_e)^3 \quad (30c)$$

An examination of Fig. 3 or the optimal design equations, Eqs. (30*a*) and (30*b*), shows that the side slope of the optimal section increases and the normal depth of the optimal section decreases with the increase in additional cost of excavation with canal depth. Accordingly, the optimal section is wider and shallower than the minimum area section. Also, the optimal dimensions are very sensitive for $c_r L/c_e > 1$.

Minimum Cost Lined Section

In most cases canals are lined, and the cost of lining and earthwork constitutes the total canal section cost. Application of the optimization algorithm with input variables $c_{r^*} \leqslant 20\, c_{l^*}$ has yielded a large number of optimal sections. An analysis similar to that of Chahar (2000) of these optimal sections has led to the following empirical equations for the parabolic canal:

$$z^* = 0.514 \frac{c_e L + 0.7626 c_r L^2 + 6.4944 c_l}{c_e L + 6.4944 c_l} \quad (31a)$$

$$y^* = 1.0806 \frac{c_e L + 6.7052 c_l}{c_e L + 0.3345 c_r L^2 + 6.7052 c_l} L \quad (31b)$$

$$C^* = 0.6307 c_r L^3 + 1.6004 c_e L^2 + 3.2421 c_l L \quad (31c)$$

These equations have been obtained by conceiving an appropriate functional form and then minimizing errors between the optimal values and the computed values from the conceived function with coefficients. For $c_r = 0$, Eqs. (31*a*) and (31*b*) reduce to equations for the minimum area section viz. Eqs. (15) and (22).

Eqs. (30*a*)–(30*c*) give nearly exact solutions and Eqs. (31*a*)–(31*c*) give close approximations within the practical limits. Alternatively, y can be determined exactly using Eq. (20*a*) once the side slope is fixed by Eqs. (30*a*) or (31*a*), because it is a function of z only. Then, C can be arrived at exactly with the help of Eq. (25).

Bounds on Canal Dimensions or Velocity

Thus far in the analysis, it has been assumed that the depth and the top width are unconstrained. However, in practice, limits are encountered on y (i.e., $y \leqslant y_L$) due to the existence of unfavorable strata and/or groundwater at a shallow depth and on T (i.e., $T \leqslant T_L$) due to restrictions on the span of bridges and cross drainage works, the width of right of way, etc. Sometimes the side slope is chosen based on the angle of repose of material for better stability or for vehicles to cross the channel during no-flow periods (i.e., $z \geqslant z_L$). The variables with subscript L denote their limiting value. Due to constraints on canal dimensions, the minimization problem becomes a nonlinear optimization problem with equality and inequality constraints. In general, a nonlinear optimization problem with equality and inequality constraints is difficult to solve because an equality constraint may make the constraint region nonconvex. Because the side slope parameter z is the governing parameter for a parabola, the optimization process will drive z toward the optimal section, but this could be prevented by the constraints on the depth of flow or on the top width, i.e., by y_L or T_L. The constraints on canal dimension become effective only if $y_L <$ unconstrained y^*, or $T_L <$ unconstrained T^*, or $z_L >$ unconstrained z^*. Otherwise, the prescribed bound on a particular parameter will be nonbinding; hence, the parameter as well as the remaining parameters will attain their unconstrained optimal values.

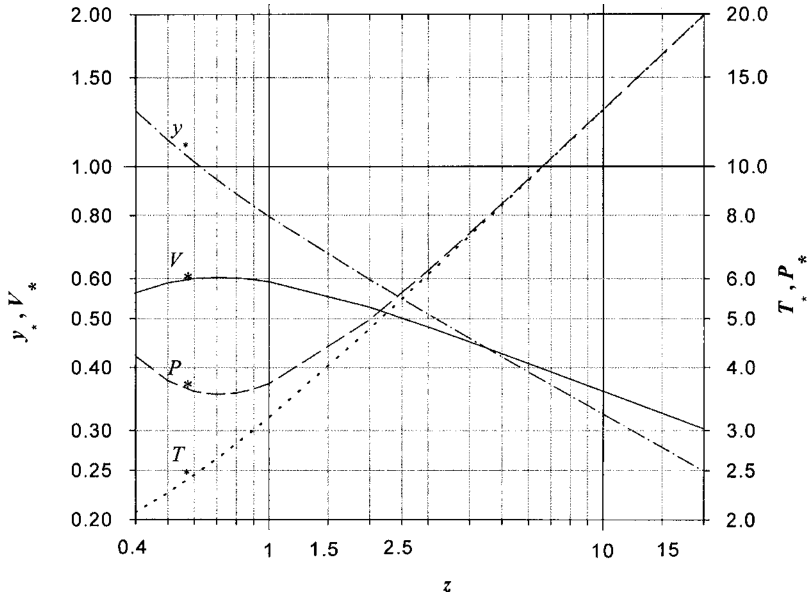

Fig. 4. Variations in nondimensional parameters with z (approximate perimeter)

The imposition of constraints y_L or T_L restrains the parabola in achieving the most preferred z value for that discharge, and it is forced to attain the z values as permitted by the limiting y_L or T_L. For example, if y_L is binding (i.e., $y_L <$ unconstrained y^*), the optimal value of depth of flow equals its bound (i.e., $y^* = y_L$) and the top width must be large enough to permit the design discharge to pass through the optimal area. For any known y, determination of T is not straightforward, because it is a function of z as given by Eq. (21*b*) and z is implicitly involved with y, as shown by Eq. (20*b*). Plotting Eq. (20*b*) for z as a dependent variable and then using regression:

$$\log z^* = 2.9493 - 8.3568 y_*^* + 8.3924 y_*^{*2} - 3.1972 y_*^{*3} \tag{32}$$

Similarly, if T_L is binding (i.e., $T_L <$ unconstrained T^*), the optimal value of top width equals its bound (i.e., $T^* = T_L$) and the depth of flow must be large enough to permit the design discharge. Solving Eq. (21*b*) for z using curve fitting:

$$z^* = 0.0998 T_*^{*2} + 0.038 T_*^* - 0.0622 \tag{33}$$

Many times the value of z is fixed if the side slope is chosen based on the angle of repose of material for better stability or for vehicles to cross the channel during no-flow periods. In such situations, z becomes binding (i.e., $z^* = z_L$) and the depth of flow has to be modified [using Eq. (20*a*)] to carry the design discharge.

Practically, restrictions can be imposed on the velocity of flow as well (i.e., $V_{\min} \leq V \leq V_L$). A classical optimal section yields the greatest flow velocity $V_{\max}$ for a given discharge, which also implies that the specification of a velocity larger than the velocity in a classical optimal section (i.e., $V_L > V_{\max}$) will result in infeasibility for that discharge. Therefore, the bound on velocity becomes effective if $V_L <$ unconstrained V^*. If the velocity is binding (i.e., $V_L <$ unconstrained optimal velocity), then $V^* = V_L$ and there will be two candidate cross sections, because the flow area increases in either direction (i.e., for increasing and decreasing values of z) from $z = 0.514$. Solving Eq. (21*d*) for z using regression yields

$$\log z^* = 5.1431 - 26.3232 V_*^* + 28.0883 V_*^{*2} \tag{34}$$

for $z < 0.514$; and

$$\log z^* = 4.6862 - 18.8135 V_*^* + 32.0774 V_*^{*2} - 23.1167 V_*^{*3} \tag{35}$$

for $z > 0.514$. As both sections have the same flow area, the wider of the two sections (i.e., $z > 0.514$) is preferred if an increase in earthwork cost with depth of excavation is the primary interest, while the narrower section (i.e., $z < 0.514$) is selected if such a cost is ignored and freeboard is to be provided.

Once the side slope is fixed based on the appropriate binding case, other parameters can easily be determined, because they are function of z only. This can be accomplished by using Eqs. (20*b*) and (21*b*)–(21*d*) or by using Fig. 2.

Approximate Perimeter

When $z \geq 1$ or $0 < 4y/T \leq 1$, Eq. (4) can satisfactorily be approximated (Chow 1973) to

$$P = 4yz + \frac{2y}{3z} = y\left(4z + \frac{2}{3z}\right) \tag{36}$$

Therefore:

$$f_z = \left(4z + \frac{2}{3z}\right) \tag{37}$$

for the approximate perimeter case. Using f_z from Eq. (37) in Eqs. (20*b*) and (21*b*)–(21*d*), a set of graphs have been plotted for variations in y_*, T_*, P_*, and V_* with z as shown in Fig. 4 for the approximate perimeter. Substituting the values from Eqs. (2) and (36) in Eq. (12):

$$\left(\frac{16}{3}yz\right)\left[\frac{2y}{3z^2}(6z^2 - 1)\right] = \left(\frac{8}{3}y^2\right)\left[\frac{2}{3z}(6z^2 + 1)\right] \tag{38}$$

and simplifying:

$$z^* = 1/\sqrt{2} = 0.707 \tag{39}$$

Substituting the value of z^* from Eq. (39) in Eqs. (20*b*) and (21*b*)–(21*d*) yields the nondimensional optimal values of other parameters as

$$y_*^* = 0.93748; \quad A_*^* = 1.65721; \quad T_*^* = 2.65159;$$
$$P_*^* = 3.53546; \quad V_*^* = 0.60342 \qquad (40)$$

Adopting a similar procedure as followed in the exact perimeter case, the optimal design equations for a minimum earthwork cost section are

$$z^* = 0.707 + 0.3048(c_rL/c_e) - 0.0198(c_rL/c_e)^2 + 2.6 \times 10^{-3}(c_rL/c_e)^3 \qquad (41a)$$

$$\frac{y^*}{L} = 0.8450 - 0.2826[\log(c_rL/c_e)] - 0.1104[\log(c_rL/c_e)]^2 - 0.0135[\log(c_rL/c_e)]^3 \qquad (41b)$$

$$\frac{C^*}{c_eL^2} = 1.6572 + 0.5573(c_rL/c_e) - 0.0121(c_rL/c_e)^2 + 2.4 \times 10^{-4}(c_rL/c_e)^3 \qquad (41c)$$

while those for a minimum cost lined section are

$$z^* = 0.707\frac{c_eL + 0.4357c_rL^2 + 6.2435c_l}{c_eL + 6.2435c_l} \qquad (42a)$$

$$y^* = 0.9375\frac{c_eL + 6.7473c_l}{c_eL + 0.1979c_rL^2 + 6.7473c_l}L \qquad (42b)$$

$$C^* = 0.5889c_rL^3 + 1.6572c_eL^2 + 3.5364c_lL \qquad (42c)$$

for the approximate perimeter case. These equations reduce to the corresponding equations for the minimum area section for $c_r=0$.

The approximate perimeter can be used when $z \geqslant 1$. Eq. (39) shows this using the approximate perimeter, $z^*=0.707$, for a classical optimal section. This implies that it is not appropriate to use the approximate perimeter in arriving at a best hydraulic section. Similarly, the binding constraints on the top width should not be dealt within an approximate perimeter because such constraints become effective only if $z<0.707$. However, the bounds on the depth of flow are binding if $z>0.707$; hence, the approximate perimeter can safely replace the exact perimeter, and in this range the regressed relation is

$$\log z^* = 3.0224 - 9.0446y_*^* + 10.1101y_*^{*2} - 4.3718y_*^{*3} \qquad (43)$$

The assumption of an approximate perimeter is appropriate in dealing with the bounds imposed on the velocity of flow in the region $z>0.707$. In this region the following regressed equation, which results in a wider section than a classical optimal section, is obtained:

$$\log z^* = 4.7213 - 19.2026V_*^* + 33.4530V_*^{*2} - 24.6559V_*^{*3} \qquad (44)$$

As with the exact perimeter case, the other parameters can be determined using Eqs. (20*b*) and (21*b*)–(21*d*) or by using Fig. 4.

Optimal Design Procedure, Examples, and Discussion

Based on the presented equations, a rigid boundary optimal parabolic channel section can be designed by adopting the following steps:

1. Choose n for a particular type of lining.
2. For a given set of data (Q and S_o) and chosen n, find L using Eq. (17).
3. Use of the appropriate optimal design equation for a particular optimization case (minimum area, minimum earthwork cost, or minimum cost lined section) results in the optimal canal side slope.
4. Using the optimal side slope, the remaining geometrical parameters in nondimensional form can be obtained with help of Eqs. (20*b*) and (21*b*)–(21*d*) or using Fig. 2. If there is no bound on canal dimension or velocity, then skip Steps 5–7.
5. Using L and given bounds on "canal dimension" or velocity, find the corresponding nondimensional parameter. Compare this nondimensional parameter with the corresponding optimal nondimensional parameter as determined in Step 4. If it becomes a binding parameter, then it should be adopted as an optimal parameter.
6. Using the adopted optimal parameter, determine z by Eqs. (32)–(35) depending upon the binding case. Alternatively, read out the value of z corresponding to the binding parameter from Fig. 2.
7. Once the side slope is fixed based on the appropriate binding case; the remaining geometrical parameters in nondimensional form can be obtained with the help of Eqs. (20*b*) and (21*b*)–(21*d*) or with the addition of Fig. 2.
8. Use of L and nondimensional parameters yields corresponding parameters for the optimal parabolic canal.
9. For the designed section, the average flow velocity V can be obtained by Eq. (2), i.e., $V=Q/A$, or with the help of Fig. 2 or Eq. (21*d*). This velocity should be greater than the nonsilting velocity but less than the limiting velocity V_L.
10. If V is greater than V_L, redesign the section with revised bed slope or surface roughness.
11. Find the minimum cost for the optimal section using the appropriate optimal cost equation [say, Eqs. (30*c*) or (31*c*)] for the case in hand. Alternatively, the cost of the section can be obtained from the cost function Eq. (25) once the section dimensions are decided.

Example 1

Design an optimal parabolic canal to carry a discharge of 50 m^3/s on a longitudinal bed slope of 0.001.

Solution

Assuming a float finished concrete lining, Manning's roughness coefficient=0.015 (Chow 1973).

From Eq. (17), $L=(50\times.015/\sqrt{0.001})^{3/8}=3.2783$ m. Using Eqs. (15) and (22) $z^*=0.514$, $y_*^*=1.08055$, $P_*^*=3.24003$, $T_*^*=2.22161$, $A_*^*=1.60036$, and $V_*^*=0.62486$; therefore, $y^*=1.08055\times3.2783=3.5424$ m, $P^*=3.24003\times3.2783=10.6218$ m, $T^*=2.22161\times3.2783=7.2831$ m, $A^*=1.60036\times(3.2783)^2=17.1995$ m^2, and $V^*=2.9071$ m/s. Alternatively, $V=Q/A=50/21.3411=2.9071$ m/s, which is less than the limiting velocity for a concrete lining (4.0 m/s).

For the approximate perimeter, the corresponding parameters are $z^*=0.707$, $y^*=3.0736$ m, $P^*=11.5903$ m, $T^*=8.6921$ m, $A^*=17.8105$ m^2, and $V^*=2.8073$ m/s<(4.0 m/s).

Example 2

Redesign a minimum cost parabolic canal section for the similar data if: (1) only earthwork cost is considered; and (2) earthwork and lining costs are taken into account. Take $c_l/c_e=13.0$ m, and $c_e/c_r=7.0$ m.

Solution

Case 1. Putting c_e/c_r=7.0 m and L=3.2783 in Eqs. (30a) and (30b): z^*=0.6745, and y^*=3.1048 m. Similarly, using Eq. (30c) yields C^*=20.063c_e $/m length of the canal. Using z and y so obtained: P^*=10.8380 m, T^*=8.3767 m, A^*=17.3387 m^2, and V^*=2.884 m/s.

The design parameters for the approximate perimeter are z^*=0.846, y^*=2.8177 m, C^*=20.587c_e $/m length of the canal, P^*=11.7556 m, T^*=9.5352 m, A^*=17.9116 m^2, and V^*=2.791 m/s.

Case 2. Dividing the numerator and denominator of the right-hand side of Eqs. (31a) and (31b) by c_e, dividing Eq. (31c) by c_e, and substituting c_l/c_e=13.0 m, c_e/c_r=7.0 m, and L=3.2783 m result in z^*=0.521, y^*=3.519 m, and C^*=158.546c_e $/m length of the canal. Using z and y so obtained, P^*=10.6224 m, T^*=7.3326 m, A^*=17.1999 m^2, and V^*=2.907 m/s.

The optimal parameters for the approximate perimeter are z^*=0.713, y^*=3.0606 m, C^*=171.488 c_e, P^*=11.5907 m, T^*=8.7289 m, A^*=17.8107 m^2, and V^*=2.807 m/s.

Example 3

Redesign an optimal parabolic canal for the data as stated in Example 1 if: (1) maximum possible depth of flow=2.0 m; (2) top width is restricted to 6.0 m; (3) side slope to be provided=1.25; and (4) permissible velocity=2.0 m/s.

Solution

L=3.2783 m, as calculated earlier.

Case 1. As y_L=2.0, y_{L^*}=2.0/3.2783=0.6101, which is less than the unconstrained optimal value of 1.08055; hence, it is binding and optimal for the present case (y_*^*=0.6101). From Eq. (32), the value of z corresponding to this value equals 1.773. For z^*=1.773, the value of T^*=14.5482 m, P^*=15.2864 m, A^*=19.8957 m^2, and V^*=2.513 m/s.

For the approximate perimeter, the optimal parameters are z^*=1.882, P^*=15.798 m, T^*=15.0877 m, A^*=20.1592 m^2, and V^*=2.480 m/s.

Case 2. If top width is restricted to 6.0 m, then T_{L^*}=6.0/3.2783=1.8302, which is less than the unconstrained optimal value of 2.22161; hence, it is binding and optimal for the present case (T_*^*=1.8302). From Eq. (33), the value of z corresponding to this value equals 0.342. For z^*=0.342, the value of y^*=4.381 m, P^*=11.0991 m, A^*=17.505 m^2, and V^*=2.856 m/s.

Case 3. If side slope z=1.25, then y^*=2.365 m, P^*=12.985 m, T^*=11.823 m, A^*=18.638 m^2, and V^*=2.683 m/s. The corresponding parameters are y^*=2.369 m, P^*=13.109 m, T^*=11.8458 m, A^*=18.7098 m^2, and V^*=2.672 m/s for considering the approximate perimeter.

Case 4. If permissible velocity V_L=2.0 m/s, then V_*=0.4299, which is less than the unconstrained optimal value of 0.6236; hence, it is binding and optimal for the present case (V_*^*=0.4299). Corresponding to this value from Eqs. (34) and (35), two values of z^* (0.104 and 4.898) are possible, out of which one leads to a wider section and the other a narrower section. Considering the lower value of z^*, i.e., 0.104, corresponding design parameters calculated from the appropriate equations are y^*=8.770 m, P^*=18.1967 m, A^*=21.3321 m^2, and T^*=3.6485 m. Similarly for the wider section, z^* is equal to 4.898 and the corresponding design parameters are y^*=1.3865 m, P^*=27.3520 m, T^*=27.1645 m, and A^*=25.1092 m^2.

When velocity is binding for the approximate perimeter case, z^*=4.895 and the other design parameters are y^*=1.3868 m, P^*=27.3433 m, T^*=27.1544 m, and A^*=25.1059 m^2.

Discussion

The cost function includes a depth-dependent excavation cost and the lining cost, assuming the water surface is at the average ground level. This assumption also implies that the canal bed is parallel to the average ground level so grading is not required and hence the cost function does not include this cost. For the canals in cutting, the earthwork and lining costs for the section above the water surface can easily be incorporated into the cost function by including the excavation area and the lined perimeter in place of the flow area and the wetted perimeter. The earthwork costs for canals in embankments or filling needs different treatment. The present worth of any cost item can be incorporated in the corresponding nondimensional cost.

To simplify the design problem, freeboard has not been included in the optimization scheme. The required freeboard may be provided over the optimal section or the optimal section may be designed for an increased discharge, leaving a margin for the freeboard, but such a section deviates from the least cost section. However, including a depth-dependent freeboard in the optimal problem formulation, the present method can be extended for designing an optimal canal section considering freeboard.

Furthermore, permissible velocities are not taken as constraints in the optimization scheme. Steep and smooth boundary channels may result in high average velocities. The average velocity in the designed section should be checked with the permissible velocities. If it is greater than the limiting velocity, then it is necessary to use a canal route with a smaller bed slope, adopt a flatter bed slope with drops, build two smaller canals, select a superior lining material having a higher limiting velocity, or use a lining material with a larger roughness. These options result in a larger value for the length scale and hence lower velocity.

Loganathan's (1991) tabular results are inconvenient to use. Direct optimization procedures require a considerable amount of programming and computation. In comparison, the proposed design equations [e.g., Eqs. (30a)–(30c)] are in explicit form and hence result in optimal design variables in single step computations. Eqs. (30a)–(30c) give nearly exact solutions, while (Eqs. (31a)–(31c) give close approximations in the range $c_{r^*} \leqslant 20\, c_{l^*}$. This range covers most of the practical cases. However, Eqs. (31a)–(31c) should not be used if $c_{r^*} \geqslant 20\, c_{l^*}$. Also, the optimal dimensions are very sensitive when $c_{r^*} > 1$ in minimum earthwork cost canal section.

The presented equations apply only in the design of non-erodible (i.e., rigid boundary) channels. For erodible channels, the principle of tractive force must be used to determine an efficient section.

It can be seen from the design examples that for $z > 1$ the approximate expression for the perimeter closely replaces the exact expression and for a large side slope ($z > 2$) both expressions yield identical results.

Conclusions

A direct optimization procedure may be adopted for the minimum cost parabolic canal sections, but they require a considerable amount of programming and computation. Using the results of a direct optimization procedure and error minimization or regression analysis, generalized explicit equations can be obtained for designing minimum earthwork cost and minimum cost lined sections of a parabolic shape. Similar equations can be obtained for bounds on canal dimensions and velocity of flow in the canal. The presented equations are convenient to use in the optimal design of a parabolic canal, because they result in canal dimensions in single step computations. The approximate perimeter assumption is not appropriate for a classical optimal parabolic section.

Acknowledgment

The writer would like to thank the anonymous reviewers for their insightful review and constructive suggestions, which resulted in a significant improvement of the manuscript.

Notation

The following symbols have been used in this paper:

A = flow area of canal [m^2];
C = cost per unit length of canal [\$/m];
C_e = cost of earthwork per unit length of canal [\$/m];
C_l = cost of lining per unit length of canal [\$/m];
c_e = cost per unit volume of earthwork at ground level [\$/m^3]
c_l = cost per unit surface area of lining [\$/m^2]
c_r = increase in unit excavation cost per unit depth [\$/m^4]
f_z = function of z [dimensionless];
L = length scale [m];
n = Manning's roughness factor [dimensionless];
P = flow perimeter of canal [m];
Q = discharge [m^3/s];
R = hydraulic radius [m];
S_f = energy or friction slope [dimensionless];
S_o = bed slope of canal [dimensionless];
T = top width at free water surface [m];
V = average velocity [m/s];
X = horizontal axis;
Y = vertical axis;
y = normal depth of flow in canal [m];
z = side slope of canal [dimensionless];
ϕ = equality constraint [dimensionless]; and
\$ = monetary unit.

Subscripts

$*$ = nondimensional; and
L = limiting value.

Superscripts

$*$ = optimal.

References

Babaeyan-Koopaei, K. (2001). "Dimensionless curves for normal depth calculations in canal sections." *J. Irrig. Drain. Eng.*, 127(6), 386–389.

Babaeyan-Koopaei, K., Valentine, E. M., and Swailes, D. C. (2000). "Optimal design of parabolic-bottomed triangle canals." *J. Irrig. Drain. Eng.*, 126(6), 408–411.

Bazaraa, M. S., and Shetty, C. M. (1979). *Nonlinear programming: Theory and algorithms*, Wiley, New York.

Bureau of Indian Standards. (1982). "Criteria for design of lined canals and guidelines for selection of type of lining." *IS:10430*, Bureau of Indian Standards, New Delhi, India.

Chahar, B. R. (2000). "Optimal design of channel sections considering seepage and evaporation losses." PhD thesis, Dept. of Civil Engineering, Univ. of Roorkee, Roorkee, India.

Chow, V. T. (1973). *Open channel hydraulics*, McGraw-Hill, New York.

Froehlich, D. C. (1994). "Width and depth constrained best trapezoidal section." *J. Irrig. Drain. Eng.*, 120(4), 828–835.

Kreyszig, E. (2001). *Advanced engineering mathematics*, Wiley, Singapore.

Loganathan, G. V. (1991). "Optimal design of parabolic canals." *J. Irrig. Drain. Eng.*, 117(5), 716–735.

Mironenko, A. P., Willardson, L. S., and Jenab, S. A. (1984). "Parabolic canal design and analysis." *J. Irrig. Drain. Eng.*, 110(2), 241–246.

Monadjemi, P. (1994). "General formulation of best hydraulic channel section." *J. Irrig. Drain. Eng.*, 120(1), 27–35.

Streeter, V. L. (1945). "Economical canal cross sections." *Trans. Am. Soc. Civ. Eng.*, 110, 421–430.

Swamee, P. K. (1995). "Discussion of 'General formulation of best hydraulic channel section,' by P. Monadjemi." *J. Irrig. Drain. Eng.*, 121(2), 222.

Swamee, P. K., Mishra, G. C., and Chahar, B. R. (2000a). "Minimum cost design of lined canal sections." *J. Water Resour. Plan. Manage.*, 14(1), 1–12.

Swamee, P. K., Mishra, G. C., and Chahar, B. R. (2000b). "Comprehensive design of minimum cost irrigation canal sections." *J. Irrig. Drain. Eng.*, 126(5), 322–327.

Swamee, P. K., Mishra, G. C., and Chahar, B. R. (2001). "Design of minimum earthwork cost sections." *J. Water Resour. Plan. Manage.*, 15(1), 17–30.

Swamee, P. K., Mishra, G. C., and Chahar, B. R. (2002). "Optimal design of transmission canal." *J. Irrig. Drain. Eng.*, 128(4), 234–243.