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Recharge/seepage from an array of rectangular channels

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Summary An exact analytical solution for the quantity of Recharge/Seepage from an array of rectangular channels underlain by a drainage layer and with pressure at a shallow/large depth has been obtained using an inverse hodograph and Schwarz–Christoffel transformation. The symmetry about the vertical axis has been utilized in obtaining the solution for half of the seepage domain only. The solution also includes relations for variation in seepage velocity along the channel perimeter and a set of parametric equations for the location of phreatic line. From this generalized case, particular solutions have also been deduced for an array of rectangular channels without pressure, without drainage layer or a single rectangular channel with drainage layer at finite/infinite depth. The solution is useful in quantifying artificial recharge and/or seepage loss from an array of rectangular channels.
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Introduction

Steadily increasing population, and expanding industries and developments are demanding more and more groundwater for irrigation, and industrial and domestic purposes. Due to extraction of groundwater at an alarming rate, the problems of groundwater depletion and their deleterious consequences have surfaced in different parts of the world. A variety of responses have been forged to mitigate or reverse these consequences.

Artificial groundwater recharging (AGR) is one of the most important responses, which is required to increase the natural supply of groundwater. The concept of artificial recharging is not new, and a lot of work has been done in this field. Various researchers have quantified the groundwater recharge analytically and numerically adopting various techniques like wells, ditches, basins etc. Artificial recharge of groundwater is an augmentation of the natural movement of the surface water into underground formations by artificially changing natural conditions. Main purposes of the artificial recharge (Todd, 1995) are: (a) to combat adverse conditions such as progressive lowering of

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Notations

The following symbols are used in this paper:

a', b', c'...	points on flow domain (dimensionless)
B	spacing between the channels (m)
b	bed width of channel (m)
C_1, C_2, C_3, ...	constants (dimensionless)
d	depth of drainage layer/aquifer (m)
d''	pressure corrected depth of drainage layer/aquifer (m)
F_s	seepage function (dimensionless)
$F(.,.)$	incomplete elliptical integral of the first kind (dimensionless)
$F_1(\tau, \alpha, \lambda)$	integral defined by Eq. (A3) (dimensionless)
$F_2(\tau, \alpha, \lambda)$	integral defined by Eq. (A12) (dimensionless)
$F(\tau, \alpha, \lambda, \gamma, \beta)$	integral defined by Eq. (A11) (dimensionless)
$f(\alpha, \lambda, \beta, \gamma)$	optimization function defined by Eq. (27) (dimensionless)
$f_1(\alpha, \lambda, \beta, \gamma)$	integral of Eq. (8) defined by Eq. (27) (dimensionless)
$f_2(\alpha, \lambda, \beta, \gamma)$	integral of Eq. (9) defined by Eq. (27) (dimensionless)
$f_3(\alpha, \lambda, \beta, \gamma)$	integral of Eq. (10) defined by Eq. (27) (dimensionless)
$f_4(\alpha, \lambda, \beta, \gamma)$	integral of Eq. (18d) defined by Eq. (27) (dimensionless)
h	potential head (m)
i	imaginary number (dimensionless)
$K(.)$	complete elliptical integral of the first kind (dimensionless)
k	hydraulic conductivity (m/s)
k'	assumed coordinate of a' in dZ/dW (s/m)
p	pressure at drainage layer/aquifer (N/m^2)
p_0	pressure head at drainage layer/aquifer (m)
q_s	seepage discharge per unit length of channel (m^2/s)
u	velocity of seeping water along X axis (m/s)
V	resultant velocity of seeping water (m/s)
v	velocity of seeping water along Y axis (m/s)
W	complex potential (m^2/s)
X	real axis of the complex plane (m)
Y	imaginary axis of the complex plane (m)
y	water depth in channel (m)
y_s	depth of point of intersection of two phreatic surfaces (m)
Z	complex plane variable (m)
α, λ, β, γ	accessory parameters (dimensionless)
ϕ	velocity potential (m^2/s)
ζ	complex variable in auxiliary plane (dimensionless)
ψ	stream function (m^2/s)
τ, t	dummy variables (dimensionless)

groundwater levels, unfavorable salt balance, and saline water intrusion; (b) to provide subsurface storage for local or imported surface waters; (c) to provide treatment and storage for reclaimed wastewater for subsequent reuse; (d) to maintain or augment natural groundwater as an economic resource, etc. Understanding the distribution and rate of groundwater recharge is a basic prerequisite for effective groundwater resource management and is one of the keys to economic development in rapidly expanding urban, industrial, and agricultural regions. The groundwater recharging is basically flow through porous media and is governed by identical principles as for seepage loss from water bodies.

The seepage or recharge from a channel is governed by hydraulic conductivity of the subsoil, channel geometry, hydraulic gradient between the channel and the aquifer underneath, and the initial and boundary conditions (ICID, 1967). Depending upon the geometry of the flow domain, one of the following boundary conditions may exist (Bouwer, 1978): (i) a thin layer of sediment or other fine material just below the channel and much lower hydraulic conductivity underlain by a uniform porous medium; (ii) porous medium underlain by a drainage layer at a finite depth, and water table is above the top of the drainage layer; (iii) porous medium underlain by a drainage layer at finite depth, and water table is below the top of the drainage layer; (iv) water table at a finite depth in a porous medium of infinite depth; and (v) porous medium of infinite depth in which water table at infinite depth or a drainage layer and water table both at infinite depth. The seepage or recharge will be highest when the channel bed approaches a drainage layer containing the water table below its top surface. On the contrary, it is least when the water table approaches the water level in the channel.

Traditionally open channel is used to convey water from a source to a destination for irrigation, industrial or domestic use. Seepage through conveyance channels was considered as a loss to the end user and hence channels were designed to minimize the seepage (Chahar, 2006, 2007; Kacimov, 1992; Swamee et al., 2000, 2002a,b). Whatever recharge was happening was taking place as incidental recharge. The quantity of seepage through channels was given by various researchers. Vedernikov (Harr, 1962) gave an exact mathematical solution to unconfined, steady-state seepage from a triangular and a trapezoidal canal in a homogeneous, isotropic, porous medium of large depth. The case of a rectangular canal was solved by Morel-Seytoux (1964). Swamee et al. (2001) obtained an analytical solution for seepage from a rectangular canal in a soil layer of finite depth overlying a drainage layer using inversion of hodograph and conformal mapping technique. Choudhary and Chahar (2006), and Bruch and Street (1967a,b) used the same method in computing seepage from trapezoidal and triangular canal underlain by a drainage layer at shallow depth, respectively. Chahar (2007) obtained an exact and exhaustive solution for quantifying seepage/recharge from a trapezoidal channel underlain by a drainage layer at shallow depth. Particular solutions for rectangular, triangular, slit and strip shaped channels were also obtained for drainage layer at shallow as well as at very large depth.

Array of channels are used to irrigate large areas of land and quantifying seepage through these channels is very important. Array of channels, also, provides a good method of groundwater recharge by providing large contact surface and less submergence as compared to basins which have problems of clogging (Bouwer, 2002). Besides recharging, the array of channels can be used to leach down the salts in areas prone to secondary salinity. Drainage from array of channels (Kapranov and Emikh, 2004; Emikh, 2006) and single channel (Polubarinova-Kochina, 1962; Aravin and Numerov, 1965) of zero depth for different boundary conditions were analysed by many Russian investigators. Hammad (1960) used an approximate method for finding out the solution of system of parallel channels having no particular geometric shape. Seepage through array of trapezoidal channel was given by El Nimr (1963). Bruch and Street (1967a) pointed out errors in solution given by both Hammad and El Nimr and also gave the solution for seepage through array of triangular channels. Seepage for rectangular array cannot be derived from the earlier solutions nor is it reported in the literature.

The present solution is given for seepage/recharge through array of rectangular channels. The inverse hodograph method has been used to derive equations for seepage/recharge through channels. The mapping adopted herein is different from that of the earlier researchers and is such that it can easily be reduced to solution for single rectangular channel.

Formulation and solution of problem

An array of rectangular channels of bed width b (m), spacing B (m) and depth of water y (m) is considered passing through a homogeneous isotropic porous medium of hydraulic conductivity k (m/s) underlain by a horizontal drainage layer at a depth d (m) below the water surface as shown in Fig. 1a. The regime corresponds to A as defined by Bouwer (1978) and pressure at the drainage layer p such that pressure head $p_0 = p/\rho g$ and $d'' = d - p_0$ The steady seepage discharge per unit length of channel q_s (m^2/s) complying with Darcy's law can be expressed in the following simplest form (Chahar, 2007; Swamee et al., 2000)

$$q_s = kyF_s \tag{1}$$

where F_s (dimensionless seepage function) is a function of channel geometry and boundary conditions.

The flow is assumed to be two-dimensional in the vertical plane as shown in Fig. 1a. The effects of capillarity, surface tension, infiltration, and evaporation are ignored. The seepage domain has symmetry about vertical axis Y so the half domain ($a'b'c'd'e'f'l'$) has been used in the analysis. Define a complex potential $W = \phi + i\psi$ (Fig. 1d) where ϕ = velocity potential (m^2/s) which is equal to k times the head h (m) and ψ = stream function (m^2/s) which is constant along streamlines. If the physical plane is defined as $Z = X + iY$, then Darcy's law yields $u = \partial\phi/\partial X = -k\partial h/\partial X$ and $v = \partial\phi/\partial Y = -k\partial h/\partial Y$; where u and v are velocity or specific discharge vectors (m/s) in X and Y directions, respectively. The hodograph $dW/dZ = u - iv$ (Fig. 1b) and the inverse hodograph dZ/dW (Fig. 1c) for half of the seepage flow domain ($a'b'c'd'e'f'l'$) have been drawn following the standard steps (Refer to Harr, 1962; Polubarinova-Kochina, 1962; or Strack, 1989). The dZ/dW plane and W plane have been mapped on the lower half ($\zeta \leqslant 0$) of an auxiliary (ζ) plane (Fig. 1(e)] using the Schwarz–Christoffel conformal transformation.

Mapping of the inverse hodograph plane on the auxiliary plane (see Appendix) is

$$\frac{dZ}{dW} = \frac{-k'F_2(\zeta,\alpha,\lambda)}{F_1(0,\alpha,\lambda)} \tag{2}$$

where α and λ = accessory parameters; $-ik'$ = unknown coordinate of point a' on inverse hodograph plane; t = dummy variable and $F_1(0,\alpha,\lambda)$ and $F_2(\tau,\alpha,\lambda)$ are defined in Eqs. (A3) and (A12), respectively, in Appendix. Whereas the W plane mapping on the auxiliary plane is

$$W = i\frac{kd''\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})}\int_\beta^\zeta \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} \tag{3}$$

where β and γ = accessory parameters; and $K(\sqrt{(\beta-\gamma)/\beta})$ = complete elliptical integral of the first kind with a modulus $(\sqrt{(\beta-\gamma)/\beta})$ (Byrd and Friedman, 1971). Using Eq. (2) and derivative of Eq. (3) and then integrating (see Appendix) result to the mapping of physical plane on the auxiliary plane as

$$Z = b/2 - iy + iy \times \int_1^\zeta F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \Big/ \int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{4}$$

where $F(\tau,\alpha,\lambda,\gamma,\beta)$ is defined in Eq. (A11) in Appendix.

The phreatic line $d'c'b'$ ($-\infty < \zeta < -\lambda$) can be located by manipulating Eq. (4) and then separating the real and imaginary parts

$$X = b/2 + y\int_{-\infty}^\zeta \left[\left(\int_\tau^{-\lambda} \frac{(t+\alpha)dt}{(1-t)^{0.5}(-t-\lambda)^{1.5}}\right) \times \frac{1}{\sqrt{(-\tau)(\beta-\tau)(\gamma-\tau)}}\right]d\tau \Big/ \int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{5}$$

$$Y = \frac{2y\pi F\left(\sin^{-1}\sqrt{\frac{\beta}{\beta-\zeta}}, \sqrt{\frac{\beta-\gamma}{\beta}}\right)}{\sqrt{\beta}\int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau} \tag{6}$$

where $F\left(\sin^{-1}\sqrt{\frac{\beta}{\beta-\zeta}}, \sqrt{\frac{\beta-\gamma}{\beta}}\right)$=elliptical integral of the first kind (Byrd and Friedman, 1971). At the intersection of the phreatic lines $X = B/2$ and $Y = y_s$. where y_s = depth of intersection of two phreatic lines (Fig. 1a). At the centre of the channel *i.e.* point f' Eq. (4) reduces to

$$b/y = 2\int_\beta^1 \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\tau-\beta)(\tau-\gamma)}}d\tau \Big/ \int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{7}$$

At the centre of drainage layer l' ($\zeta = \gamma$; $Z = -id$) Eq. (4) reduces to

$$d/y = 1 + \int_\gamma^\beta \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\beta-\tau)(\tau-\gamma)}}d\tau \Big/ \int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{8}$$

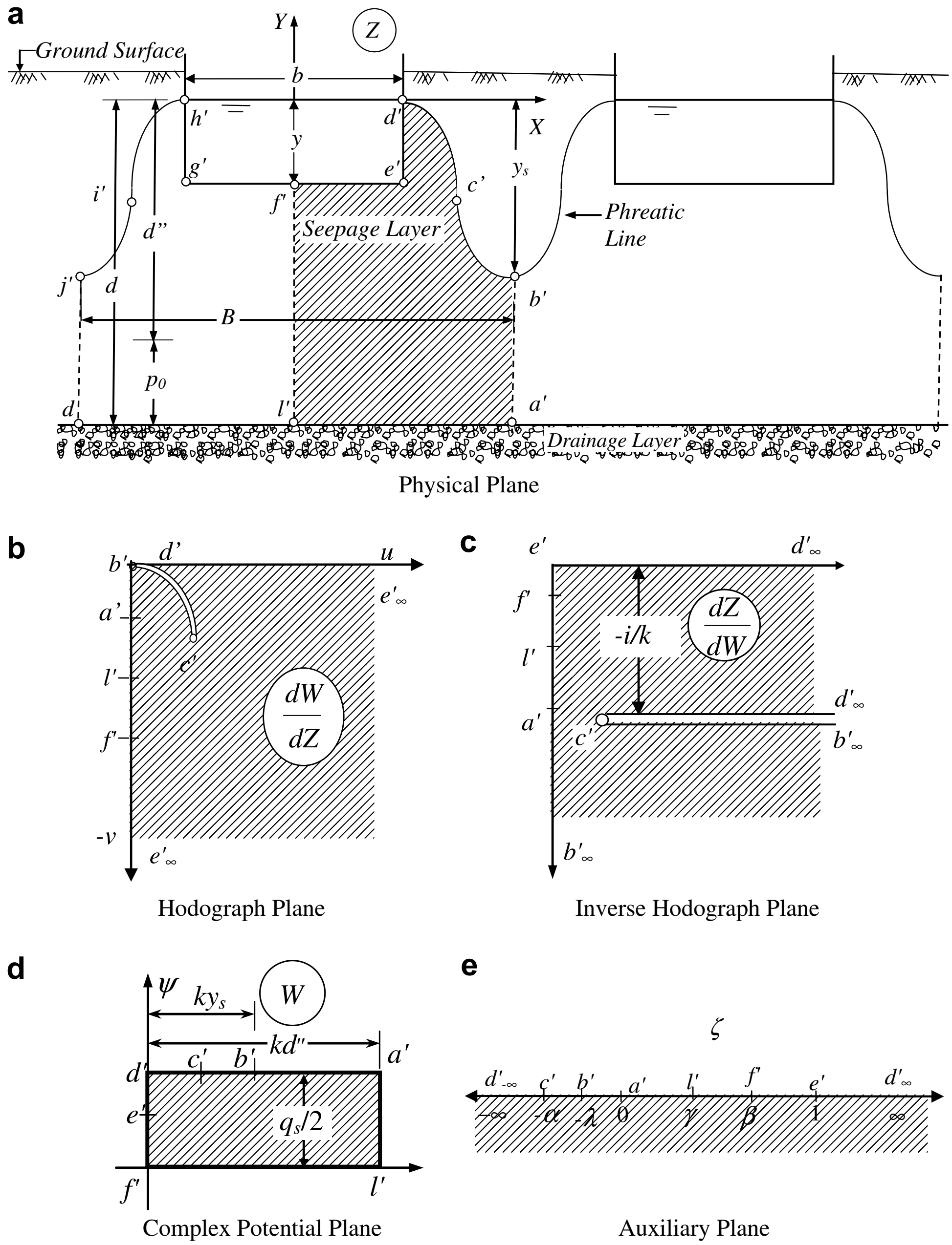

Figure 1 Seepage from an Array of Rectangular channels.

At point a' ($\zeta = 0$; $Z = B/2 - id$) it yields

$$B/y = 2\int_0^{\gamma} \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\beta-\tau)(\gamma-\tau)}}\mathrm{d}\tau \bigg/ \int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)\mathrm{d}\tau \qquad (9)$$

and at b' ($\zeta = -\lambda$; $Z = B/2 - iy_s$) Eq. (4) results

$$y_s/y = d/y - \int_{-\lambda}^{0} \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{(-\tau)(\beta-\tau)(\gamma-\tau)}}\mathrm{d}\tau \bigg/ \int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)\mathrm{d}\tau \qquad (10)$$

The seepage velocity along the channel perimeter can be obtained by manipulating Eq. (2) as

$$\frac{\mathrm{d}Z}{\mathrm{d}W} = \frac{1}{u - iv} = \frac{u + iv}{u^2 + v^2} = \frac{-k'F_2(\zeta,\alpha,\lambda)}{F_1(0,\alpha,\lambda)} \qquad (11)$$

Equating the real and imaginary parts and then squaring and adding them along the bed of the channel $f'e'$ ($\beta \leqslant \zeta \leqslant 1$) results

$$\frac{V}{k} = \frac{F_1(0,\alpha,\lambda)}{k \cdot k'F_1(\zeta,\alpha,\lambda)} \qquad (12)$$

where $V = \sqrt{u^2 + v^2}$ is the resultant velocity of seeping water. The denominator is zero when $\zeta = 1$; hence at the corner of the channel, the seepage velocity becomes infinite. The minimum seepage velocity along the bed of the channel occurs at the centre f' ($\zeta = \beta$) and is equal to

$$\frac{V}{k} = \frac{F_1(0,\alpha,\lambda)}{k \cdot k'F_1(\beta,\alpha,\lambda)} \qquad (13)$$

Relationship for the seepage velocity along the side of the channel $e'd'$ ($1 \leqslant \zeta \leqslant \infty$) is

$$\frac{V}{k} = \frac{F_1(0,\alpha,\lambda)}{k \cdot k' F_2(\zeta,\alpha,\lambda)} \tag{14}$$

which becomes zero at the water surface d' ($\zeta = \infty$).

Finally, the expression for the quantity of seepage can be obtained by using the values at the point a' ($\zeta = 0$; $W = kd' + iq_s/2$) in Eq. (3)

$$kd'' + \frac{q_s}{2}i = \frac{ikd''\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})}\int_\beta^0 \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} \tag{15}$$

which leads to

$$q_s = 2kd''K\left(\sqrt{\gamma/\beta}\right)\Big/K\left(\sqrt{(\beta-\gamma)/\beta}\right) \tag{16}$$

Equating Eq. (1) with Eq. (16) gives

$$F_s = 2\frac{d''}{y}K\left(\sqrt{\gamma/\beta}\right)\Big/K\left(\sqrt{(\beta-\gamma)/\beta}\right) \tag{17}$$

Further using the values at the point b' ($\zeta = -\lambda$; $W = ky_s + iq_s/2$) in Eq. (3) gives

$$y_s = d''\left(\frac{\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})}\int_{-\infty}^{-\lambda}\frac{dt}{\sqrt{(-t)(\gamma-t)(\beta-t)}}\right) \tag{18a}$$

After rearranging the terms and dividing both sides by y

$$\frac{d''}{y} = \frac{y_s}{y}\left(\frac{K(\sqrt{(\beta-\gamma)/\beta})}{F\left(\sin^{-1}\sqrt{\frac{\beta}{\beta+\lambda}},\sqrt{\frac{\beta-\gamma}{\beta}}\right)}\right) \tag{18b}$$

Putting y_s/y from Eq. (10) in the above equation

$$\frac{d''}{y} = \left(d/y - \frac{\int_{-\lambda}^0 \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{(-\tau)(\beta-\tau)(\gamma-\tau)}}d\tau}{\int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau}\right)\frac{K(\sqrt{(\beta-\gamma)/\beta})}{F\left(\sin^{-1}\sqrt{\frac{\beta}{\beta+\lambda}},\sqrt{\frac{\beta-\gamma}{\beta}}\right)} \tag{18c}$$

By substituting $d'' = d - p_0$

$$\frac{p_0}{y} = \frac{d}{y} - \left(d/y - \frac{\int_{-\lambda}^0 \frac{F_1(\tau,\alpha,\lambda)}{\sqrt{(-\tau)(\beta-\tau)(\gamma-\tau)}}d\tau}{\int_1^\infty F(\tau,\alpha,\lambda,\gamma,\beta)d\tau}\right)\left(\frac{K(\sqrt{(\beta-\gamma)/\beta})}{F\left(\sin^{-1}\sqrt{\frac{\beta}{\beta+\lambda}},\sqrt{\frac{\beta-\gamma}{\beta}}\right)}\right) \tag{18d}$$

Simultaneous solution of Eqs. (7), (8), (9) and (18d) for the given dimensions of rectangular array (B, b, y and p) and depth of the drainage layer (d) results in parameters α, λ, β and γ. Using these values in Eqs. (5) and (6) gives the shape of phreatic surface. Putting value of ($\zeta = -\alpha$) in Eqs. (5) and (6) gives point of inflection. At ($\zeta = -\lambda$) in Eqs. (5) and (6) gives the values of $B/2$ and y_s, which can be used for cross checking. Moreover, Eqs. (12) and (14) result into the seepage velocity, while Eq. (16) into the quantity of seepage. However these equations involve complicated integrals with implicit accessory parameters. These integrals (incomplete elliptical integrals, and remaining improper integrals) can be evaluated using numerical integration (Press et al., 1992) after converting the improper integrals into proper integrals.

Special cases

Drainage layer and water table at infinite depth

When the drainage layer and water table both lie at infinite depth there can be two cases: (i) pressure is infinite and (ii) pressure is kept constant. Since $d'' = d - p_0$, in the first case when $d \to \infty$ and $p_0 \to \infty$ such that d'' = *constant* then there is no change in the complex potential plane and it remains a rectangle. Hence the analysis remains unchanged. In the second case when $d \to \infty$ and p_0 = *constant* the complex potential plane degenerates into a semi infinite strip. Therefore both the points l' and a' approach each other in the hodograph and inverse hodograph mapping planes. When p_0 = zero, at infinite depth the velocity potential becomes infinite, so the mapping in the W plane converts to a semi-infinite strip. With these adjustments, the accessory parameter γ vanishes from the transformation after attaining a value equal to zero and the changed relations turn into

$$W = \frac{2iq_s\tan^{-1}\sqrt{(\zeta-\beta)/\beta}}{2\pi} \tag{19}$$

$$Z = b/2 - iy + iy\int_1^\zeta F_3(\tau,\alpha,\lambda,\beta)d\tau\Big/\int_1^\infty F_3(\tau,\alpha,\lambda,\beta)d\tau \tag{20}$$

$$b/y = 2\int_\beta^1 \frac{F_1(\tau,\alpha,\lambda)}{\tau\sqrt{(\tau-\beta)}}d\tau\Big/\int_1^\infty F_3(\tau,\alpha,\lambda,\beta)d\tau \tag{21}$$

$$B/y = b/y + 2\int_{-\infty}^\zeta\left[\left(\int_\tau^{-\lambda}\frac{(t+\alpha)dt}{(1-t)^{0.5}(-t-\lambda)^{1.5}}\right)\times\frac{1}{(-\tau)\sqrt{(\beta-\tau)}}\right]d\tau\Big/\int_1^\infty F_3(\tau,\alpha,\lambda,\beta)d\tau \tag{22}$$

$$y_s = \frac{q_s\log\left(\frac{\sqrt{\beta+\lambda}-\sqrt{\beta}}{\sqrt{\beta+\lambda}+\sqrt{\beta}}\right)}{2k\pi} \tag{23}$$

$$y_s/y = \frac{\pi\log\left(\frac{\sqrt{\beta+\lambda}-\sqrt{\beta}}{\sqrt{\beta+\lambda}+\sqrt{\beta}}\right)}{\sqrt{\beta}\int_1^\infty F_3(\tau,\alpha,\lambda,\beta)d\tau} \tag{24}$$

and

$$q_s = 2\pi^2 ky\Big/\sqrt{\beta}\int_1^\infty F_3(\tau,\alpha,\lambda,\beta)d\tau \tag{25}$$

where

$$F_3(\tau,\alpha,\lambda,\beta) = \frac{F_2(\tau,\alpha,\lambda)}{\tau\sqrt{(\tau-\beta)}} \tag{26}$$

The expressions for the variation in the seepage velocity remains unaltered because the inverse hodograph mapping is identical in both cases except for the location of l' which does not take part in Schwarz–Christoffel transformation due to vertex angle being equal to π in the drainage layer for the shallow depth case.

Single rectangular channel

When the spacing between the channels is increased *i.e.* when B increases point c' approaches b' and point b' approaches a' in the physical plane. When B exceeds a certain value the channels will not interact each other, thus act as single independent channel. In this limiting case the saturated plume of each channel is swallowed by drainage layer and the points c' and b' coincide with a' and the image of these points in hodograph plane lies on ordinate axis. The region from d' to c' is represented by semi circle in hodograph plane. The inverse of this hodograph plane is represented by a semi infinite strip and the case is described as a case for seepage through single rectangular channel solved by Chahar (2007). For a single rectangular channel, α and λ are zero and the relevant equations transform to those given by Chahar (2007). Above discussion is true only when p_0 is zero.

Example

As an example, we can compute the quantity of recharge from a rectangular array of channels having a bed width = 3.0 m, a depth of flow = 2 m, and spacing between two channels = 5 m, passing through a porous medium having hydraulic conductivity = 3×10^{-6} m/s and underlain by a highly pervious drainage layer at a depth of 4.0 m with zero pressure.

Solution: For the given data $d/y = 2$, $b/y = 1.5$, $p = 0$ and $B/y = 2.5$. Eqs. (7), (8), (9) and (18d) should be solved simultaneously to get α, λ, β and γ. Since these equations are highly nonlinear and contain improper integrals; an indirect method has been used to find α, λ, β and γ. The method consists of minimization of an objective function by Powell's conjugate search method (Press et al., 1992). The objective function is defined as

$$f(\alpha,\lambda,\beta,\gamma) = \left(\frac{B}{y} - f_1(\alpha,\lambda,\beta,\gamma)\right)^2 + \left(\frac{d}{y} - f_2(\alpha,\lambda,\beta,\gamma)\right)^2 + \left(\frac{d}{y} - f_3(\alpha,\lambda,\beta,\gamma)\right)^2 + \left(\frac{p}{y} - f_4(\alpha,\lambda,\beta,\gamma)\right)^2 \quad (27)$$

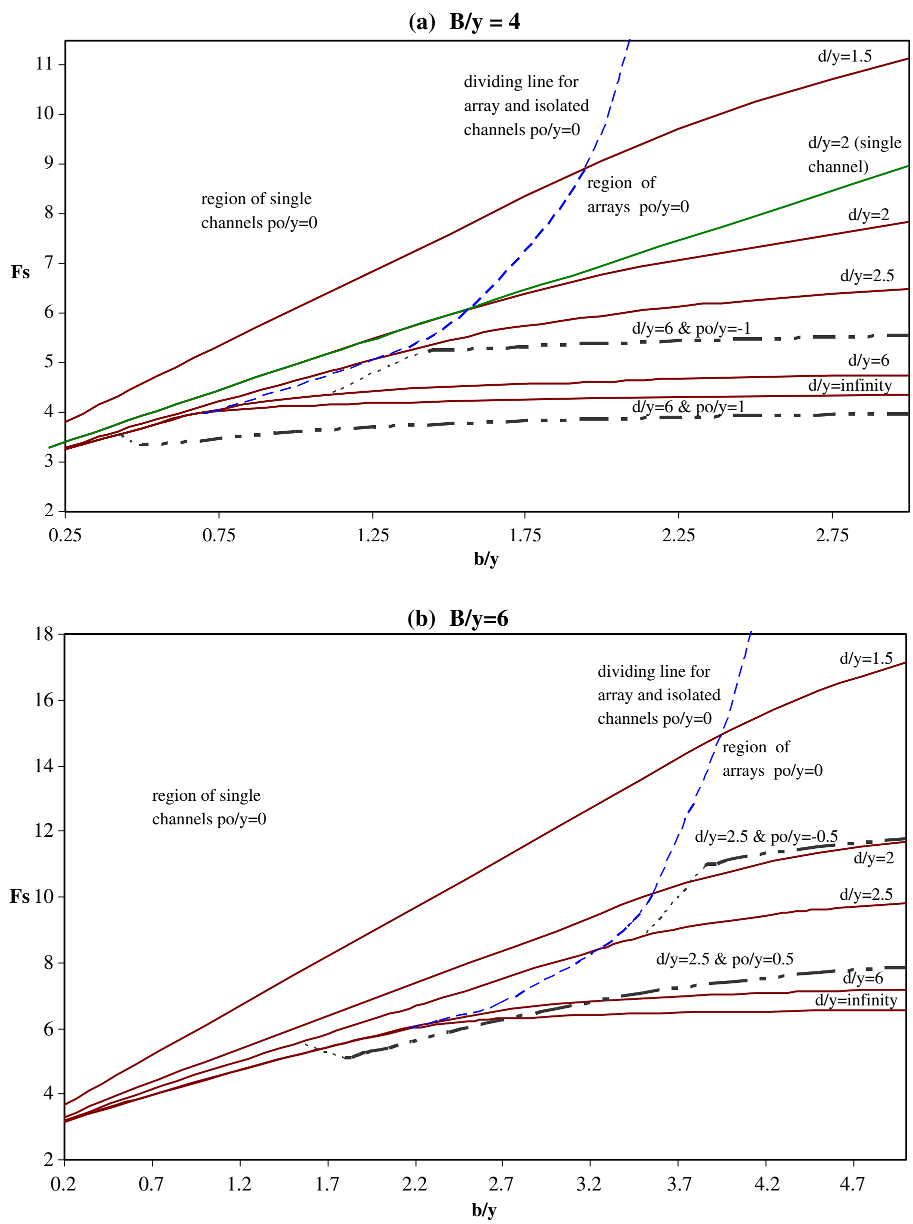

Figure 2 Variation of Seepage function with b/y for different values of d/y and B/y.

where $f_1(\alpha,\lambda,\beta,\gamma)$, $f_2(\alpha,\lambda,\beta,\gamma)$, $f_3(\alpha,\lambda,\beta,\gamma)$, and $f_4(\alpha,\lambda,\beta,\gamma)$ are right hand sides of Eqs. (8), (9), (10) and (18d) respectively. Since minimum of this function is zero, which can be attained only when all the parts of the function reach zero values and hence satisfy Eqs. (7), (8), (9) and (18d). After removing singularities (Chahar, 2005) and using Gaussian quadratures (96 points for weights and abscissa for both inner and outer integrals) for numerical integration (Abramowitz and Stegun, 1972), the function has been minimized for α, γ, β and γ for given set of B/y, b/y and d/y to get

$\alpha = 2191.56$, $\lambda = 182.427$, $\beta = 0.242451$ and $\gamma = 0.162265$

Making use of these values in Eqs. (17), (6) and (13)

$F_s = 4.69291$, $y_s/y = 0.04$,
$V/k = 2.0547$ (at centre of channel)

and finally from Eq. (1)

$q_s = 2.8157 \times 10^{-5}\,\mathrm{m}^3/\,\mathrm{s}$ per meter run of the channel.

If the depth of the drainage layer is increased to 12.0 m (i.e., $d/y = 6$), $F_s = 2.96116$, $y_s/y = 0.0228$, $V/k = 1.29252$ (at centre of channel) and $q_s = 1.776696 \times 10^{-5}\ \mathrm{m}^3/\mathrm{s/m}$.

Discussion

Following the procedure explained above, graphs have been plotted for seepage function against b/y for different values of d/y as shown in Fig. 2a and Fig. 2b for $B/y = 4$ and $B/y = 6$, respectively. The graphs for zero pressure are presented by continuous line. Additional graphs for $p_0/y = \pm 1$ counterpart to $d/y = 6$ in Fig. 2a and graphs for $p_0/y = \pm 0.5$ counterpart to $d/y = 2.5$ in Fig. 2b have, also, been drawn with bold chain lines. Another curve with dashed line has been added to both figures, which divides the region in two parts, *i.e.* interfering array of channels and isolated channels. Similar graphs can be prepared for other values of B/y. Fig. 2 shows that the quantity of seepage is very sensitive to the presence of a drainage layer at a shallow depth (i.e., $d/y < 3$) while the quantity of seepage varies only a little with change in position of the drainage layer at large depth (i.e., $d/y > 5$). For b/y small in comparison to B/y the channels acts as independent and hence the seepage function is same as single channel. For large b/y phreatic lines intersect each other i.e., interference begins and seepage function decreases in contrast to single channel. To show this an additional graph for single channel corresponding to $d/y = 2$ has been added in Fig. 2a. Further the seepage function approaches to b/y value as $b/y \to B/y$. Once $b/y \geqslant B/y$ the array cease to exist and acts as a pond of depth y resulting seepage function = b/y. The plots corresponding to positive pressure show that the seepage reduces due to increase in overlapping of phreatic surfaces thus decrease in y_s. The effect of positive pressure is to decrease the isolated channel region and to increase the interfering array region. Hence, a positive value of p_0 (upper limit being d) can always be applied for any finite value of B for the channels to interact. The consequence of the negative pressure is just opposite to that of positive pressure. Therefore, any interacting channels can be made to behave independently by applying a negative pressure.

Conclusions

An exact analytical solution for the quantity of recharge/seepage from an array of rectangular channels underlain by a drainage layer at a finite depth and with pressure can be obtained using an inverse hodograph and Schwarz–Christoffel transformation for one half of the seepage domain. From this general solution, special cases like an array of rectangular channels without pressure, without a drainage layer, or a rectangular channel underlain by a drainage layer at a shallow depth can be deduced. The analysis also gives solutions for the variation in the seepage velocity along the channel perimeter. However the solutions contain improper integrals which can only be evaluated by numerical integration.

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Appendix. Mapping details

Mapping of inverse hodograph plane

Mapping of $\mathrm{d}Z/\mathrm{d}W$ plane on the ζ plane results in

$$\frac{\mathrm{d}Z}{\mathrm{d}W} = C_1 \int_0^{\zeta} \frac{(t+\alpha)\mathrm{d}t}{(t-1)^{0.5}(t+\lambda)^{1.5}} + C_2 \tag{A1}$$

where C_1 and C_2 = constants. These constants can be found by using values of $\mathrm{d}Z/\mathrm{d}W$ and ζ at two points in $\mathrm{d}Z/\mathrm{d}W$ plane and ζ plane. Using the values at point e' ($\zeta = 1$; $\mathrm{d}Z/\mathrm{d}W = 0$) and at the point a' ($\zeta = 0$; $\mathrm{d}Z/\mathrm{d}W = -\mathrm{i}k'$) in Eq. (A1) and solving simultaneously

$$C_1 = \mathrm{i}k'\mathrm{i} \Bigg/ \int_0^1 \frac{(t+\alpha)\mathrm{d}t}{(1-t)^{0.5}(t+\lambda)^{1.5}} = -k'/F_1(0,\alpha,\lambda) \tag{A2a}$$

$$C_2 = -\mathrm{i}k' \tag{A2b}$$

where

$$\begin{aligned} F_1(\tau,\alpha,\lambda) &= \int_\tau^1 \frac{(t+\alpha)\mathrm{d}t}{(1-t)^{0.5}(t+\lambda)^{1.5}} \\ &= \left[-\frac{2\sqrt{1-t}(\alpha-\lambda)}{(1+\lambda)\sqrt{t+\lambda}} - 2\tan^{-1}\left(\frac{-\sqrt{t+\lambda}}{\sqrt{1-t}}\right)\right]_\tau^1 \\ &= \pi + \frac{2\sqrt{1-\tau}(\alpha-\lambda)}{(1+\lambda)\sqrt{\tau+\lambda}} + 2\tan^{-1}\left(\frac{-\sqrt{\tau+\lambda}}{\sqrt{1-\tau}}\right) \end{aligned} \tag{A3}$$

so that

$$F_1(0,\alpha,\lambda) = \pi + \frac{2(\alpha-\lambda)}{(1+\lambda)} - 2\tan^{-1}\left(\sqrt{\lambda}\right) \tag{A3a}$$

Substitution of C_1 and C_2 in Eq. (A1) gives Eq. (2).

Mapping of complex potential plane

The W plane mapping on the ζ plane is

$$W = C_3 \int_0^{\zeta} \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} + C_4 \tag{A4}$$

Using the values at points f' ($\zeta = \beta$; $W = 0$) and l' ($\zeta = \gamma$; $W = kd''$). The constants

$$C_4 = kd'' + ikd''K\left(\sqrt{\gamma/\beta}\right)\Big/K\left(\sqrt{(\beta-\gamma)/\beta}\right) \tag{A5a}$$

and

$$C_3 = -ikd''\sqrt{\beta}\Big/2K\left(\sqrt{(\beta-\gamma)/\beta}\right) \tag{A5b}$$

since

$$\int_{\beta}^{\gamma} \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} = i\int_{\gamma}^{\beta} \frac{dt}{\sqrt{t(t-\gamma)(\beta-t)}} = i\frac{2}{\sqrt{\beta}}K\left(\sqrt{(\beta-\gamma)/\beta}\right) \tag{A6}$$

After substituting C_3 and C_4, Eq. (A4) results in Eq. (3). Differentiating Eq. (3) with respect to ζ gives

$$\frac{dW}{d\zeta} = i\frac{kd''\sqrt{\beta}}{2K\left(\sqrt{(\beta-\gamma)/\beta}\right)} \frac{1}{\sqrt{\zeta(\zeta-\gamma)(\zeta-\beta)}} \tag{A7}$$

Mapping of physical plane

Since $\frac{dZ}{d\zeta} = \frac{dZ}{dW}\frac{dW}{d\zeta}$, substitution of dZ/dW from Eq. (2) and $dW/d\zeta$ from Eq. (A7) results in

$$\frac{dZ}{d\zeta} = C_1C_3\frac{F_2(\zeta,\alpha,\lambda)}{\sqrt{\zeta(\zeta-\beta)(\zeta-\gamma)}} \tag{A8}$$

Integrating Eq. (A8) we get

$$Z = C_5\int_0^{\zeta} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau + C_6 \tag{A9}$$

where

$$C_5 = C_1C_3 \tag{A10}$$

$$F(\tau,\alpha,\lambda,\gamma,\beta) = \frac{F_2(\tau,\alpha,\lambda)}{\sqrt{\tau(\tau-\beta)(\tau-\gamma)}} \tag{A11}$$

$$F_2(\tau,\alpha,\lambda) = \int_1^{\tau}\frac{(t+\alpha)dt}{(t-1)^{0.5}(t+\lambda)^{1.5}} = \left[\frac{2\sqrt{t-1}(\alpha-\lambda)}{(1+\lambda)\sqrt{t+\lambda}} + 2\tanh^{-1}\left(\frac{\sqrt{t-1}}{\sqrt{t+\lambda}}\right)\right]_1^{\tau} = \frac{2\sqrt{\tau-1}(\alpha-\lambda)}{(1+\lambda)\sqrt{\tau+\lambda}} + 2\tanh^{-1}(\sqrt{\tau-1}/\sqrt{\tau+\lambda}) \tag{A12}$$

Using the values at points e' ($\zeta = 1$; $Z = b/2 - iy$) and d' ($\zeta = \infty$; $Z = b/2$), the constants C_5 and C_6 have been determined as

$$C_6 = b/2 - iy\int_0^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A13}$$

and

$$C_5 = iy\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A14}$$

Substituting values of C_1, C_3 and C_5 in Eq. (A10)

$$k' = \frac{1}{k}\frac{2K\left(\sqrt{(\beta-\gamma)/\beta}\right)F_1(0,\alpha,\lambda)}{(d''/y)\sqrt{\beta}\int_1^{\infty}F(\tau,\alpha,\lambda,\gamma,\beta)d\tau} \tag{A15}$$

After substituting C_5 and C_6, Eq. (A9) results in Eq. (4). The phreatic line $d'c'b'$ ($-\infty < \zeta < -\lambda$) can be located by manipulating Eq. (4) as

$$Z = b/2 + iy\int_{-\infty}^{\zeta} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A16}$$

and separating the real and imaginary parts leads to Eqs. (5) and (6), in which

$$\int_{-\lambda}^{1}\frac{(t+\alpha)dt}{(1-t)^{0.5}(t+\lambda)^{1.5}} = \pi$$

Along the bed of the channel $e'f'$ ($\beta \leqslant \zeta \leqslant 1$) Eq. (4) becomes

$$Z = b/2 - iy - y\int_{\zeta}^{1}\frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\tau-\beta)(\tau-\gamma)}}d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A17}$$

hence Eq. (A17) at point f' ($\zeta = \beta$ and $Z = -iy$) becomes Eq. (7). Along the centre line of channel $f'l'$ ($\gamma \leqslant \zeta \leqslant \beta$), Eq. (4) becomes

$$Z = -iy - y\int_{\zeta}^{\beta}\frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\tau-\beta)(\tau-\gamma)}}d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A18}$$

Therefore Eq. (A18) can be rewritten as Eq. (8) for ($\zeta = \gamma$; $Z = -id$). Along the drainage layer $l'a'$ ($0 \leqslant \zeta \leqslant \gamma$), Eq. (4) becomes

$$Z = -id + y\int_{\zeta}^{\gamma}\frac{F_1(\tau,\alpha,\lambda)}{\sqrt{\tau(\beta-\tau)(\gamma-\tau)}}d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau \tag{A19}$$

so that at a' ($\zeta = 0$; $Z = -id + B/2$) this reduces to Eq. (9). Further, along the line $a'b'$ ($-\lambda \leqslant \zeta \leqslant 0$) Eq. (4) becomes

$$Z = -iy\int_{\zeta}^{0}\frac{F_1(\tau,\alpha,\lambda)}{\sqrt{(-\tau)(\beta-\tau)(\gamma-\tau)}}d\tau\Big/\int_1^{\infty} F(\tau,\alpha,\lambda,\gamma,\beta)d\tau - id + B/2 \tag{A20}$$

and at the point b' ($\zeta = -\lambda$ and $Z = B/2 - iy_s$) Eq. (A20) yields Eq. (10) .

For single rectangular channel (with $\alpha = \lambda = 0$) Eq. (A15) becomes

$$kk' = \frac{2\pi K(\sqrt{(\beta-\gamma)/\beta}}{\sqrt{\beta}\left\{\int_1^\infty \frac{2\log(\sqrt{t-1}+\sqrt{t})}{\sqrt{t(t-\beta)(t-\gamma)}}dt + \int_\gamma^\beta \frac{\pi+2\tan^{-1}(-\sqrt{t/(1-t)})}{\sqrt{t(\beta-t)(t-\gamma)}}dt\right\}} \tag{A21}$$

Numerical integration of RHS for all values of γ and β is always unity, therefore,

$$k' = \frac{1}{k} \tag{A22}$$

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