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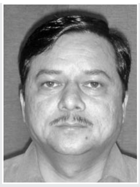

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Steady subsurface drainage of homogeneous soils by ditches

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Subsurface drainage of agricultural land is a method for improved production, a safeguard for sustainable investment in irrigation and a tool for conservation of land resources. An analytical solution for the problem of subsurface drainage of a ponded surface by an array of parallel ditches has been obtained using velocity hodograph and Schwarz–Christoffel transformation. The presented solution is applicable for the two-dimensional steady drainage from a horizontal ponded surface to an array of ditches in a homogeneous and isotropic porous medium. The solution includes equations for the quantity of drainage and the variation in seepage velocity. Thus the solution is a generalisation of Youngs solution. From this generalised solution, the particular solutions have also been deduced for different cases.

NOTATION

A, B, C	dimensionless points on flow domain
b	width of ditch (m)
C_1, C_2	constants
d	depth of ditch (m)
I_1, I_2, I_3	dimensionless integrals
i	imaginary number
k	hydraulic conductivity (m/s)
q	seepage discharge from one side of ditch per unit length (m^2/s)
q_D	seepage discharge from one side of ditch below water surface (m^2/s)
S	spacing (half) between ditches (m)
t, τ	dummy variables
u	velocity of seeping water along X axis (m/s)
V	resultant velocity of seeping water (m/s)
v	velocity of seeping water along Y axis (m/s);
W	$\phi + i\psi$ complex potential (m^2/s)
X	real axis of the complex plane (m)
Y	imaginary axis of the complex plane (m)
y	water depth in ditch (m)
y'	velocity reversal point along ditch water depth (m)
Z	$X + iY$ complex plane variable (m)
α, β, γ, δ	transformation or accessory parameters
ζ	complex variable in auxiliary plane
ϕ	velocity potential (m^2/s)
ψ	stream function (m^2/s)

1. INTRODUCTION

Irrigation is widely practiced in many parts of arid and semi-arid regions of the world to enhance agricultural production as a result of the uncertainty in rain-fed agriculture.[1] Irrigation, however, has negative impacts such as waterlogging and increases in soil and groundwater salinity. In irrigated lands, depending on the extent and duration of waterlogging, the crop yield may reduce substantially or even reach zero.[2] Problems of waterlogging and soil salinisation are known to have developed in irrigated agriculture since the first human civilisation in Mesopotamia. The fall of this ancient civilization is attributed to both waterlogging and soil salinity.[3] The expansion of waterlogging and soil salinity has been widely reported by many workers in India[4] and varies from 4·75 to 16 million ha and 3·3 to 10·9 million ha, respectively.[5] Waterlogging is also a problem experienced on cricket grounds, golf courses, race courses, parks and other amenities.

Problems such as water logging and soil salinity can be alleviated by application of a proper subsurface drainage design which helps in preventing the rise of the water table above a critical level.[6] Drainage maintains the productive capacity of soil by removing excess water, improving the soil moisture, improving the air circulation and reducing salt content and erosion. Ditches or tiles are used for the subsurface drainage. Compared with tiles, ditches do not require very accurate levelling, construction of gravel packs or geotextile wrapping, casing, anti-scaling, or anti-clogging measures.[7] Subsurface drainage has a long history: the oldest known systems date back some 9000 years in Mesopotamia. Subsurface drainage is a form of drainage that was widely introduced in Europe and North America in the 20th century. Among the developing countries, Egypt is the country with the largest area provided with subsurface drainage, about 2·5 million ha,[8] while countries such as Pakistan, China, Turkey and India are providing subsurface drainage to large tracts of their irrigated lands.[9,10] Out of the 1500 million ha of irrigated and rain-fed cropped lands of the world, only about 14% is provided with some form of drainage. The total area in need of artificial drainage can be roughly estimated to be 300 million ha, mainly in the arid and tropical humid zones of the developing countries.[11] Thus the analysis of subsurface drainage is important to ensure productivity of agricultural land and reclamation of land and for conservation of land resources.

The theory of two-dimensional groundwater flow has been considered by many investigators to analyse the ditch drainage system for different boundary conditions using different methods.[4,7,12–28] Drainage of a homogeneous and isotropic porous medium with ponded water was analysed by Youngs[29] using conformal transformation, but the solution is restricted to a single ditch and is applicable only for the case of negligible width and zero ponded depth. In the present study, a general solution to the problem of drainage of a ponded field in a homogeneous and isotropic soil medium by an array of ditches is obtained using velocity hodograph and Schwarz–Christoffel conformal mapping[12,30,31] (refer to Appendix for details). The usefulness of conformal mapping in finding analytical solution to two-dimensional flow problems stems from the fact that solutions of Laplace's equation remain solutions when subjected to conformal transformation. The solution of a two-dimensional groundwater problem could be reduced to one seeking the solution of Laplace's equation subject to certain boundary conditions within a region R in the Z plane. Unless the region R is of a very simple shape, an analytical solution to Laplace's equation is generally very difficult. By means of conformal mapping, however, it is often possible to transform the region R into a simpler region R_1 wherein Laplace's equation can be solved subject to the transformed boundary conditions. Once the solution has been obtained in region R_1 it can be carried back by the inverse transformation to the region R of the original problem. Hence the crux of the problem is finding a series of transformations that will map a region R conformally into a region R_1 so that R_1 will be of a simple shape, such as a rectangle or a circle. The Z (physical) plane in the problem of subsurface drainage of a ponded area by an array of ditches is a polygon; its velocity hodogragh mapping is also a polygon. So both the planes have been mapped through Schwarz–Christoffel conformal transformation into an auxiliary plane to yield the desired analytical solution. The solution takes into account the effect of depth of water in ditches as well as width and spacing of ditches. The proposed solution includes expressions for the quantity of drainage and the velocity distribution of draining water.

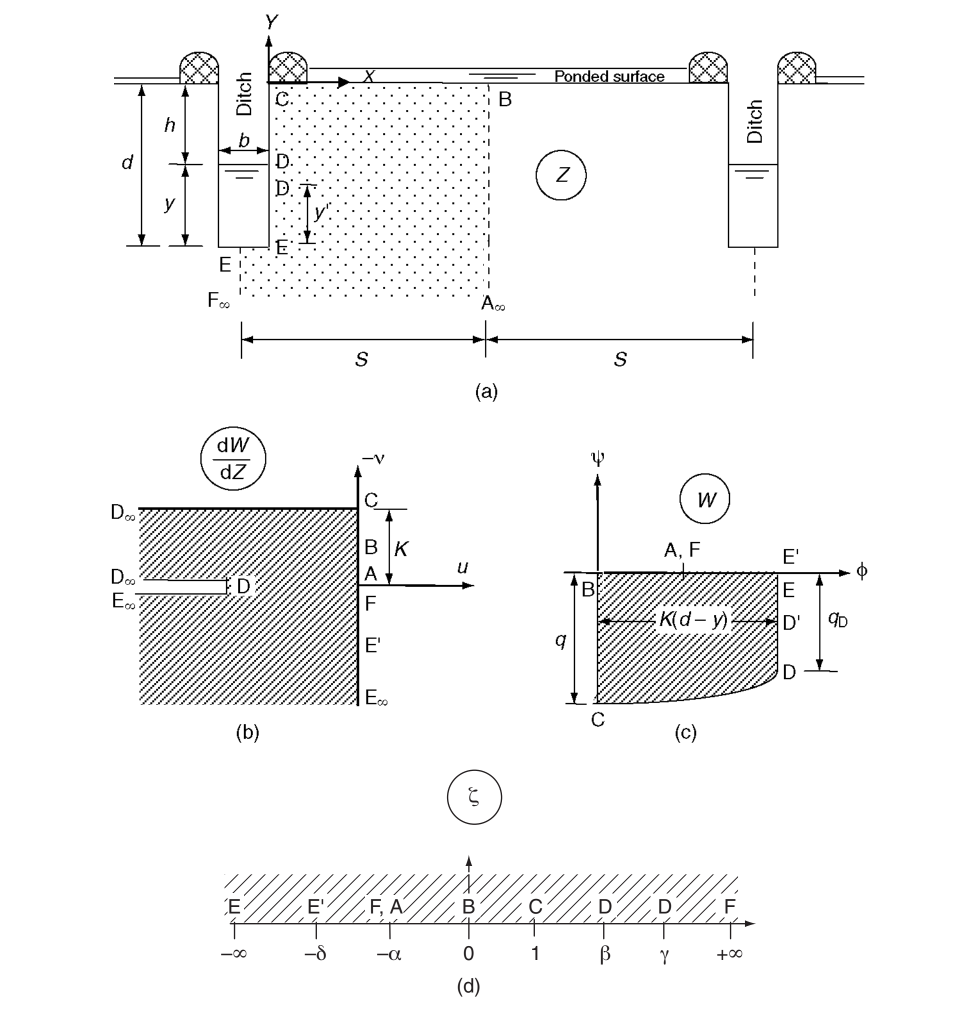

Fig. 1. Drainage from ponded surface to an array of ditches: (a) physical plane; (b) hodograph plane; (c) complex potential plane; (d) auxiliary plane

2. ANALYTICAL SOLUTION

The physical domain of drainage from a ponded surface by an array of ditches of depth d (m) having spacing $2S$ (m) and water depth y (m) is shown in Fig. 1(a). The ditches partially penetrate into homogeneous and isotropic porous medium of infinite extent. Generally the ponded water is drained by surface drainage. The remaining water is drained by subsurface drainage; therefore the depth of ponded water is assumed to be very small. It is also assumed that the flow is steady and satisfies Darcy's law. In view of the significant length of the ditches, the flow in the porous medium can be considered two-dimensional in the vertical plane, thus the solution of a two-dimensional Laplace equation with boundary conditions answers this problem. Owing to vertical symmetry, the solution for the half domain (ABCDEE$'$F) is sought. Let complex potential $W = \phi + i\psi$, where $\phi =$ velocity potential (m^2/s) which is equal to the negative of hydraulic conductivity K (m/s) multiplied by the head h (m) and $\psi =$ stream function (m^2/s) which is constant along streamlines. If the physical plane is defined as $Z = X + iY$ then Darcy's law yields $u = \partial\phi/\partial X = -K\,\partial h/\partial X$ and $v = \partial\phi/\partial Y = -K\,\partial h/\partial Y$, where u and v are velocity or specific discharge vectors in X and Y directions, respectively. The velocity hodograph plane ($\mathrm{d}W/\mathrm{d}Z = u - iv$) (Fig. 1(b)) and the complex potential W (Fig. 1(c)) for half of the physical flow domain have been drawn following the standard steps.[30,31] The $\mathrm{d}W/\mathrm{d}Z$ plane and Z plane have been mapped on to the upper half of an auxiliary ($\mathrm{Im}\,\zeta > 0$) plane (Fig. 1(d)) using the Schwarz–Christoffel conformal transformation.[30,31]

2.1. Mapping in different planes

Mapping of Z plane on to ζ plane results in

$$Z = C_1 \int_t^{\zeta} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} + C_2 \tag{1}$$

where t = dummy variable; α and δ = accessory parameters; and C_1 and C_2 = constants. The constants can be found by using values of Z and ζ at two points in the Z plane and the ζ plane. Using the values at points C ($\zeta = 1$; $Z = 0$) and E ($\zeta = \infty$; $Z = -id$) and then solving

$$Z = \frac{-id}{I_1} \int_1^{\zeta} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} \tag{2}$$

where

$$I_1 = \int_1^{\infty} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} \tag{3}$$

Mapping of hodograph ($\mathrm{d}W/\mathrm{d}Z$) plane on to ζ plane gives

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = C_3 \int_0^{\zeta} \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}} + C_4 \tag{4}$$

Using the values at points C ($\zeta = 1$; $\mathrm{d}W/\mathrm{d}Z = iK$) and A ($\zeta = -\alpha$; $\mathrm{d}W/\mathrm{d}Z = 0$) and solving the two equations simultaneously

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -\frac{K}{I_2} \int_{-\alpha}^{\zeta} \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}} \tag{5}$$

where

$$I_2 = \int_{-\alpha}^{1} \frac{(\gamma-t)\,\mathrm{d}t}{(\beta-t)\sqrt{(1-t)}} \tag{6}$$

Since

$$\frac{\mathrm{d}W}{\mathrm{d}\zeta} = \frac{\mathrm{d}W}{\mathrm{d}Z}\frac{\mathrm{d}Z}{\mathrm{d}\zeta} \tag{7}$$

Using equation (5) and derivative of equation (2) and subsequently integrating, noting that at B, $\zeta = 0$ and $W = 0$

$$W = \frac{iKd}{I_1 I_2} \int_0^{\zeta} \frac{\displaystyle\int_{-\alpha}^{\tau} \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}}}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}}\,\mathrm{d}\tau \tag{8}$$

An alternative expression for W can be obtained by finding constants from W plane at B ($\zeta = 0$; $W = 0$) and at E′ ($W = K(d-y)$; $\zeta = -\delta$) as

$$W = \frac{iK(d-y)}{I_3} \int_0^{\zeta} \frac{\displaystyle\int_{-\alpha}^{\tau} \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}}}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}}\,\mathrm{d}\tau \tag{9}$$

where

$$I_3 = \int_0^{\delta} \frac{\displaystyle\int_{\alpha}^{\tau} \frac{(t+\gamma)\,\mathrm{d}t}{(t+\beta)\sqrt{(1+t)}}}{(\tau-\alpha)\sqrt{\tau(1+\tau)(\delta-\tau)}}\,\mathrm{d}\tau \tag{10}$$

2.2. Drainage velocity

The drainage velocities along the water divide line AB and the ponded surface BC are vertically downwards and can be obtained from equation (5) as

$$v = -\frac{K}{I_2} \int_{-\alpha}^{\zeta} \frac{(\gamma-t)}{(\beta-t)\sqrt{(1-t)}}\,\mathrm{d}t \tag{11}$$

Therefore v_B (m/s), the velocity at B ($\zeta = 0$; $\mathrm{d}W/\mathrm{d}Z = v_B$)

$$v_B = -\frac{K}{I_2} \int_{-\alpha}^{0} \frac{(\gamma-t)}{(\beta-t)\sqrt{(1-t)}}\,\mathrm{d}t \tag{12}$$

and at C, ($\zeta = 1$; $\mathrm{d}W/\mathrm{d}Z = ik$), $v_c = -K$.

The seepage velocities along the seepage surface (CD) have a horizontal component only, given by

$$u = -\frac{K}{I_2} \int_{1}^{\zeta} \frac{(\gamma-t)}{(\beta-t)\sqrt{(t-1)}}\,\mathrm{d}t \tag{13}$$

Finally the velocity at E′ ($\zeta = -\delta$; $\mathrm{d}W/\mathrm{d}Z = v_E$) is given by

$$v_{E'} = -\frac{K}{I_2} \int_{\alpha}^{\delta} \frac{(\gamma+t)\,\mathrm{d}t}{(\beta+t)\sqrt{(1+t)}} \tag{14}$$

2.3. Determination of accessory parameters

Applying equation (2) at the points D ($\zeta = \beta$; $Z = -i(d-y)$), D′ ($\zeta = \gamma$; $Z = -i(d-y')$), B ($\zeta = 0$; $Z = (S - b/2)$), E′ ($\zeta = -\delta$; $Z = -id - b/2$), and A ($\zeta = -\alpha$; $Z = -ih + (S - b/2)$), and manipulating results in

$$\frac{y}{d} = \frac{1}{I_1} \int_{\beta}^{\infty} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} \tag{15}$$

$$\frac{b}{d} = \frac{2}{I_1} \int_{\delta}^{\infty} \frac{\mathrm{d}t}{(t-\alpha)\sqrt{t(1+t)(t-\delta)}} \tag{16}$$

$$\frac{S}{d} = \frac{\displaystyle\int_0^1 \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(1-t)(t+\delta)}} + \int_{\delta}^{\infty} \frac{\mathrm{d}t}{(t-\alpha)\sqrt{t(1+t)(t-\delta)}}}{I_1} \tag{17}$$

$$\frac{y'}{d} = \frac{1}{I_1} \int_{\gamma}^{\infty} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} \tag{18}$$

where y' is position of point D′ as shown in Fig. 1(a), such that there is reversal in the drainage velocity in the hodograph plane (Fig. 1(b)). Comparing equations (8) and (9) and simplifying gives

$$I_3 = I_2 \int_1^{\beta} \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)(t+\delta)}} \quad (19)$$

Equations (15), (16), (17) and (19) can be solved simultaneously for α, β, γ and δ for given values of y, b, S and d. Subsequently equation (18) will locate the position of D′.

When the transformation parameters have been determined, the drainage velocities at key points can be determined by equations (12) to (14). Furthermore, equation (2) can be used to find ζ at the desired Z and then equation (11) ascertains v at that location. The variations in the drainage velocity along the ponded surface and water divide line are shown in Fig. 2 for different water depths in ditches and $S/d = 1{\cdot}5$. It can be seen that the drainage velocities are higher near the ditches and for small water depths in the ditches.

2.4. Quantity of seepage/drainage

The steady drainage from one side into the ditch in the hydrogeological conditions of Fig. 1(a) can be obtained by using equation (9) at the point C ($\zeta = 1$; $W = -iq$)

$$q = \frac{K(d-y)}{I_3} \int_0^1 \frac{\int_{-\alpha}^{\tau} \frac{(\gamma - t)\,\mathrm{d}t}{(\beta - t)\sqrt{(1-t)}}}{(\tau+\alpha)\sqrt{\tau(1-\tau)(\tau+\delta)}} \mathrm{d}\tau \quad (20)$$

where q = seepage discharge from one side of the ditch (m^2/s per unit length), so total seepage into the ditch is $2q$. Fig. 3 shows the variation in the drainage quantity with the depth of water in ditches for a selected ditch width ($b/d = 0{\cdot}45$) for different spacing between the ditches. The drainage rate is less sensitive to change in spacing if the half spacing between the ditches is greater than five times the ditch depth.

Using equation (9) at the point D, ($W = -iq_{\mathrm{D}} + K(d-y)$, $\zeta = \beta$) to find the drainage quantity through the seepage face CD or non-seepage face DE (water depth portion of the ditch)

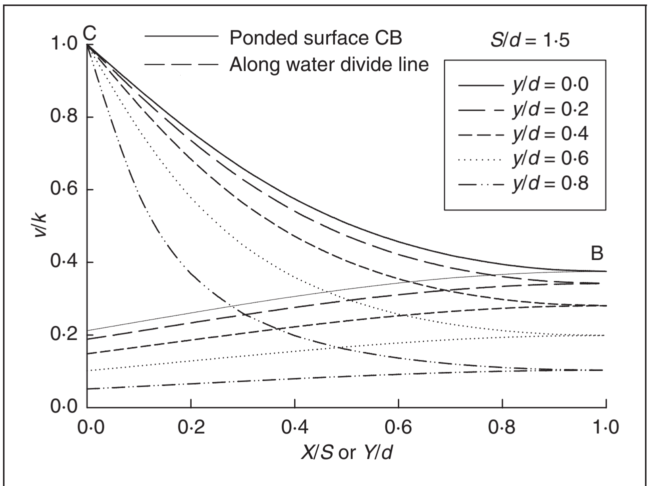

Fig. 2. Velocity distribution along ponded surface for $S/d = 1{\cdot}5$

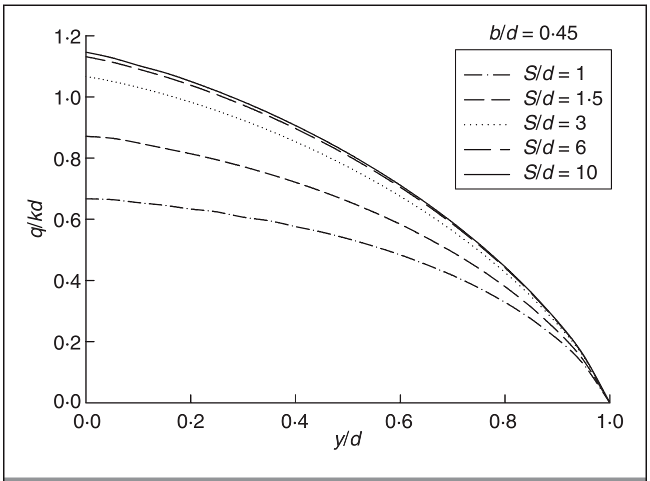

Fig. 3. Variation of the drainage quantity with depth of water for $b/d = 0{\cdot}45$

$$q_{\mathrm{CD}} = \frac{K(d-y)}{I_3} \int_1^{\beta} \frac{\int_1^{\tau} \frac{(\gamma - t)\,\mathrm{d}t}{(\beta - t)\sqrt{(t-1)}}}{(\tau+\alpha)\sqrt{\tau(\tau-1)(\tau+\delta)}} \mathrm{d}\tau \quad (21)$$

where q_{D} = drainage (m^2/s per m of length of ditch) received from one side below the water surface in the ditch (DE part), therefore $q - q_{\mathrm{D}} = q_{\mathrm{CD}}$ = drainage contribution from the seepage face part of the ditch (CD portion).

3. SPECIAL CASES

The above general solution involves width of ditches, depth of water in ditches and spacing between ditches. The following sections describe the particular solutions for the selected special cases.

3.1. Single ditch

As the spacing between the ditches increases, the points A and F approach B in the physical plane, thus $\alpha \to 0$. With $\alpha = 0$, the transformations in the hodograph plane remain unchanged but the physical plane and the W plane convert to

$$Z = \frac{-id}{I_{11}} \int_1^{\zeta} \frac{\mathrm{d}t}{t\sqrt{t(t-1)(t+\delta)}} \quad (22)$$

and

$$W = \frac{iK(d-y)}{I_{31}} \int_0^{\zeta} \frac{\int_0^{\tau} \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}}}{\tau\sqrt{\tau(\tau-1)(\tau+\delta)}} \mathrm{d}\tau \quad (23)$$

respectively, where

$$I_{11} = \int_1^{\infty} \frac{\mathrm{d}t}{t\sqrt{t(t-1)(t+\delta)}} \quad (24)$$

$$I_{31} = \int_0^{\delta} \frac{\int_0^{\tau} \frac{(t+\gamma)\,\mathrm{d}t}{(t+\beta)\sqrt{(1+t)}}}{\tau\sqrt{\tau(1+\tau)(\delta-\tau)}} \mathrm{d}\tau \quad (25)$$

Quantity of drainage is therefore given by

$$q = \frac{K(d-y)}{I_{31}} \int_0^1 \frac{\int_0^\tau \frac{(\gamma - t)\,\mathrm{d}t}{(\beta - t)\sqrt{(1-t)}}}{\tau\sqrt{\tau(1-\tau)(\tau+\delta)}}\mathrm{d}\tau \tag{26}$$

and the other relations become

$$\frac{y}{d} = \frac{1}{I_{11}} \int_\beta^\infty \frac{\mathrm{d}t}{t\sqrt{t(1-t)(t+\delta)}} \tag{27}$$

$$\frac{b}{d} = \frac{2}{I_{11}} \int_\delta^\infty \frac{\mathrm{d}t}{t\sqrt{t(t+1)(t-\delta)}} \tag{28}$$

$$\frac{y'}{d} = \frac{1}{I_{11}} \int_\gamma^\infty \frac{\mathrm{d}t}{t\sqrt{t(t-1)(t+\delta)}} \tag{29}$$

$$q_\mathrm{D} = q - \frac{K(d-y)}{I_{31}} \int_1^\beta \frac{\int_1^\tau \frac{(\gamma - t)\,\mathrm{d}t}{(\beta - t)\sqrt{(t-1)}}}{\tau\sqrt{\tau(\tau-1)(\tau+\delta)}}\mathrm{d}\tau \tag{30}$$

As the hodograph plane mapping remains unchanged, the expressions for drainage velocity do not alter, apart from $\alpha = 0$. The effect of the ditch width on the quantity of drainage is shown in Fig. 4, which reveals that the effect of ditch width on the drainage quantity is not substantial.

When the effect of width of the ditch is not taken into consideration, the point E′ approaches to E ($\delta \to \infty$) and therefore the seepage quantity becomes

$$q = \frac{K(d-y)}{I_{32}} \int_0^1 \frac{\int_0^\tau \frac{(\gamma - t)\,\mathrm{d}t}{(\beta - t)\sqrt{(1-t)}}}{\tau\sqrt{\tau(1-\tau)}}\mathrm{d}\tau \tag{31}$$

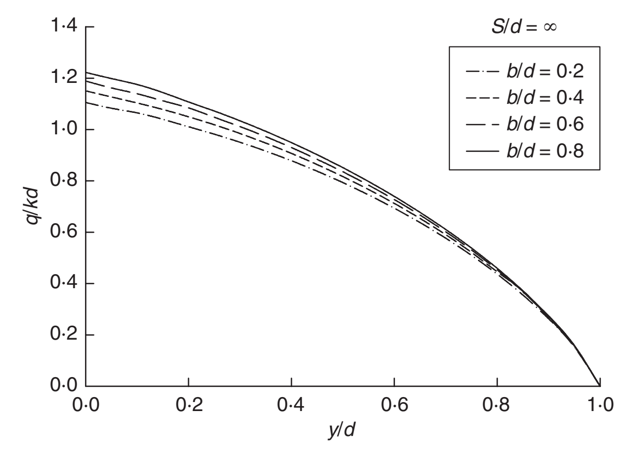

Fig. 4. Effect of ditch width on the quantity of drainage for a single ditch

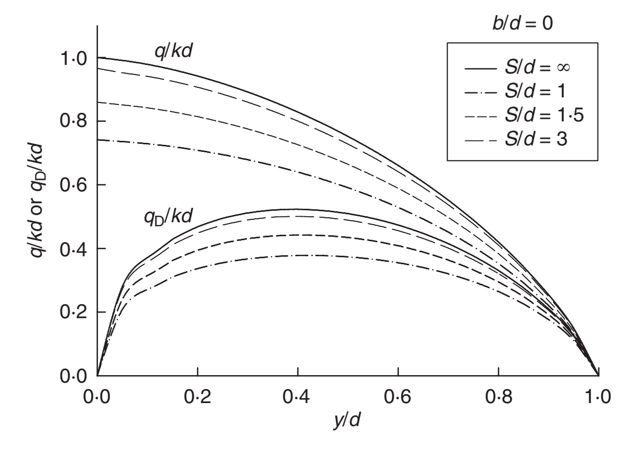

Fig. 5. Effect of depth on drain discharges for different spacing of ditch

where

$$I_{32} = \int_0^\infty \frac{\int_0^\tau \frac{(t+\gamma)\,\mathrm{d}t}{(t+\beta)\sqrt{(1+t)}}}{\tau\sqrt{\tau(1+\tau)}}\mathrm{d}\tau \tag{32}$$

For this case Fig. 5 shows the variation in the drainage quantity and drainage from the non-seepage face with the depth of water in ditches, for different cases of ditch spacing. It is interesting to note that the drainage from the non-seepage face is at a maximum when the water depth in the ditch is 40% of the ditch depth, irrespective of the spacing between the ditches.

Further, if the water depth in the ditch is zero as well ($\beta = \gamma = \infty$), then

$$q = \frac{Kd}{\int_0^\infty \frac{\int_0^\tau \left(1/\sqrt{(t+1)}\right)\mathrm{d}t}{\tau\sqrt{\tau(1+\tau)}}\mathrm{d}\tau} \int_0^1 \frac{\int_0^\tau \left(1/\sqrt{(1-t)}\right)\mathrm{d}t}{\tau\sqrt{\tau(1-\tau)}}\mathrm{d}\tau = kd \tag{33}$$

which is the same as given by Youngs,[29] that is the drainage rate is the product of the hydraulic conductivity and the depth of the ditch in the case of an infinitely deep soil for the empty ditch condition.

3.2. Width of ditch very small

When the effect of width of the ditch is not taken into consideration, the point E′ approaches to E ($\delta \to \infty$) and with $\delta = \infty$, the hodograph remains unchanged while the physical plane and the W plane relations become

$$Z = \frac{-id}{I_{12}} \int_1^\zeta \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)}} \tag{34}$$

$$W = \frac{iK(d-y)}{I_{33}} \int_0^\zeta \frac{\int_{-\alpha}^\tau \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}}}{(\tau+\alpha)\sqrt{\tau(\tau-1)}}\mathrm{d}\tau \tag{35}$$

where

$$I_{12} = \int_1^\infty \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)}} = \frac{2\sinh^{-1}\sqrt{\alpha}}{\sqrt{\alpha(1+\alpha)}} \quad (36)$$

$$I_{33} = \int_0^\infty \frac{\int_\alpha^\tau \frac{(t+\gamma)\,\mathrm{d}t}{(t+\beta)\sqrt{(1-t)}}}{(\tau-\alpha)\sqrt{\tau(1+\tau)}}\mathrm{d}\tau \quad (37)$$

Alternate W plane mapping similar to equation (9) is given by

$$W = \frac{iK(d-y)}{I_2 I_{12}} \int_0^\zeta \frac{\int_{-\alpha}^\tau \frac{(t-\gamma)\,\mathrm{d}t}{(t-\beta)\sqrt{(t-1)}}}{(\tau+\alpha)\sqrt{\tau(\tau-1)}}\mathrm{d}\tau \quad (38)$$

The expressions for drainage velocity remain unchanged, but other relations for accessory parameters and drainage discharge (q) deduced from equations (34), (35) and (38) are

$$\frac{y}{d} = \frac{1}{I_{12}}\int_\beta^\infty \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(t-1)}} = \frac{\tanh^{-1}\sqrt{\alpha/(1+\alpha)} - \tanh^{-1}\sqrt{\alpha(\beta-1)/(1+\alpha)/\beta}}{\sinh^{-1}\sqrt{\alpha}} \quad (39)$$

$$\frac{S}{d} = \frac{1}{I_{12}}\int_0^1 \frac{\mathrm{d}t}{(t+\alpha)\sqrt{t(1-t)}} = \frac{\pi}{2\sinh^{-1}\sqrt{\alpha}} \quad (40)$$

or

$$\alpha = \sinh^2(\pi d/2S) \quad (40a)$$

$$I_{33} = I_2 I_{12} \quad (41)$$

$$q = \frac{K(d-y)}{I_{33}}\int_0^1 \frac{\int_{-\alpha}^\tau \frac{(\gamma-t)\,\mathrm{d}t}{(\beta-t)\sqrt{(1-t)}}}{(\tau+\alpha)\sqrt{\tau(1-\tau)}}\mathrm{d}\tau \quad (42)$$

Further if the spacings between the ditches are very large ($\alpha = 0$), then the present solution gets converted into a particular case of a single ditch and the quantity of drainage is expressed by equation (31).

3.3. Array of empty ditches

Drainage efficiency increases as the water depth in ditches decreases. If the water depth in ditches is maintained at zero for maximum efficiency, then the points D and D$'$ approach E ($\beta \to \infty$ and $\gamma \to \infty$), thus the slit DD$'$E in the hodograph plane will disappear to give

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = -\frac{K}{\int_{-\alpha}^1 \left(1/\sqrt{(1-t)}\right)\mathrm{d}t}\int_{-\alpha}^\zeta \left(1/\sqrt{(t-1)}\right)\mathrm{d}t = K\frac{i\sqrt{1+\alpha}-\sqrt{\zeta-1}}{\sqrt{1+\alpha}} \quad (43)$$

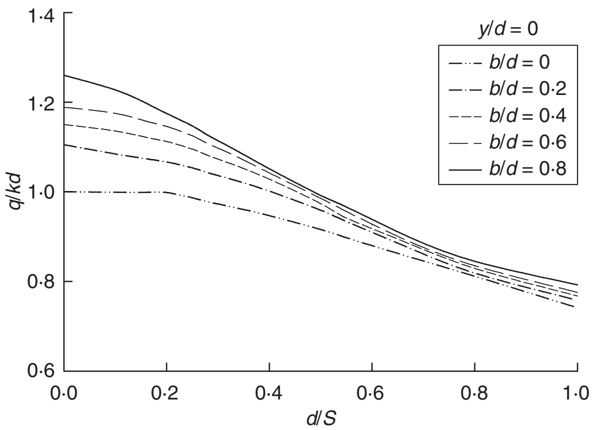

Fig. 6. Variation of drain discharge with spacing in an array of empty ditches

The physical plane mapping does not alter but the W plane becomes

$$W = \frac{iKd}{I_{34}}\int_0^\zeta \frac{\sqrt{\tau-1} - i\sqrt{1+\alpha}}{(\tau+\alpha)\sqrt{\tau(\tau-1)(\tau+\delta)}}\mathrm{d}\tau \quad (44)$$

where

$$I_{34} = \int_0^\delta \frac{\int_\alpha^\tau \left(1/\sqrt{(1+t)}\right)\mathrm{d}t}{(\tau-\alpha)\sqrt{\tau(1+\tau)(\delta-\tau)}}\mathrm{d}\tau = \int_0^\delta \frac{\sqrt{\tau+1}-\sqrt{\alpha+1}}{(\tau-\alpha)\sqrt{\tau(1+\tau)(\delta-\tau)}}\mathrm{d}\tau \quad (45)$$

Therefore

$$q = \frac{Kd}{I_{34}}\int_0^1 \frac{\int_{-\alpha}^\tau \left(1/\sqrt{(1-t)}\right)\mathrm{d}t}{(\tau+\alpha)\sqrt{\tau(1-\tau)(\tau+\delta)}}\mathrm{d}\tau = \frac{Kd}{I_{34}}\int_0^1 \frac{\sqrt{1-\tau}-\sqrt{1+\alpha}}{(\tau+\alpha)\sqrt{\tau(1-\tau)(\tau+\delta)}}\mathrm{d}\tau \quad (46)$$

$$v_\mathrm{B}/K = 1 - \frac{1}{\sqrt{1+\alpha}} \quad (47)$$

The accessory parameters α and δ are computed using equations (16) and (17). Fig. 6 represents equation (46) and shows the effect of spacing on drain discharge in an empty ditch. The variation in q with ditch width is shown in Fig. 7 for empty ditches.

Furthermore, if the ditch width is very small ($\delta = \infty$), then the hodograph mapping and physical mapping remains the same as equation (43) and equation (34) respectively but the W-plane mapping becomes

$$W = \frac{iKd}{I_{35}}\int_0^\zeta \frac{\sqrt{\tau-1}-\sqrt{-\alpha-1}}{(\tau+\alpha)\sqrt{\tau(\tau-1)}}\mathrm{d}\tau \quad (48)$$

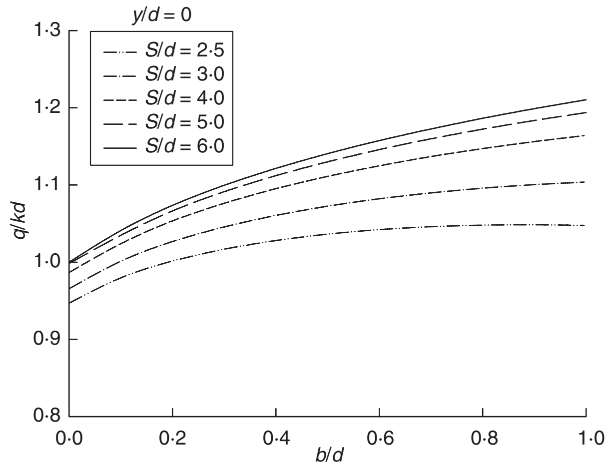

Fig. 7. Variation of drain discharge with ditch width for an array of empty ditches

where

$$I_{35} = \int_0^{\infty} \frac{\sqrt{\tau+1} - \sqrt{\alpha+1}}{(\tau-\alpha)\sqrt{\tau(1+\tau)}}\,\mathrm{d}\tau \tag{49}$$

The quantity of drainage is given by

$$q = \frac{Kd}{I_{35}} \int_0^{1} \frac{\sqrt{1-\tau} - \sqrt{1+\alpha}}{(\tau+\alpha)\sqrt{\tau(1-\tau)}}\,\mathrm{d}\tau \tag{50}$$

and the relation for finding the single accessory parameter α is given by equation (40) or equation (40a). Moreover, if the spacing is very large then it becomes a single ditch and the quantity of drainage is given by equation (33).

4. EXAMPLE

Consider drainage of a homogeneous and isotropic porous medium of a ponded area 2·5 m deep and parallel ditches 0·6 m wide, spaced at 5·0 m. The water depth in the ditches is maintained at 0·6 m.

For the given data $b/d = 0{\cdot}24$, $S/d = 1{\cdot}0$ and $y/d = 0{\cdot}24$, assuming negligible ponded water and using equations (15) to (17) and (19), the accessory parameters are $\alpha = 13{\cdot}8159$; $\beta = 7{\cdot}4796$; $\delta = 59{\cdot}8989$; and $\gamma = 24{\cdot}2498$. With these parameters: $q/Kd = 0{\cdot}6465$; $q_D/Kd = 0{\cdot}3424$; $v_B/K = 0{\cdot}6651$; and $y'/d = 0{\cdot}2083$.

If the ditch is empty then $\alpha = 10{\cdot}9233$ and $\delta = 50{\cdot}0746$, so $q/Kd = 0{\cdot}7151$ and $v_B/K = 0{\cdot}7104$. Also if the spacing is very large, then $q/Kd = 0{\cdot}9579$ and $q_D/Kd = 0{\cdot}2702$.

5. DISCUSSION AND LIMITATIONS

Subsurface drainage may be achieved by adopting tile drains or ditch drains. One of the main drawbacks in the installation of a tile/pipe-type subsurface drainage system is high initial investment. The important advantages of open ditches are that they are easy to construct, have low initial cost and have the ability to carry large quantities of water; the disadvantages are interference with farming operations, removal of land, requirement for regular maintenance and poor side slope stability.[23] In spite of these limitations an array of ditches may present an economical method of subsurface drainage. An array of ditches method of subsurface drainage is advantageous for various playgrounds, golf courses, parks and also for orchard plantations where there are few farming operations.

The presented analytical solution has been derived for an idealised porous medium and boundary conditions. There may be great variations in soil and boundary conditions in an actual field drainage case. The actual field problem has to be simplified to make it possible to obtain an analytical solution. As the solution only approximates field conditions, it is necessary to examine the assumptions made in its derivation before applying it to a particular field problem. In most cases the assumptions will not exactly correspond to the situation encountered in the field. In some cases the analytical solution may work quite well, in other cases it may be useful only as a first approximation to the proper design. In any case the analytical solutions provide a great deal of understanding of the problem in a rational way through their functional relationship between influencing parameters.

Also, the present solution is developed with an assumption that the ponded water will initially be drained out by surface field drains. At a later stage for small ponded water depth, the subsurface drainage by ditches will take place. Furthermore, if the porous medium is non-homogeneous then it should be transformed into an equivalent homogeneous medium[30] and thereafter the presented solution becomes applicable on the transformed problem.

6. CONCLUSIONS

Application of velocity hodograph and Schwarz–Christoffel transformation make it possible to obtain an exact analytical solution for the problem of subsurface drainage by an array of parallel ditches. The solution is applicable for an idealised porous medium and boundary conditions such as steady state subsurface drainage of a ponded surface in an homogeneous porous medium of infinite extent. The actual field problem may not satisfy these assumptions, hence the solution may work quite well in some cases and in other cases may be useful only as a first approximation. From the general solution, various particular solutions can easily be deduced for an array of ditches with very small ditch width and the empty ditch condition. The solution estimates drainage rate and drainage velocity in an array of ditches in ponded land. It may therefore be useful to practising engineers in controlling waterlogging and secondary salinisation in irrigated lands, cricket grounds, golf courses, race courses, parks and others amenities.

APPENDIX

Velocity hodograph[30,31]

Let the complex potential $W = \phi + i\psi$ be an analytical function of the complex variable Z, as $W = f(Z)$. Differentiation of W with respect to Z, yields

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = \frac{\partial\phi}{\partial X} + i\frac{\partial\psi}{\partial X} = \frac{\partial\psi}{\partial Y} - i\frac{\partial\phi}{\partial Y} = u - iv \tag{51}$$

The transformation of the region of flow from Z plane into the $\mathrm{d}W/\mathrm{d}Z$ plane is called the velocity hodograph. The utility of the hodograph stems from the fact that, although the shape of the free surface and the limit of the surface of seepage are not known initially in the Z plane, their hodographs are completely defined in the $\mathrm{d}W/\mathrm{d}Z$ plane.

Generally, the various boundaries of a flow region in the Z plane are transformed first into the $u + iv$ plane, then the mirror reflections about the u axis result into the velocity hodograph, that is the $u - iv$ plane. The various boundary relations of the hodograph are

(*a*) at an impervious boundary the velocity vector is in the direction of the boundary; in the u–v plane a straight line passing through the origin and parallel to the impervious boundary represents the impervious boundary
(*b*) since a boundary of reservoir is an equipotential line, the velocity vector is perpendicular to the boundary, hence in the u–v plane a straight line passing through the origin and normal to the reservoir boundary represents the reservoir boundary
(*c*) a line of seepage (phreatic line) is a stream-line and along it $\phi + KY =$ constant, in the u–v plane a circle ($u^2 + v^2 + Kv = 0$) passing through the origin, with radius $K/2$ and centre at $(0, -K/2)$ represents the phreatic line
(*d*) a surface of seepage in the Z plane, which is neither a stream-line nor an equipotential line, is represented by a straight line normal to the seepage surface and passing through the point $(0, -K)$ in the u–v plane.

Schwarz–Christoffel transformation[30,31]

The Schwarz–Christoffel transformation is a method of mapping a polygon consisting of straight-line boundaries, from one plane on to the upper/lower half of another plane. The transformation can be considered as the mapping of a polygon from the Z plane on to a similar polygon in the ζ plane in such a manner that the polygon in the Z plane is opened at some convenient point and then extended on one side to $\zeta = -\infty$ and the other to $\zeta = +\infty$ along the real axis of the ζ plane. Thus the transformation maps conformally the region interior to the polygon into the entire upper/lower half of the auxiliary ζ plane. For the polygon ABCDEE′F located in the Z plane (Fig. 1(a)), the transformation that maps it conformally on to the upper half of the auxiliary ζ plane (Fig. 1(d)) is

$$\frac{\mathrm{d}Z}{\mathrm{d}\zeta} = C_1(\zeta-\alpha)^{A_1-1}(\zeta-\beta)^{A_2-1}(\zeta-\delta)^{A_3-1}\ldots \tag{52}$$

$$Z = C_1\int_0^{\zeta}\frac{\mathrm{d}t}{(t-\alpha)^{1-A_1}(t-\beta)^{1-A_2}(t-\delta)^{1-A_3}\ldots} + C_2 \tag{53}$$

where $A_1, A_2, A_3, \ldots$ are the interior angles (fraction of π) of the polygon in the Z plane, and $\alpha, \beta, \delta, \ldots$ $(-\infty < \alpha < \beta < \delta < \ldots < \infty)$ are the points on the real axis of the ζ plane corresponding to the respective vertices. Any three of the values $\alpha, \beta, \delta, \ldots$ can be chosen arbitrarily to correspond to three of the vertices of the given polygon. The $(N - 3)$ remaining values must then be determined so as to satisfy conditions of similarity. The interior angle at the point of opening may be regarded as π, hence it takes no part in the transformation. Also the vertex of the polygon placed at infinity in the ζ plane does not appear in the transformation. Thus by mapping a vertex of the flow region into one at infinity in the auxiliary plane omits the vertex factor from the transformation and reduces one unknown. Using values of $A_1, A_2, A_3, \ldots$ from Fig. 1(a) and $\alpha, \beta, \delta, \ldots$ from Fig. 1(a), equation (53) reduces to equation (1).

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