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Optimal design of curved bed trapezoidal canal sections

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Optimal design equations for circular and parabolic bed trapezoidal canal sections are presented in this paper. A cost function involving earthwork cost and lining cost has been considered and Manning's equation has been used as a constraint function. The resulting optimisation problem involving a non-linear cost function with the non-linear equality constraint has been converted in non-dimensional form and has then been solved by Lagrange's method of undetermined multipliers for the minimum area section and by applying Solver by MS Excel for minimum cost sections. The study also addresses the bounds on the canal dimensions. The design equations for a constrained section, a minimum earthwork cost section and a minimum cost lined section have been obtained through minimisation of errors or regression analysis. The proposed design equations are in explicit form and result in optimal dimensions of a canal in single step computations. A design example involving different cases has been presented to demonstrate the simplicity of the method.

NOTATION

A	flow area of canal (m^2)
C	cost per unit length of canal (\$/m)
c_e	cost per unit volume of earthwork at ground level (\$/m^3)
c_l	cost per unit surface area of lining (\$/m^2)
c_r	increase in unit excavation cost per unit depth (\$/m^4)
f_z	function of z
L	length scale (m)
n	Manning's roughness factor
P	flow perimeter of canal (m)
Q	discharge (m^3/s)
R	hydraulic radius (m)
S_f	energy or friction slope
S_0	bed slope of canal
T	top width at free water surface (m)
V	average velocity (m/s)
X	horizontal axis
Y	vertical axis
y	normal depth of flow in canal (m)
z	side slope of canal
ϕ	equality constraint

Subscripts

$*$	non-dimensional
L	limiting value
s	specified
u	unconstrained

Superscript

$*$	optimal

1. INTRODUCTION

Networks of canals are used to convey, distribute and apply water to the land. A canal in a network may be either unlined or lined. River beds, unlined canals and irrigation furrows all tend to approximate a stable parabolic shape.[1] Therefore, unlined canals can be made more hydraulically stable by initially constructing them in a curved shape. As the channel side slopes along the cross-section are always less than the maximum allowable side slope at the water surface, parabolic channels are physically more stable.[1–3] A lined parabolic channel has no sharp angles of stress concentration where cracks may occur, and can be prefabricated in moulded sections. Small parabolic ditches can be constructed by bulldozers and other types of earth-moving equipment.[1] On the other hand, the most commonly used channel shape is trapezoidal owing to the ease of construction. Therefore, it is more advantageous to use a combination of curved bottom section and trapezoidal upper section. Such combined (circular bed trapezoid and rounded corner trapezoid) sections are the recommended sections for lined canals by the Bureau of Indian Standards.[4]

Irrigation canals are lined for several purposes.[5] Lined canals are designed for uniform flow considering hydraulic efficiency, practicability and economy.[6] A maximum hydraulic radius results in a section of minimum excavation area and best hydraulic design. Monadjemi[7] and others[8–12] have presented a fundamental approach for determining the best hydraulic section based on Lagrange's method of undetermined multipliers.

When an open channel is constructed, the excavation and the lining constitute major cost. Obviously it is desirable to keep these costs at a minimum by adopting the most economical canal cross-section. Several investigators have considered cost aspects in the design of canal sections.[5,8–10,13–18] Explicit equations are not available for designing minimum earthwork cost sections and minimum cost lined sections of circular and parabolic bed trapezoid canals. Also, explicit equations are not available for setting bounds to these canal dimensions. The present study is an attempt to address these design problems in such canals.

Starting from the geometric properties of circular and parabolic bed trapezoidal sections and the governing uniform flow equation, the optimisation method for a minimum area section has been described. The resultant optimal parameters have been obtained in the non-dimensional form. The bounds on the canal dimensions have also been treated. The next part proposes design equations for the minimum cost sections, which have been obtained by formulating a cost function involving earthwork cost and lining cost. The optimisation problem involving a non-linear cost function with the non-linear equality constraint has been converted in non-dimensional form and minimised using Solver (MS Excel). The proposed design equations have been obtained in explicit form through minimisation of errors or regression analysis.

2. GEOMETRIC PROPERTIES OF COMBINED SECTIONS

2.1. Circular bed trapezoidal section

Geometric elements of a circular bed trapezoidal section (Figure 1) can be defined as follows

1a–c
$$\theta = \cot^{-1} z; \quad \sin\theta = 1/\sqrt{1+z^2}; \quad \cos\theta = z/\sqrt{1+z^2}$$

1d
$$y = h_1 + h_2 = r(1-\cos\theta) + \frac{(T - 2r\sin\theta)}{2z}$$

1e
$$T = 2r\sin\theta + 2h_2 z$$

1f
$$r = \left(\frac{T}{2z} - y\right)\frac{z}{\sqrt{1+z^2} - z}$$

1g
$$h_1 = \left(\frac{T}{2z} - y\right)\frac{z}{\sqrt{1+z^2}}$$

1h
$$h_2 = \frac{2y - T\left(\sqrt{1+z^2} - z\right)}{2\sqrt{1+z^2}\left(\sqrt{1+z^2} - z\right)}$$

1i
$$A = A_1 + A_2 = \frac{T^2}{4z} - \frac{r^2}{z}\left(1 - z\cot^{-1} z\right) = \frac{T^2}{4z} - z\left(1 - z\cot^{-1} z\right)\left(\frac{T/2z - y}{\sqrt{1+z^2} - z}\right)^2$$

1j
$$P = 2r\theta + 2h_2\sqrt{(1+z^2)} = \frac{T}{z}\sqrt{(1+z^2)} - \frac{2r}{z}\left(1 - z\cot^{-1} z\right) = \frac{T}{z}\sqrt{(1+z^2)} - 2\left(1 - z\cot^{-1} z\right)\left(\frac{T/2z - y}{\sqrt{1+z^2} - z}\right)$$

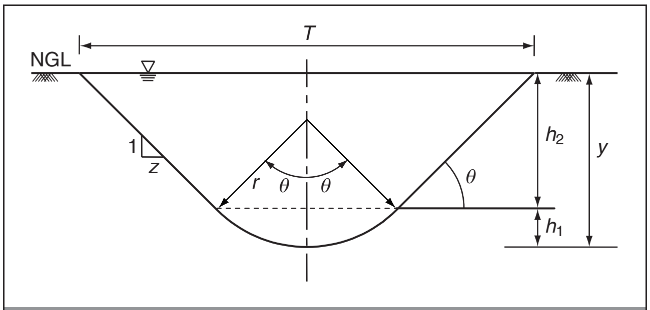

Figure 1. Cross-section of a circular bed trapezoid canal section

where A is the flow area of channel section (m^2); A_1 is the area of the circular portion (m^2); A_2 is the area of the trapezoidal portion (m^2); y is the depth of flow (m); h_1 is the depth of the circular part (m); h_2 is the depth of the trapezoidal part (m); r is the radius of the bed section or circular part (m); T is the top width of the channel (m); z is the side slope of the channel (H:V) at the water surface; and θ is the side slope angle.

2.2. Parabolic bed trapezoidal section

The following expressions describe the geometric elements of a parabolic bed trapezoid channel (Figure 2)

2a,b
$$h_1 = \left(\frac{T}{2z} - y\right); \quad h_2 = \left(2y - \frac{T}{2z}\right)$$

2c
$$T = 4zh_1 + 2zh_2 = 4z\left(\frac{T}{2z} - y\right) + 2z\left(2y - \frac{T}{2z}\right)$$

2d
$$A = \frac{8h_1^2 z}{3} + (4h_1 + h_2)zh_2 = \frac{4Ty}{3} - \frac{T^2}{12z} - \frac{4y^2 z}{3}$$

2e
$$P = 2z^2\left[\frac{1}{z}\sqrt{1+\frac{1}{z^2}} + \ln\left(\frac{1}{z} + \sqrt{1+\frac{1}{z^2}}\right)\right]h_1 + 2h_2\sqrt{1+z^2}$$

2f
$$P = f_z\left(\frac{T}{2z} - y\right) + 2\left(2y - \frac{T}{2z}\right)\sqrt{1+z^2}$$

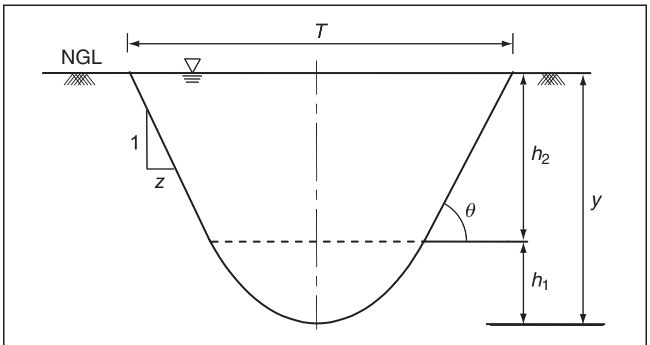

Figure 2. Cross-section of a parabolic bed trapezoid canal section

where f_z is a function of z only[8] defined as

$$f_z = 2z^2\left[\frac{1}{z}\sqrt{1+\frac{1}{z^2}} + \ln\left(\frac{1}{z}+\sqrt{1+\frac{1}{z^2}}\right)\right] \tag{2g}$$

3. CANAL SECTION COST

Considering depth-dependent earthwork cost and the cost of lining, the unit length canal section cost was given by Chahar[8] and Swamee *et al.*[5,17] as

$$C = c_e A + c_r A\bar{y} + c_l P \tag{3}$$

where C is the canal cost in monetary unit per unit length, (for example \$/m); c_e is the cost per unit volume of earthwork at ground level (\$/m^3); c_r is the additional cost per unit volume of excavation per unit depth (\$/m^3 per m of depth); c_l is the cost per unit surface area of lining (\$/m^2); and $\bar{y}$ is the depth (m) of the centroid of the area of excavation from the ground surface, which is given by

$$\bar{y} = \frac{r^2[(r/3)(3\sin\theta - 3\theta\cos\theta - \sin^3\theta) + h_2(\theta - \sin\theta\cos\theta)] + (h_2^2/3)(zh_2 + 3r\sin\theta)}{[(T^2/4z) - (r^2/z)(1 - z\cot^{-1} z)]} \tag{4}$$

for circular bed section and

$$\bar{y} = \frac{16h_1^3 + 40h_1^2 h_2 + 30h_1 h_2^2 + 5h_2^3}{40h_1^2 + 60h_1 h_2 + 15h_2^2} \tag{5}$$

for parabolic bed section.

4. OPTIMAL CANAL SECTION

A lined canal section is designed for uniform flow. The most commonly used uniform flow formula is the Manning equation.[19] The uniform flow rate or discharge $\boldsymbol{Q}$ (m^3/s) in a canal by Manning's equation is

$$\boldsymbol{Q} = A\boldsymbol{V} = \frac{1}{n}AR^{2/3}S_f^{1/2} = \frac{1}{n}A\left(\frac{A}{P}\right)^{2/3}S_f^{1/2} = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S_0^{1/2} \tag{6}$$

where $\boldsymbol{V}$ is the mean velocity of uniform flow (m/s); R is the hydraulic radius (m), defined as the ratio of flow area to the flow perimeter; n is the Manning's roughness coefficient; S_f is the energy slope (dimensionless); and S_0 is the bed slope of the canal (dimensionless). For uniform flow $S_f = S_0$.

The design of an optimal canal section involves minimisation of canal section cost subject to flow requirements. Mathematically, it could be stated as follows:

Minimise

$$C = c_e A + c_r A\bar{y} + c_l P \tag{7a}$$

Subject to

$$\phi = \boldsymbol{Q} - \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S_0^{1/2} = 0 \tag{7b}$$

Assuming a length scale L (m)

$$L = \left(\boldsymbol{Q}n/\sqrt{S_0}\right)^{3/8} \tag{8}$$

the following non-dimensional variables have been defined

$$y_* = y/L; \quad r_* = r/L; \quad T_* = T/L; \quad P_* = P/L \tag{9a–d}$$

$$A_* = A/L^2; \quad C_* = \frac{C}{c_e}\frac{1}{L^2}; \quad c_{r*} = \frac{c_r}{c_e}L; \quad c_{l*} = \frac{c_l}{c_e}\frac{1}{L} \tag{9e–h}$$

where subscript * denotes a non-dimensional parameter. As c_e/c_r and c_l/c_e have length dimensions (m), they remain unaffected by the monetary unit chosen. These ratios can be obtained for various types of linings and soil strata using appropriate unit rates.[16] Using these parameters, the problem in non-dimensional form becomes

Minimise

$$C_* = A_* + c_{r*}A_*\bar{y}_* + c_{l*}P_* \tag{10a}$$

Subject to

$$\phi_* = A_*^{5/3} - P_*^{2/3} = 0 \tag{10b}$$

This optimisation problem has been solved for various cases.

4.1. Best hydraulic section

When lining- and depth-dependent excavation costs are not considered (i.e. $c_l = c_r = 0$), the problem in Equations 10a and 10b reduces to the following area minimisation case.

Minimise

$$A_* = A(y_*, T_*, z) \tag{11a}$$

Subject to

$$\phi_* = A_*^{5/3} - P_*^{2/3} = \phi(A_*, P_*) = \phi(y_*, T_*, z) = 0 \tag{11b}$$

where y, T and z have been chosen as independent (decision) variables. The minimum area section is the best hydraulic section as well. Applying Lagrange's method of undetermined multipliers[20] for three variables

$$\frac{\partial A_*}{\partial T_*}\frac{\partial P_*}{\partial y_*} = \frac{\partial A_*}{\partial y_*}\frac{\partial P_*}{\partial T_*} \tag{12a}$$

$$\frac{\partial A_*}{\partial T_*}\frac{\partial P_*}{\partial z} = \frac{\partial A_*}{\partial z}\frac{\partial P_*}{\partial T_*} \tag{12b}$$

$$\frac{\partial A_*}{\partial y_*}\frac{\partial P_*}{\partial z} = \frac{\partial A_*}{\partial z}\frac{\partial P_*}{\partial y_*} \tag{12c}$$

Refer to Chahar[8] or Monadjemi[7] for derivation of the above equations. Equations 12a–c give only two independent conditions, as the third is a simple combination of the remaining two. Applying Equations 12a and 12b on a circular bed section

$$T_* = 2r_*\sqrt{1+z^2}; \quad T_* = 2r_*/\sqrt{1+z^2} \tag{13a,b}$$

Solving Equations 13a,b

$$z_u^* = 0; \quad \theta_u^* = \pi/2 \tag{14a,b}$$

where subscript u and superscript * denote unconstrained and optimum values respectively. The condition in Equation 14a implies that the resultant optimal section is a semicircular section. Solving non-dimensional Manning Equation 10b with this condition

$$y_{u*}^* = r_{u*}^* = T_{u*}^*/2 = 2/(2\pi)^{3/8} \approx 1{\cdot}004 \tag{15a}$$

Therefore

$$A_*^* = (2\pi)^{1/4} \approx 1{\cdot}5834; \quad P_*^* = (2\pi)^{5/8} \approx 3{\cdot}1542 \tag{15b,c}$$

Applying Equations 12a and 12b on a parabolic bed section

$$T_* = \left(\frac{16z\sqrt{1+z^2}}{3f_z - 4\sqrt{1+z^2}}\right)y_*; \quad T_* = 4zy_* \tag{16a,b}$$

Solving Equations 16a and 16b

$$z_u^* = 0{\cdot}514 \tag{17}$$

This shows that the optimal section for this case is the best parabolic section[8] and thus

$$y_{u*}^* = 1{\cdot}08055; \quad T_{u*}^* = 2{\cdot}22161; \quad P_*^* = 3{\cdot}24003; \quad A_*^* = 1{\cdot}60036 \tag{18a–d}$$

4.2. Optimal section with constraints on canal dimensions

Constraints on the section geometry (side slope, depth, or top width) may exist owing to various reasons. Such constraints have already been considered in the optimal section design.[8,9,11,15,21] In practice, restrictions can be put on the canal depth (i.e. $y_s \leqslant y_L$) owing to the existence of unfavourable strata and/or groundwater at a shallow depth and on the top width (i.e. $T_s \leqslant T_L$) owing to restrictions on the span of bridges and cross drainage works, or on the width of right of way, and so on. Generally the side slope is chosen based on the angle of repose of material for better stability (see Table 1 for side slopes recommended by the Bureau of Indian Standards) or for vehicles to cross the channel during no-flow periods (i.e. $z_s \geqslant z_L$). The variables with subscript s and L denote their specified and limiting values respectively. The constraints on canal dimension become effective only if $y_L \leqslant y_u^*$, or $T_L \leqslant T_u^*$, or $z_L \geqslant z_u^*$. Otherwise the prescribed bound on a particular parameter will be non-binding, and hence the parameter as well as the remaining parameters will attain their unconstrained optimal values.

S. No.	Type of soil	Side slopes (horizontal:vertical)	
		In cutting	In embankment
i	Very light loose sand to average sandy soil	2:1 to 3:1	2:1 to 3:1
ii	Sandy loam	1·5:1 to 2:1	2:1
iii	Sandy gravel/murum	1·5:1	1·5:1 to 2:1
iv	Black cotton	1·5:1 to 2·5:1	2:1 to 3·5:1
v	Clayey soils	1·5:1 to 2:1	1·5:1 to 2·5:1
vi	Rock	0·25:1 to 0·5:1	0·25:1 to 0·5:1

Table 1. Recommended side slopes by IS 10430: 2000[4]

4.2.1. Limits on side slope. If restrictions are put on the canal side slope (i.e. $z_s \geqslant z_L$), then the optimisation problem in Equations 11a and 11b reduces to a case of two variables (y and T) – that is, minimise $A_* = A(y_*, T_*)$, subject to $\phi(y_*, T_*) = 0$. Lagrange's method for two variables (y and T) gives Equation 12a, which consequently yields Equation 13a as the optimal condition for a circular bed section, as follows

$$T_* = 2r_*\sqrt{1+z_s^2} \tag{19a}$$

This implies that

$$y_*^* = r_*^*; \quad h_{1*}^* = y_*\left(1 - \frac{z_s}{\sqrt{1+z_s^2}}\right); \quad h_{2*}^* = y_*\frac{z_s}{\sqrt{1+z_s^2}} \tag{19b–d}$$

$$A_*^* = y_*^2(z_s + \cot^{-1}z_s); \quad P_*^* = 2y_*(z_s + \cot^{-1}z_s) \tag{19e–f}$$

This is the recommended section by the Bureau of Indian Standards.[4] Solving the non-dimensional Manning's equation with Equations 12a and 19a for z_s ranging from 0·0 to 3·5, a set of values for y_* has been generated.[22] These values of y_* have been regressed through MS Excel. The regressed equation (regression coefficient = 0·9995) for y_* is

$$y_*^* = 0{\cdot}0139z_s^3 - 0{\cdot}0723z_s^2 + 0{\cdot}011z_s + 1{\cdot}004 \tag{20}$$

Once the value of y_* is obtained from Equation 20 for a specified value of z the remaining optimal parameters of the Indian standard section can be obtained by Equations 19a–f.

For a parabolic bed section the optimal condition is described by Equation 16a. Using Manning's equation along with Equation 16a, y_* has been obtained for a given value of z. For these values of y_* and z_s, Equation 16a has been used to find T_*. Following the similar procedure[4] T_* and y_* have been obtained for z_s ranging from 0·51 to 3·0. The resultant values have been regressed to obtain the following equations

$$T_*^* = 0{\cdot}0023z_s^2 + 1{\cdot}0385z_s + 1{\cdot}6872 \tag{21a}$$

$$y_*^* = -0{\cdot}0239z_s^3 + 0{\cdot}1539z_s^2 - 0{\cdot}4157z_s + 1{\cdot}2568 \tag{21b}$$

The regression coefficients (RC) of the above equations are 0·9996 and 0·9957 respectively.[22] Equations 2a–e can be used to fix other parameters for specified z and corresponding T_* from Equation 21a and y_* from Equation 21b.

4.2.2. Limits on flow depth. If restrictions are put on the canal depth (i.e. $y_s \leqslant y_L$), then the optimisation problem becomes: minimise $A_* = A(T_*, z)$, subject to $\phi(T_*, z) = 0$. The optimal condition is given by Equation 12b and thus by Equation 13b, which implies that the optimal channel section is a part of circular section only (i.e. the trapezoidal part is absent). Regressed equation (RC = 0·9994) satisfying flow equation and Equation 13b for prescribed y_* is

$$z^* = -21{\cdot}173y_{s*}^3 + 58{\cdot}377y_{s*}^2 - 56{\cdot}446y_{s*} + 19{\cdot}255 \quad (22)$$

For specified y_s and known z, the remaining optimal parameters of the section can be obtained by

$$T_*^* = \frac{2y_{s*}}{(\sqrt{1+z^2} - z)}; \quad r_*^* = \frac{\sqrt{1+z^2}}{(\sqrt{1+z^2} - z)}y_{s*} \quad (23a,b)$$

$$A_*^* = y_{s*}^2 \frac{(1+z^2)\cot^{-1} z - z}{\left(\sqrt{1+z^2} - z\right)^2}; \quad P_*^* = 2y_{s*}\frac{\sqrt{1+z^2}\cot^{-1} z}{(\sqrt{1+z^2} - z)} \quad (23c,d)$$

For a parabolic section the optimal condition is given by Equation 12b and thus by Equation 16b, which implies that the optimal channel section is a part of a parabolic section only (i.e. the trapezoidal part is absent). Regressed equation (RC = 0·9989) satisfying flow equation and Equation 16b for prescribed y_* is

$$z^* = -16{\cdot}18y_{s*}^3 + 47{\cdot}768y_{s*}^2 - 48{\cdot}167y_{s*} + 17{\cdot}201 \quad (24)$$

Once the value of z is obtained for a given value of y_* the remaining optimal parameters for the section can be obtained by

$$T_*^* = 4y_{s*}z; \quad P_*^* = f_z y_{s*}; \quad A_*^* = 8y_{s*}^2 z/3 \quad (25a\text{–}c)$$

4.2.3. Limits on top width. If there are restrictions on the canal top width, then the optimisation problem becomes: minimise $A_* = A(y_*, z)$, subject to $\phi(y_*, z) = 0$. The optimal condition is given by Equation 12c and then by Equation 13b for a circular bed section or by Equation 16b for a parabolic bed section. These are identical to the depth constraint case and hence the optimal channel section is a part of circular section or a part of parabolic section only. For prescribed T_* the fitted equation (RC = 0·9999) for a parabolic section is

$$z^* = 0{\cdot}0427T_{s*}^2 + 0{\cdot}2834T_{s*} - 0{\cdot}3267 \quad (26)$$

The following equations give the remaining optimal parameters

$$y_*^* = T_{s*}/4z; \quad P_*^* = f_z T_{s*}/4z; \quad A_*^* = T_{s*}^2/6z \quad (27a\text{–}c)$$

4.3. Minimum cost canal section

Although the best hydraulic section has the minimum flow area and the minimum wetted perimeter, it is not necessarily the most economical section.

4.3.1. Minimum earthwork cost canal section. A canal passing through hard/firm strata may be kept unlined but it is designed as a rigid boundary channel. Sometimes canals are lined with low-cost lining materials, in which case cost of the earthwork is more significant than the cost of lining. Neglecting the lining cost, the optimisation problem converts to the following:

Minimise

$$C_* = A_* + c_{r*}A_*\bar{y}_*$$

Subject to

$$\phi_* = A_*^{5/3} - P_*^{2/3} = 0$$

Solver (MS Excel) has been used to solve this minimisation problem with appropriate constraints on the decision and state variables. The solution with input variables $0 \leqslant c_{r*} \leqslant 10$ has yielded a large number of optimal sections.[22] The equations of regressed curves (RC > 0·98) for the optimal parameters are

$$T_*^* = 0{\cdot}0059c_{r*}^3 - 0{\cdot}0813c_{r*}^2 + 0{\cdot}7884c_{r*} + 2{\cdot}008 \quad (28a)$$

$$z^* = 0{\cdot}0053c_{r*}^3 - 0{\cdot}0675c_{r*}^2 + 0{\cdot}5654c_{r*} \quad (28b)$$

$$y_*^* = -0{\cdot}0015c_{r*}^3 + 0{\cdot}028c_{r*}^2 - 0{\cdot}18613c_{r*} + 1{\cdot}004 \quad (28c)$$

for a circular bed trapezoidal canal section and

$$T_*^* = 0{\cdot}0102c_{r*}^3 - 0{\cdot}146c_{r*}^2 + 1{\cdot}1335c_{r*} + 2{\cdot}221 \quad (29a)$$

$$z^* = 0{\cdot}0127c_{r*}^3 - 0{\cdot}1612c_{r*}^2 + 1{\cdot}1395c_{r*} + 0{\cdot}514 \quad (29b)$$

$$y_*^* = 0{\cdot}0032c_{r*}^2 - 0{\cdot}0866c_{r*} + 1{\cdot}081 \quad (29c)$$

for a parabolic bed trapezoidal canal section. Other optimal parameters can be evaluated by geometric element Equations 1 or 2 according to the case in hand.

4.3.2. Minimum cost lined section. In most cases canals are lined, and lining and earthwork costs constitute the total canal section cost. The solution of general optimisation problem stated as in Equations 10a and 10b through Solver with input variables $c_{r*} \leqslant c_{l*}$ has yielded a large number of optimal sections.[22] An analysis similar to Chahar[8] of these optimal sections has led to the following empirical equations

$$T_*^* = 2{\cdot}008\left(\frac{1 + 0{\cdot}11c_{r*} + c_{l*}}{1 + c_{l*}}\right) \quad (30a)$$

30b	$$y_*^* = 1{\cdot}004\left(\frac{1 + 1{\cdot}05c_{l*}}{1 + 0{\cdot}07c_{r*} + 1{\cdot}05c_{l*}}\right)$$

for a circular bed trapezoidal canal section and

31a	$$T_*^* = 2{\cdot}221\left(\frac{1 + 0{\cdot}09c_{r*} + 0{\cdot}29c_{l*}}{1 + 0{\cdot}29c_{l*}}\right)$$

31b	$$y_*^* = 1{\cdot}081\left[\frac{1 + 0{\cdot}0009c_{r*}(c_{l*} - 0{\cdot}68c_{r*}) + 0{\cdot}01c_{l*}}{1 + 0{\cdot}01c_{l*}}\right]$$

for a parabolic bed trapezoidal canal section. The optimal side slope (z^*) can be calculated by solving Manning's equation with T^* and y^*. Equations 30a,b and 31a,b have been obtained by conceiving an appropriate functional form and then minimising errors between the optimal values and the computed values from the conceived function with coefficients.

Equations 28–31 for minimum cost sections are valid without constraints on the side slope, depth, or top width. For $c_r = 0$, Equations 28a–c, 29a–c, 30a,b and 31a,b reduce to corresponding equations for the minimum area section, namely Equations 14a,b, 15a, 17 and 18a,c.

5. EXAMPLE AND DISCUSSION

5.1. Example

For design, consider a canal to carry a discharge of 3 m^3/s on a longitudinal bed slope of 0·001. Design steps suggested by Chahar[8,9] have been adopted. Assuming a float-finished concrete lining,[19] Manning's roughness coefficient = 0·015. From Equation 8

$$L = \left(3 \times 0{\cdot}15/\sqrt{0{\cdot}001}\right)^{3/8} = 1{\cdot}1414\,\text{m}$$

Case 1 (*unconstrained section*). Using Equation 14a and 15a–c for circular bottom section $z^* = 0$; $\theta^* = \pi/2$; $y_*^* = r_*^* = T_*^*/2 = 1{\cdot}004$; $A_*^* = 1{\cdot}5834$; and $P_*^* = 3{\cdot}1542$; therefore $y^* = r^* = 1{\cdot}004 \times 1{\cdot}14144 = 1{\cdot}146$ m; $T^* = 2{\cdot}292$ m; $A^* = 1{\cdot}5834 \times (1{\cdot}14144)^2 = 2{\cdot}063\ \text{m}^2$; and $P^* = 3{\cdot}1542 \times 1{\cdot}1414 = 3{\cdot}60$ m. Average velocity of flow $\boldsymbol{V} = \boldsymbol{Q}/A = 3/2{\cdot}063 = 1{\cdot}454$ m/s, which is less than the permissible velocity for a concrete lining.[18]

Similarly using Equations 17 and 18a–d for parabolic bottom section $z^* = 0{\cdot}514$; $y^* = 1{\cdot}2333$ m; $T^* = 2{\cdot}5358$ m, $A^* = 2{\cdot}0849\ \text{m}^2$ and $P^* = 3{\cdot}6983$ m.

Case 2 (specified side slope). If the side slope is restricted to $z_L = 1{\cdot}5$, then using corresponding equations for circular bottom and parabolic bottom sections the optimal parameters are listed in Table 2. The final values of various parameters in Table 2 have been rounded off up to two decimal points for all the cases.

Case 3 (specified depth). Let $y_L = 1{\cdot}0$ m so $y_{L*} = 1/1{\cdot}1414 = 0{\cdot}8761$, which is less than the unconstrained optimal values for circular and parabolic bed sections; hence it is binding and optimal for both type of sections. See Table 2 for optimal values.

Case 4 (*specified top width*). If the top width is restricted to $T_L = 2{\cdot}5$ m, then $T_{L*} = 2{\cdot}5/1{\cdot}1414 = 2{\cdot}1903$, which is binding for the parabolic bed section. For a circular bed section the specified top width is impractical.

Case 5 (*minimum earthwork cost section*). Adopting $c_e/c_r = 7{\cdot}0$ m,[16] $c_{r*} = 0{\cdot}1631$ from Equation 9g. Using Equations 28a–c for the circular bottom section and Equations 29a–c for the parabolic bottom section optimal parameters are tabulated.

Case 6 (*minimum cost lined section*). Assuming $c_l/c_e = 10{\cdot}0$ m,[16] $c_{l*} = 8{\cdot}7609$. Equations 30a,b and Equations 31a,b result into optimal parameters for the circular bottom section and the parabolic bottom section respectively as listed in Table 2.

5.2. Discussion

Table 2 shows that for specified side slope, a circular bed section is a combined section, otherwise it is a part of a circle only, whereas a parabolic bed section is a combined section for specified side slope as well as corresponding to minimum earthwork cost section and minimum cost lined section. A circular bed section is more efficient (hydraulically or economically) than a parabolic bed section for all cases, although for large specified side slopes they are indistinguishable.

The binding constraint on the top width of a circular bed section ($T_L \leqslant T_u^*$) is not practical as it results in a negative z –

Case	Type of bed	z	h_1: m	y: m	T: m	A: m^2	P: m	C/c_e
Case 1: unconstrained	Circular	0·00	1·15	1·15	2·29	2·06	3·60	38·21
	Parabolic	0·51	1·23	1·23	2·54	2·09	3·70	42·75
Case 2: specified side slope	Circular	1·50	0·17	1·03	3·72	2·23	4·31	45·48
	Parabolic	1·50	0·21	1·03	3·71	2·21	4·29	45·30
Case 3: specified depth	Circular	0·37	1·00	1·00	2·88	2·09	3·73	39·52
	Parabolic	0·79	1·00	1·00	3·14	2·10	3·86	43·65
Case 4: specified top width	Circular	–	–	–	–	–	–	–
	Parabolic	0·50	1·25	1·25	2·50	2·09	3·71	42·84
Case 5: minimum earthwork cost section	Circular	0·22	1·11	1·11	2·44	2·07	3·61	38·42
	Parabolic	0·70	0·75	1·22	2·74	2·18	3·81	41·53
Case 6: minimum cost lined section	Circular	0·028	1·14	1·14	2·30	2·04	3·54	37·55
	Parabolic	0·57	1·01	1·24	2·55	3·06	3·71	41·41

Table 2. Comparison of circular and parabolic bed trapezoid sections

that is, a closing circular section (like sewer sections), which is not used for channels. For a similar reason $y_s > y_u^*$ is impractical. However, if a top width more than T_u^* is desired, then the resultant section will not be better than the unconstrained optimal section and the corresponding side slope for prescribed T^* can be estimated directly from the following regressed equation ($R^2 = 0{\cdot}9996$) without the need of solution of Manning's equation

$$z = 0{\cdot}0083T_{s*}^2 + 0{\cdot}6396T_{s*} - 1{\cdot}3176 \tag{32a}$$

The remaining parameters can be obtained by

$$y_* = \left(\sqrt{1+z^2} - z\right)T_{s*}/2; \quad r_* = \sqrt{1+z^2}\,T_{s*}/2 \tag{32b,c}$$

$$A_* = \left[\left(1+z^2\right)\cot^{-1} z - z\right]T_{s*}^2/4; \quad P_* = T_{s*}\sqrt{1+z^2}\cot^{-1} z \tag{32d, e}$$

For a parabolic bed section, if the channel depth is fixed greater than y_u^*, then

$$\log z = -0{\cdot}0039y_{s*}^3 + 0{\cdot}1678y_{s*}^2 - 1{\cdot}0159y_{s*} + 0{\cdot}6079 \tag{33}$$

results into the corresponding side slope and subsequently Equations 25a–c determine T, P and A. On the other hand, if $T_s > T_u^*$ then the following equation ($R^2 = 0{\cdot}9999$) gives side slope for specified T_*, satisfying Manning's equation

$$z = 0{\cdot}0878T_{s*}^2 + 0{\cdot}082T_{s*} - 0{\cdot}1005 \tag{34}$$

and afterwards Equations 27a–c find out T, P and A.

6. CONCLUSIONS

Circular and parabolic bed trapezoidal canal sections are advantageous and their optimal (hydraulic efficient section) design equations can be obtained by Lagrange's method of undetermined multipliers. Optimal sections for unconstrained circular bed trapezoid and parabolic bed trapezoid are the semicircle and parabola (side slope = 0·514) respectively. Also the optimal sections corresponding to specified flow depth or top width are the curved portion only (trapezoidal part absent). For specified side slope these are combined sections. Design of minimum cost circular and parabolic bed trapezoidal canal sections involves a non-linear cost function with a non-linear equality constraint, which can be converted into non-dimensional form and minimised using the Solver (MS Excel). Minimum earthwork cost section and minimum cost lined section are a part of a circle only for circular bed channels whereas they are combined sections for parabolic bed channels. Also, a circular bed section is more efficient than a parabolic bed section for all cases. The design equations for a best hydraulic section, a minimum earthwork cost section and a minimum cost lined section can be obtained in explicit form through minimisation of errors or regression analysis. The proposed equations are simple as they result in optimal dimensions of a canal in single-step computations.

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