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Seepage from a special class of a curved channel with drainage layer at shallow depth

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[1] In the present study an inverse method has been used to obtain an exact solution for seepage from a curved channel passing through a homogeneous isotropic porous medium underlain by a drainage layer at shallow depth whose boundary maps along a circle onto the hodograph plane. The solution involves inverse hodograph and Schwarz-Christoffel transformation. The solution also includes a set of parametric equations for the shape of the channel contour and loci of phreatic line. Variation in the seepage velocity along the channel contour is presented as well. All these expressions involve improper integrals along with accessory parameters. A particular solution corresponding to the water table below the top of the drainage layer has also been deduced from the general solution.

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1. Introduction

[2] Study of seepage from curved channels is important because of its applications in areas of irrigation engineering, hydrology, reservoir management, and groundwater recharge. Seepage from curved channels has been estimated by several investigators for different boundary conditions using analytical methods [*Aravin and Numerov*, 1965; *Ilyinskii and Kacimov*, 1984; *Polubarinova Kochina*, 1962]. Using an inverse method, *Chahar* [2006] presented an exact solution for seepage from a curved channel whose boundary maps along a circle onto the hodograph plane. The resulting channel possesses many interesting and useful properties. The channel shape is an approximate semiellipse with top width as major axis and twice of depth of flow as minor axis and vice versa. Actually, it always lies between an ellipse and a parabola and its any coordinate is nearly exact to the average of coordinates of corresponding ellipse and parabola. Also, this shape is non self intersecting and is feasible from a slit (very narrow and deep channel) to a strip (very wide and shallow channel) unlike the Kozeny's trochoid shape, which is self intersecting for top width to depth ratio greater than 1.14 [*Kacimov*, 2003]. The seepage function of this channel is a linear combination of seepage functions for a slit and a strip [*Chahar*, 2001, 2007a]. A semicircle is a special case of a semiellipse, consequently at top width to depth ratio equal to 2 the curved channel can be approximated into a semicircular channel. Moreover, the quantity of seepage from this class of curvilinear channel is simply top width times the maximum velocity at the center of the channel. Additionally, *Chahar and Vadodaria* [2008] observed that the variation in the drainage quantity from a ponded surface to an array of empty ditches is like the shape of this curved channel. The hydraulic properties of this type of channel and sections corresponding to minimum area, minimum seepage and maximum hydraulic efficiency have been investigated by *Chahar* [2007b]. The solutions given by *Chahar* [2006, 2007b] are applicable for the curvilinear bottomed channel passing through a homogeneous and isotropic porous medium of infinite extent. However, in most alluvial plains the soil is stratified. In many cases, highly permeable layers of sand and gravel underlie the top low permeable layer of finite depth. In that case the lower layer of sand and gravel acts as a free drainage layer for the top seepage layer. The seepage from a canal running through such stratified strata depends on the position of water table with respect to the drainage layer. If the water table is below the top of the drainage layer then the seepage is much more than that in homogeneous medium of very large depth. The difference in quantity of seepage becomes appreciable when the drainage layer lies at a depth less than twice the depth of water in the canal [*Chahar*, 2007a]. On the other hand, if the water table is above the top of the drainage layer then the seepage is less depending on the head difference between the water table and the water surface in the channel. For that reason it is pertinent to investigate the seepage from a curved section whose boundary maps along a circle onto the hodograph plane and passing through a porous medium of finite depth underlain by a drainage layer. Using the inverse method, presented herein is an exact solution for seepage from the above mentioned curvilinear channel.

2. Analytical Solution

[3] Consider seepage from a curvilinear bottomed symmetrical channel of top width T (m) and water depth y (m) passing through a homogeneous isotropic porous medium of hydraulic conductivity k (m/s) underlain by a drainage layer at a depth d (m) below the water surface is shown in Figure 1a. The position of the water table far from the channel is d_w (m) below the water surface and $d_w < d$. Thus

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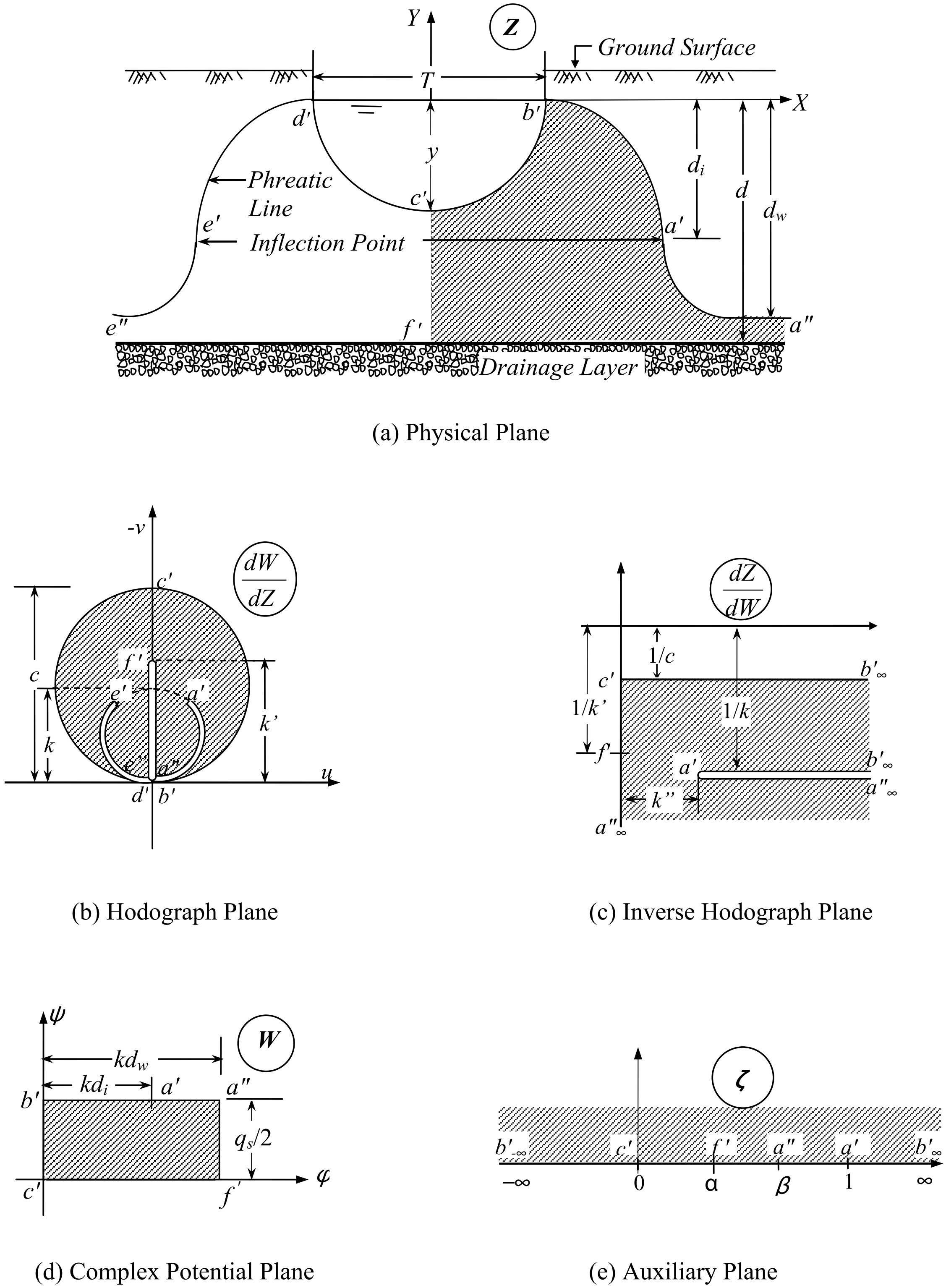

Figure 1. Seepage from a curvilinear bottomed channel with drainage layer at shallow depth and the water table above the top of the drainage layer.

the boundary condition corresponds to condition A as defined by *Bouwer* [1978] since the water table undisturbed by seepage from the channel is above the top of drainage layer or pressure head at the drainage layer is $d - d_w$ (m) above atmospheric pressure. The pattern of seepage from the channel is shown in Figure 1a, where a' is the point of inflection or maximum velocity point on the phreatic line $b'a'a''$ at a depth d_i (m) below the water surface. The effects of capillarity, accretion due to infiltration, and evaporation are ignored. In view of the significant length of the channel, the seepage flow can be considered 2-D in the vertical plane. Because of vertical symmetry, the solution for the half domain ($c'f'a''a'b'c'$) is sought.

[4] Let the complex potential be defined as $W = \varphi + i\psi$ where φ = velocity potential (m^2/s) which is equal to negative of hydraulic conductivity times the head h (m) plus C_0 (a constant) (i.e., $\varphi = -kh + C_0$) and ψ = stream function (m^2/s) which is constant along streamlines. If the physical plane is defined as $Z = X + iY$ then Darcy's law yields $u = \partial\phi/\partial X = -k\partial h/\partial X$ and $v = \partial\phi/\partial Y = -k\partial h/\partial Y$; where u and v are velocity or specific discharge vectors in X and Y directions, respectively. In the velocity hodograph plane ($dW/dZ = u - iv$), the phreatic line $b'a'a''$ will map along a double arc of circle of diameter k with a' as reversal point. The channel boundary $b'c'd'$ is an equipotential line, so the seepage velocity is normal to the boundary and in hodograph plane it will map a curvilinear path. However, the exact shape of the curve is not known. It is assumed that the channel boundary maps along a circle of diameter c in the hodograph plane and $c > k$. The top of the drainage layer $a''f'e''$ will represent a cut of length k' in the hodograph plane as shown in Figure 1b. It should be noted that the ordinate of f' and the position of point a' or e' on the circle of diameter k are unknown in the hodograph plane. Thus there are three unknown parameters c, k' and k'' in the hodograph plane, which have to be established. The inverse hodograph dZ/dW (Figure 1c) for half of the physical flow domain has been drawn following the standard steps [*Harr*, 1962; *Strack*, 1989]. The complex potential W (Figure 1d) for half of the physical flow domain has been drawn assuming $C_0 = 0$, so that $\varphi = 0$ along the channel contour $c'b'$ and kd_w along the water table. The dZ/dW plane and W plane have been mapped onto the upper half of an auxiliary (Im $\zeta > 0$) plane (Figure 1e) using the Schwarz-Christoffel conformal transformation [*Harr*, 1962; *Polubarinova-Kochina*, 1962].

2.1. Mapping in Various Planes

[5] Mapping of dZ/dW plane on the ζ plane (refer to Appendix A for details) results in

$$\frac{dZ}{dW} = \left(\frac{c - k'}{ck'F_1(\alpha,\beta)}\right)\int_0^{\zeta}\frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} - \frac{i}{c} \tag{1}$$

where t = dummy variable; and α, β = accessory parameters. The W plane mapping on the ζ plane is

$$W = \frac{kd_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\int_0^{\zeta}\frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} \tag{2}$$

where $K(\sqrt{\alpha/\beta})$ = complete elliptical integral of the first kind with modulus ($\sqrt{\alpha/\beta}$) [*Byrd and Friedman*, 1971]. Consequently the physical plane mapping becomes

$$Z = \frac{kd_w\sqrt{\beta}}{2cK\left(\sqrt{\alpha/\beta}\right)}\left(\frac{i\,(c-k')}{k'F_1(\alpha,\beta)}\int_0^{-\zeta}F_4(\tau,\alpha,\beta)d\tau + \int_0^{-\zeta}\frac{d\tau}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}}\right) - i\,y \tag{3}$$

where τ = dummy variable; and $F_1(\alpha, \beta)$ and $F_4(\tau, \alpha, \beta)$ are defined by equation (A9) and equation (A17), respectively.

2.2. Shape of Channel

[6] Separating real and imaginary parts in equation (3) gives

$$X = \frac{kd_w}{cK\left(\sqrt{\alpha/\beta}\right)}F\left(\sin^{-1}\sqrt{\frac{-\zeta}{-\zeta+\alpha}},\sqrt{\frac{\beta-\alpha}{\beta}}\right) \tag{4a}$$

$$Y = \frac{k(c-k')d_w\sqrt{\beta}}{2ck'F_1(\alpha,\beta)K\left(\sqrt{\alpha/\beta}\right)}\int_0^{-\zeta}F_4(\tau,\alpha,\beta)\,d\tau - iy \tag{4b}$$

where $F\left(\sin^{-1}\sqrt{-\zeta/(-\zeta+\alpha)}, \sqrt{(\beta-\alpha)/\beta}\right)$ = incomplete elliptical integral of the first kind with modulus $\sqrt{(\beta-\alpha)/\beta}$ and amplitude $\sin^{-1}\sqrt{-\zeta/(-\zeta+\alpha)}$ [*Byrd and Friedman*, 1971]. At the point b' ($\zeta = -\infty$; $Z = T/2$), equations (4a) and (4b) yield

$$c = \frac{2kd_wK\left(\sqrt{(\beta-\alpha)/\beta}\right)}{T\,K\left(\sqrt{\alpha/\beta}\right)} \tag{5}$$

$$\frac{(c-k')}{ck'F_1(\alpha,\beta)} = \frac{2yK\left(\sqrt{\alpha/\beta}\right)}{kd_w\sqrt{\beta}}\Big/\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau \tag{6}$$

and

$$\frac{1}{k'} = \frac{K\left(\sqrt{\alpha/\beta}\right)}{kd_w}\cdot\left(\frac{T}{2K\left(\sqrt{(\beta-\alpha)/\beta}\right)} + \frac{2yF_1(\alpha,\beta)}{\sqrt{\beta}}\Big/\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau\right) \tag{7}$$

Substitution of c and k' in equations (4a) and (4b) results in

$$\frac{X}{T} = \frac{1}{2K\left(\sqrt{(\beta-\alpha)/\beta}\right)}F\left(\sin^{-1}\sqrt{-\zeta/(-\zeta+\alpha)},\sqrt{(\beta-\alpha)/\beta}\right) \tag{8a}$$

and

$$\frac{Y}{y} = -1 + \int_0^{-\zeta}F_4(\tau,\alpha,\beta)\,d\tau\Big/\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau \tag{8b}$$

Equations (8a) and (8b) in parametric form can be used to plot the channel shape by varying parameter ζ in the range $-\infty < \zeta \leq 0$.

2.3. Position of Phreatic Line and Inflection Point

[7] The physical plane equation yields the following parametric equations for the phreatic line $b'a'a''$ ($\beta \leq \zeta \leq \infty$):

$$X = \frac{T}{2} + y\int_{\zeta}^{\infty}F_5(\tau,\alpha,\beta)\,d\tau\Big/\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau \tag{9a}$$

and

$$Y = -\frac{d_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\int_{\zeta}^{\infty}\frac{d\tau}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}} = -\frac{d_w}{K\left(\sqrt{\alpha/\beta}\right)}F\left(\sin^{-1}\sqrt{\beta/\zeta},\sqrt{\alpha/\beta}\right) \quad (9b)$$

where $F_5(\tau, \alpha, \beta)$ is defined by equation (A26). At the inflection point a' ($\zeta = 1$; $Z = X - id_i$), equation (9b) results in

$$d_i = \frac{d_w}{K\left(\sqrt{\alpha/\beta}\right)}F\left(\sin^{-1}\sqrt{\beta},\sqrt{\alpha/\beta}\right) \quad (10a)$$

Utilizing equation (2) at the point a' ($\zeta = 1$; $W = kd_i + iq_s/2$) to get

$$d_i = \frac{d_w}{K\left(\sqrt{\alpha/\beta}\right)}\left(K\left(\sqrt{\alpha/\beta}\right) - F\left(\sin^{-1}\sqrt{(1-\beta)/(1-\alpha)},\sqrt{\alpha/\beta}\right)\right) \quad (10b)$$

Equating equations (10a) and (10b) gives an identity

$$F\left(\sin^{-1}\sqrt{\beta},\sqrt{\alpha/\beta}\right) + F\left(\sin^{-1}\sqrt{(1-\beta)/(1-\alpha)},\sqrt{\alpha/\beta}\right) = K\left(\sqrt{\alpha/\beta}\right) \quad (11)$$

Equation (1) of the inverse hodograph plane at the inflection point a' ($\zeta = 1$; $dZ/dW = k'' - i/k$) after substituting c and k' yields

$$kk'' = \left(\frac{4yK\left(\sqrt{\alpha/\beta}\right)}{d_w\sqrt{\beta}}\right)\cdot\left(\ln\left(\frac{1+\sqrt{1-\beta}}{\sqrt{\beta}}\right) + \frac{\sqrt{1-\beta}}{\beta}\right)\Bigg/\int_0^{\infty}F_4(\tau,\alpha,\beta)d\tau \quad (12)$$

2.4. Variation in Seepage Velocity

[8] The distribution of the velocity of seeping water normal to the channel contour can be found by using equation (1) along $c'b'$ ($-\infty < \zeta \leq 0$) as

$$\frac{V}{k} = \frac{1}{K\left(\sqrt{\alpha/\beta}\right)}\frac{d_w}{y}\Bigg/\sqrt{\left(\frac{2F_2(\zeta,\beta)}{\sqrt{\beta}}\Bigg/\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau\right)^2 + \left(\frac{T}{2y}\Bigg/K\left(\sqrt{\frac{\beta-\alpha}{\beta}}\right)\right)^2} \quad (13)$$

This equation reduces to equation (4) for $\zeta = 0$ (at the center of the channel c') yielding maximum velocity ($V_{\max}$ = assumed c).

2.5. Quantity of Seepage

[9] The steady seepage discharge per unit length of channel q_s (m^2/s) can be expressed in the following simplest form:

$$q_s = k(T + Ay) = kyF_s \quad (14)$$

where A = Vedernikov's parameter [*Harr*, 1962] and F_s = seepage function [*Chahar*, 2001, 2006], which is a dimensionless function of channel geometry and boundary conditions [*Chahar*, 2007a]. Applying equation (2) at the point a'' ($\zeta = \beta$; $W = kd_w + iq_s/2$) gives

$$q_s = 2kd_w\frac{K\left(\sqrt{(\beta-\alpha)/\beta}\right)}{K\left(\sqrt{\alpha/\beta}\right)} \quad (15)$$

Therefore the seepage function is

$$F_s = \frac{q_s}{ky} = 2\frac{d_w}{y}\frac{K\left(\sqrt{(\beta-\alpha)/\beta}\right)}{K\left(\sqrt{\alpha/\beta}\right)} \quad (16)$$

Operating equations (5) and (15) results in

$$q_s = Tc \quad (17)$$

This equation reveals that the seepage from this class of curvilinear channels is simply top width times the seepage velocity at the center of the channel. The same was true for homogeneous medium of infinite depth (no drainage layer) case [*Chahar*, 2006].

2.6. Accessory Parameters

[10] Overall there are two unknowns i.e., accessory parameters α and β. Other parameters c, k', and k'' are part of solution as established by equations (5), (7), and (12), respectively. Two equations are required to determine these accessory parameters. One equation can be obtained from the physical plane. The physical plane equation at the point f' ($\zeta = \alpha$; $Z = -id$) gives

$$\frac{d}{y} = \frac{\int_0^{\alpha}F_6(\tau,\alpha,\beta)d\tau}{\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau} + \frac{T}{y}\frac{K\left(\sqrt{\alpha/\beta}\right)}{2\,K\left(\sqrt{(\beta-\alpha)/\beta}\right)} + 1 \quad (18)$$

See equation (A24) for the definition of $F_6(\tau, \alpha, \beta)$. For the second equation, the property of the slope of phreatic line at the inflection point can be used (see Appendix A). The slope at the point of inflection is maximum, hence kk'' is maximum. Therefore the determination of the accessory parameters α and β involves maximization of right-hand side of equation (12) subject to equation (18). Thus it is a nonlinear optimization problem subjected to a nonlinear equality constraint. Once α and β are known for specified d/y, d_w/y and T/y, the relevant relations can be used to find c/k, k'/k, kk'', the variation in the seepage velocity, the quantity of seepage, the loci of phreatic lines, the position of inflection point and the shape of the channel contour.

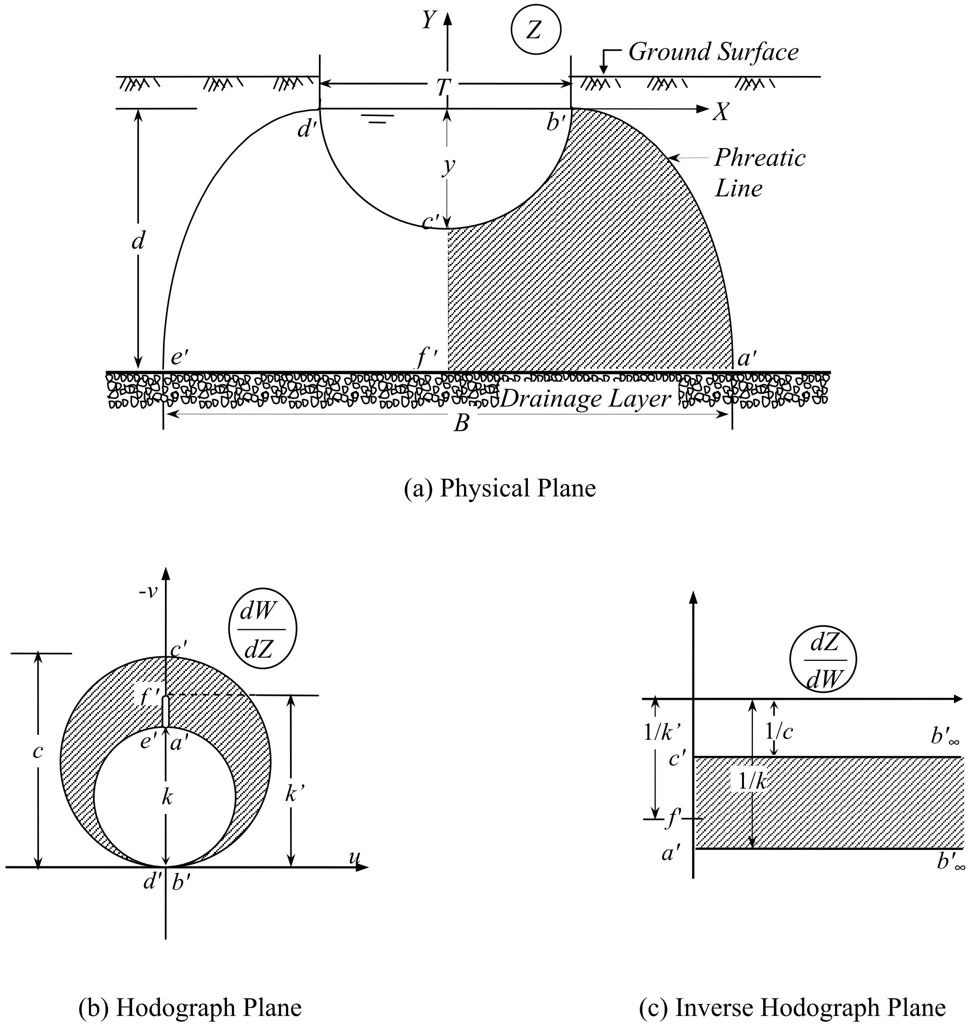

Figure 2. Seepage from a curvilinear bottomed channel with drainage layer at shallow depth and the water table below the top of the drainage layer.

2.7. Special Case: Condition A′

[11] When the water table is below the top of the drainage layer ($d_w \geq d$) then atmospheric pressure prevails there leading to condition A′ [*Bouwer*, 1978]. Figure 2 shows the altered physical, hodograph, and inverse hodograph planes. For this condition $\beta = 1$ and hence, the mappings in various planes become

$$\frac{dZ}{dW} = \frac{1}{\pi}\left(\frac{1}{k} - \frac{1}{c}\right)\int_0^{\zeta} \frac{dt}{\sqrt{t(t-1)}} - \frac{i}{c} \tag{19}$$

$$W = \frac{kd}{2K(\sqrt{\alpha})}\int_0^{\zeta} \frac{dt}{\sqrt{t(t-\alpha)(t-1)}} \tag{20}$$

$$Z = \frac{kd}{2K(\sqrt{\alpha})} \times \int_0^{\zeta} \left(\frac{1}{\pi}\left(\frac{1}{k} - \frac{1}{c}\right)\int_0^{t} \frac{d\tau}{\sqrt{\tau(\tau-1)}} - \frac{i}{c}\right)\frac{dt}{\sqrt{t(t-\alpha)(t-1)}} - iy \tag{21}$$

The Z plane mapping at the point b' gives

$$c = \frac{2kd\,K\left(\sqrt{1-\alpha}\right)}{T\,K(\sqrt{\alpha})} \tag{22a}$$

$$\frac{1}{\pi}\left(\frac{1}{k} - \frac{1}{c}\right) = yK\left(\sqrt{\alpha}\right)\Big/ 2kd\int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha + \sinh^2\tau}} \tag{22b}$$

and

$$\frac{2}{K(\sqrt{\alpha})}\frac{d}{y} - \frac{1}{K\left(\sqrt{1-\alpha}\right)}\frac{T}{y} = \pi\Big/\int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha + \sinh^2\tau}} \tag{23}$$

Equation (23) gives accessory parameter α for specified d/y and T/y.

[12] From equation (21) in the range $-\infty < \zeta \le 0$, the parametric equations for the shape of the channel contour are

$$\frac{X}{T} = \frac{1}{2K\left(\sqrt{1-\alpha}\right)} F\left(\sin^{-1}\sqrt{-\zeta/(-\zeta+\alpha)}, \sqrt{1-\alpha}\right) \quad (24a)$$

and

$$\frac{Y}{y} = -1 + \int_0^{\sinh^{-1}\sqrt{-\zeta}} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \Bigg/ \int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \quad (24b)$$

The real and imaginary parts of equation (21) in the range $1 \le \zeta < \infty$, give the following parametric equations for the phreatic line:

$$X = \frac{T}{2} + y \int_{\cosh^{-1}\sqrt{\zeta}}^{\infty} \frac{\tau\, d\tau}{\sqrt{\cosh^2\tau - \alpha}} \Bigg/ \int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \quad (25a)$$

and

$$Y = \frac{-d}{K(\sqrt{\alpha})} F\left(\sin^{-1}\sqrt{1/\zeta}, \sqrt{\alpha}\right) \quad (25b)$$

At the drainage layer a' ($\zeta = 1$; $X = B/2$), equation (25a) yields

$$\frac{B}{y} = \frac{T}{y} + 2\int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\cosh^2\tau - \alpha}} \Bigg/ \int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \quad (26)$$

The seepage velocity for this condition reduces to

$$\frac{V}{k} = \frac{\pi K\left(\sqrt{1-\alpha}\right) + \frac{T}{y}\int_0^{\infty}\left(\tau/\sqrt{\alpha+\sinh^2\tau}\right) d\tau}{\sqrt{\left(2K\left(\sqrt{1-\alpha}\right)\sinh^{-1}\sqrt{-\zeta}\right)^2 + \left(\frac{T}{y}\int_0^{\infty}\left(\tau/\sqrt{\alpha+\sinh^2\tau}\right) d\tau\right)^2}} \quad (27)$$

Therefore the maximum velocity ($V_{\max}$ = assumed c) at the center point c' ($\zeta = 0$) is

$$V_{\max} = k + \pi k \frac{y}{T} K\left(\sqrt{1-\alpha}\right) \Big/ \int_0^{\infty}\left(\tau\Big/\sqrt{\alpha+\sinh^2\tau}\right) d\tau \quad (28)$$

On substituting y from equation (23) in equation (28), it becomes identical to equation (22a).

[13] Adopting $\zeta = \alpha$ and $dZ/dW = -i/k'$ in equation (19) to get k' or seepage velocity at the point f' as

$$\frac{k}{k'} = \frac{TK(\sqrt{\alpha})}{2dK\left(\sqrt{1-\alpha}\right)}\left(1 + \frac{2\sin^{-1}\sqrt{\alpha}}{\pi}\right) - \frac{2\sin^{-1}\sqrt{\alpha}}{\pi} \quad (29)$$

Using $\beta = 1$ in equation (15) gives the quantity of seepage:

$$q_s = 2kd\frac{K\left(\sqrt{1-\alpha}\right)}{K(\sqrt{\alpha})} \quad (30)$$

Therefore the seepage function from equations (23) and (30) is

$$F_s = \frac{T}{y} + \pi K\left(\sqrt{1-\alpha}\right) \Big/ \int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \quad (31)$$

and Vedernikov's parameter is

$$A = F_s - \frac{T}{y} = \pi K\left(\sqrt{1-\alpha}\right) \Big/ \int_0^{\infty} \frac{\tau\, d\tau}{\sqrt{\alpha+\sinh^2\tau}} \quad (32)$$

2.8. Special Case: Condition A′ With Infinite Depth of Drainage Layer and Water Table

[14] When both the depth of the drainage layer and the position of water table lie at very large depth ($d \to \infty$ and $d_w \to \infty$), then $\alpha = \beta = 1$ and the above expressions reduce to the solution obtained by *Chahar* [2006] for homogeneous and isotropic porous medium of infinite extent.

2.9. Example

[15] Consider a curved channel having $T = 2y$; $d = 2y$ and $d_w = 1.5y$. The determination of the accessory parameters α and β consists of maximization of kk'' subject to equation (18). This constrained optimization problem has been converted into unconstrained minimization problem by penalty function to have augmented function as

$$F(\alpha, \beta) = -\left(\frac{4yK\left(\sqrt{\alpha/\beta}\right)}{d_w\sqrt{\beta}}\right)\left(\ln\left(\frac{1+\sqrt{1-\beta}}{\sqrt{\beta}}\right) + \frac{\sqrt{1-\beta}}{\beta}\right) \Bigg/ \int_0^{\infty} F_4(\tau,\alpha,\beta)\, d\tau + \lambda\, abs \cdot \left(1 - \frac{d}{y} + \frac{\int_0^{\alpha} F_6(\tau,\alpha,\beta) d\tau}{\int_0^{\infty} F_4(\tau,\alpha,\beta)\, d\tau} + \frac{T}{y}\frac{K\left(\sqrt{\alpha/\beta}\right)}{2K\left(\sqrt{(\beta-\alpha)\beta}\right)} + 1\right) \quad (33)$$

where λ = penalty function. Minimization function *fminsearch* of MATLAB [*The MathWorks*, 2007] has been used to minimize equation (33) with respect to α and β for $T/y = 2$; $d/y = 2$; $d_w/y = 1.5$; and $\lambda = 10$. The resultant values have been $\alpha = 0.0179$ and $\beta = 0.1388$. With these values of accessory parameters in equations (5), (7), (12), (10a), and (15) give $c = 2.2689k$; $k' = 1.3618k$; $k'' = 0.8874/k$; depth

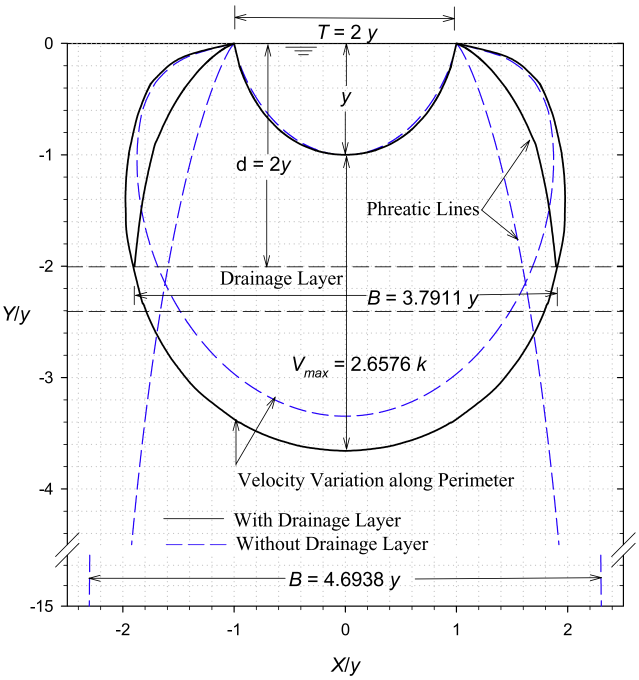

Figure 3. Comparison of channels with and without the drainage layer.

of inflection point = 0.3534y; and seepage discharge = 4.5378ky, respectively.

[16] If $d < d_w$ then solving equation (23) with *fzero* function of MATLAB [*The MathWorks*, 2007], the accessory parameter = 0.2182, therefore equations (26), (28), and (30) yield: $B = 3.7911y$; $V_{\max} = 2.6576\,k$; and $q_s = 5.3152ky$, respectively. Thus the seepage is 17.13% higher than the previous case.

[17] If both the drainage layer and water table were at infinite depth then $B = 4.6938y$; $V_{\max} = 2.3469k$; and $q_s = 4.6938ky$. Figure 3 shows a comparison between the last two cases; that is, the water table is below the top of the drainage layer ($d < d_w$) and both the drainage layer and water table are at infinite depth ($d \to \infty$ and $d_w \to \infty$). It can be observed that the channel without drainage layer is inscribed in the corresponding channel with drainage layer, this is true for all T/y and d/y ratios. The same thing is applicable for the phreatic lines, and the velocity distribution curves. Though the perimeters do not differ very much but the phreatic lines and the velocity curves do. The differences increase with increase in T/y or decrease in d/y. For same T/y and d/y ratios, $q_s = 6.9702ky$ and $B = 4.4410y$ in a rectangular channel, $q_s = 5.5028ky$ and $B = 3.7884y$ in a trapezoidal channel and $q_s = 4.2593ky$ and $B = 3.3126y$ in a triangular channel [*Chahar*, 2007a]. The comparison shows that the seepage from the present curved channel is higher (19.87%) than from the inscribing triangular channel and less (31.14%) than from the comprising rectangular channel; thus it complies with the comparison theorem [*Kacimov*, 2003].

3. Conclusions

[18] An exact analytical solution for the quantity of seepage from a curved channel passing through a porous medium underlain by a drainage layer at shallow depth whose boundary maps along a circle onto the hodograph plane can be obtained using an inverse method along with inverse hodograph and Schwarz-Christoffel transformation. The exact analytical expressions for the quantity of seepage, the shape of the channel contour, the variation in the seepage velocity normal to the channel contour, and loci of the phreatic lines involve improper integrals along with the accessory parameters. The quantity of seepage from a channel whose boundary maps along a circle onto the hodograph plane is simply top width times the seepage velocity at the center of the channel irrespective of it passes through a homogeneous medium of infinite depth (no drainage layer case) or it underlain by a drainage layer with water table above or below the top of the drainage layer.

Appendix A: Mapping Details

A1. Mapping of Inverse Hodograph Plane

[19] Mapping of dZ/dW plane on the ζ plane results in

$$\frac{dZ}{dW} = C_1 \int_0^{\zeta} \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} + C_2 \tag{A1}$$

where C_1 and C_2 = constants. Using the values at point c' ($\zeta = 0$; $dZ/dW = -i/c$) gives $C_2 = -i/c$ and then at the point f' ($\zeta = \alpha$; $dZ/dW = -i/k'$) yields

$$C_1 = i\left(\frac{1}{c} - \frac{1}{k'}\right) \Big/ \int_0^{\alpha} \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} = \left(\frac{c-k}{ck'}\right) \Big/ F_1(\alpha,\beta) \tag{A2}$$

Substitution of C_1 and C_2 in equation (A1) gives equation (1), where for the range $0 < \zeta \le \beta$,

$$\int_0^{\zeta} \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} = -i \int_0^{\zeta} \frac{(1-t)dt}{\sqrt{t}\,(\beta-t)^{1.5}} = -iF_1(\zeta,\beta) = -i2\left(\sin^{-1}\sqrt{\frac{\zeta}{\beta}} + \frac{1-\beta}{\beta}\sqrt{\frac{\zeta}{\beta-\zeta}}\right) \tag{A3}$$

$$\frac{dZ}{dW} = -i\left(\frac{c-k}{ck'}\right)\frac{F_1(\zeta,\beta)}{F_1(\alpha,\beta)} - \frac{i}{c} \tag{A4}$$

for the range $-\infty < \zeta \le 0$,

$$\int_0^{\zeta} \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} = \int_0^{-\zeta} \frac{(1+t)dt}{\sqrt{t}\,(t+\beta)^{1.5}} = F_2(\zeta,\beta) = 2\ln\left(\sqrt{\frac{-\zeta}{\beta}} + \sqrt{\frac{\beta-\zeta}{\beta}}\right) + 2\frac{1-\beta}{\beta}\sqrt{\frac{-\zeta}{\beta-\zeta}} \tag{A5}$$

$$\frac{dZ}{dW} = \left(\frac{c-k'}{ck'}\right)\frac{F_2(\zeta,\beta)}{F_1(\alpha,\beta)} - \frac{i}{c} \tag{A6}$$

and otherwise,

$$\int_\beta^\zeta \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} = F_3(\zeta,\beta) = 2\ln\left(\sqrt{\frac{\zeta}{\beta}} + \sqrt{\frac{\zeta-\beta}{\beta}}\right) + 2\frac{1-\beta}{\beta}\sqrt{\frac{\zeta}{\zeta-\beta}} \quad \text{(A7)}$$

$$\frac{dZ}{dW} = \left(\frac{c-k'}{ck'}\right)\frac{F_3(\zeta,\beta)}{F_1(\alpha,\beta)} - \frac{i}{k} \quad \text{(A8)}$$

Therefore

$$F_1(\alpha,\beta) = \int_0^\alpha \frac{(1-t)dt}{\sqrt{t}\,(\beta-t)^{1.5}} = 2\left(\sin^{-1}\sqrt{\frac{\alpha}{\beta}} + \frac{1-\beta}{\beta}\sqrt{\frac{\alpha}{\beta-\alpha}}\right) \quad \text{(A9)}$$

The branch cuts in equations (A2)–(A9) are selected such that both $\sqrt{t}$ and $\sqrt{t-\beta}$ are negative.

A2. Mapping of Complex Potential Plane

[20] The W plane mapping on the ζ plane is

$$W = C_3\int_0^\zeta \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} + C_4 \quad \text{(A10)}$$

Using the values of W and ζ at points c' ($\zeta = 0$; $W = 0$) and f' ($\zeta = \alpha$; $W = kd_w$), the constants $C_4 = 0$ and

$$C_3 = kd_w \Big/ \int_0^\alpha \frac{dt}{\sqrt{t(\alpha-t)(\beta-t)}} = 0.5kd_w\sqrt{\beta}/K\left(\sqrt{\alpha/\beta}\right) \quad \text{(A11)}$$

where

$$\int_0^\alpha \frac{dt}{\sqrt{t(\alpha-t)(\beta-t)}} = \frac{2}{\sqrt{\beta}}K\left(\sqrt{\alpha/\beta}\right) \quad \text{(A12)}$$

The branch cuts are selected such that $\sqrt{t-\alpha}$ is positive while both $\sqrt{t}$, and $\sqrt{t-\beta}$ are negative as before. After substituting C_3 and C_4, equation (A10) results in equation (2). Differentiating equation (2) with respect to ζ gives

$$\frac{dW}{d\zeta} = \frac{kd_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\frac{1}{\sqrt{\zeta(\zeta-\alpha)(\zeta-\beta)}} \quad \text{(A13)}$$

A3. Mapping of Physical Plane

[21] Since

$$\frac{dZ}{d\zeta} = \frac{dZ}{dW}\frac{dW}{d\zeta} \quad \text{(A14)}$$

Substitution of dZ/dW from equation (1) and $dW/d\zeta$ from equation (A13) results in

$$\frac{dZ}{d\zeta} = \frac{kd_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\frac{1}{\sqrt{\zeta(\zeta-\alpha)(\zeta-\beta)}} \cdot \left(\frac{c-k'}{ck'F_1(\alpha,\beta)}\int_0^\zeta \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}} - \frac{i}{c}\right) \quad \text{(A15)}$$

Integrating equation (A15) along the channel contour $c'b'$ ($-\infty < \zeta \le 0$), gives equation (3) wherein

$$\int_0^\zeta\left(\int_0^\tau \frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}}\frac{1}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}}\right)d\tau = \int_0^{-\zeta}\left(\int_0^{-\tau} \frac{(1+t)dt}{\sqrt{t}\,(t+\beta)^{1.5}}\frac{1}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}}\right)d\tau \quad \text{(A16)}$$

$$F_4(\tau,\alpha,\beta) = \int_0^{-\tau} \frac{(1+t)dt}{\sqrt{t}\,(t+\beta)^{1.5}}\frac{1}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}} = \frac{F_2(\tau,\beta)}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}} \quad \text{(A17)}$$

Equation (3) is the mapping along the channel contour $c'b'$. The real part of it is

$$X = \frac{kd_w\sqrt{\beta}}{2cK\left(\sqrt{\alpha/\beta}\right)}\int_0^{-\zeta}\frac{d\tau}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}} \quad \text{(A18)}$$

where

$$\int_0^{-\zeta}\frac{dt}{\sqrt{t(t+\alpha)(t+\beta)}} = \frac{2}{\sqrt{\beta}}F\left(\sin^{-1}\sqrt{\frac{-\zeta}{-\zeta+\alpha}}, \sqrt{\frac{\beta-\alpha}{\beta}}\right) \quad \text{(A19)}$$

Plugging equation (A19) in equation (A18) results in equation (4a). At the point b' ($\zeta = -\infty$; $Z = T/2$) equations (4a) and (4b) give

$$\frac{T}{2} = \frac{kd_w\sqrt{\beta}}{2cK\left(\sqrt{\alpha/\beta}\right)}\int_0^\infty\frac{d\tau}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}} = \frac{kd_wK\left(\sqrt{(\beta-\alpha)/\beta}\right)}{cK\left(\sqrt{\alpha/\beta}\right)} \quad \text{(A20)}$$

$$y = \frac{k(c-k')d_w\sqrt{\beta}}{2ck'F_1(\alpha,\beta)K\left(\sqrt{\alpha/\beta}\right)}\int_0^\infty F_4(\tau,\alpha,\beta)\,d\tau \quad \text{(A21)}$$

respectively, where

$$\int_0^\infty\frac{d\tau}{\sqrt{\tau(\tau+\alpha)(\tau+\beta)}} = \frac{2}{\sqrt{\beta}}K\left(\sqrt{(\beta-\alpha)/\beta}\right) \quad \text{(A22)}$$

Manipulation of equations (A20) and (A21) yields equations (5), (6), and (7).

[22] Physical plane equation along the centerline $c'f'$ ($0 \le \zeta \le \alpha$) converts to

$$Z = \frac{-i\,kd_w\sqrt{\beta}}{2cK(\sqrt{\alpha/\beta})}\left(\frac{c-k'}{k'F_1(\alpha,\beta)}\int_0^{\zeta} F_6(\tau,\alpha,\beta)d\tau + \int_0^{\zeta}\frac{d\tau}{\sqrt{\tau(\alpha-\tau)(\beta-\tau)}}\right) - iy \quad \text{(A23)}$$

where

$$F_6(\tau,\alpha,\beta) = \int_0^{\tau}\frac{(1-t)dt}{\sqrt{t}\,(\beta-t)^{1.5}}\frac{1}{\sqrt{\tau(\alpha-\tau)(\beta-\tau)}} = \frac{F_1(\tau,\beta)}{\sqrt{\tau(\alpha-\tau)(\beta-\tau)}} \quad \text{(A24)}$$

Equation (A23) at the point f' ($\zeta = \alpha$; $Z = -id$) gives equation (18).

A4. Position of Phreatic Line and Inflection Point

[23] For the phreatic line segment $b'a'a''$ ($\beta \le \zeta \le \infty$), the Z plane mapping equation (3) changes to

$$Z = \frac{T}{2} + \frac{kd_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\left(\frac{(c-k')}{ck'F_1(\alpha,\beta)}\int_{\zeta}^{\infty} F_5(\tau,\alpha,\beta)d\tau - \frac{i}{k}\int_{\zeta}^{\infty}\frac{d\tau}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}}\right) \quad \text{(A25)}$$

where

$$F_5(\tau,\alpha,\beta) = \int_{\beta}^{\tau}\frac{(t-1)dt}{\sqrt{t}\,(t-\beta)^{1.5}}\frac{1}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}} = \frac{F_3(\tau,\beta)}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}} \quad \text{(A26)}$$

Separating real and imaginary parts of equation (A25) along with values of c and k' results in equations (9a) and (9b) where

$$\int_{\zeta}^{\infty}\frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} = \frac{2}{\sqrt{\beta}}F\left(\sin^{-1}\sqrt{\beta/\zeta}, \sqrt{\alpha/\beta}\right) \quad \text{(A27)}$$

A5. Inflection Point

[24] The phreatic line equation (9b) at the inflection point ($\zeta = 1$) yields equation (10a) where

$$\int_{1}^{\infty}\frac{d\tau}{\sqrt{\tau(\tau-\alpha)(\tau-\beta)}} = \frac{2}{\sqrt{\beta}}F\left(\sin^{-1}\sqrt{\beta}, \sqrt{\alpha/\beta}\right) \quad \text{(A28)}$$

In the hodograph plane, the slit $a''a'b'$ at point a' on the circumference of the circle of diameter k is a consequence of the velocity at the point of inflection representing the maximum velocity along the phreatic line, thus the slope at the inflection point is a maximum. The slope at any point on the phreatic line can be obtained as

$$\frac{dY}{dX} = \frac{d\phi/dX}{d\phi/dY} = \frac{u}{v} = \frac{u/(u^2+v^2)}{v/(u^2+v^2)} = k\frac{u}{u^2+v^2} \quad \text{(A29)}$$

wherein $u^2 + v^2 - kv = 0$ along the phreatic line [*Harr*, 1962]. In the inverse hodograph plane the abscissa axis is $u/u^2 + v^2$) and the ordinate axis is $v/(u^2 + v^2)$. It is evident from equation (A29) that the slope at any point on the phreatic line is simply k times the abscissa value at that point in the inverse hodograph plane, therefore the slope at the inflection point is

$$\left(\frac{dY}{dX}\right)_{\text{inflection Point}} = kk'' \quad \text{(A30)}$$

where k'' = location of the inflection point a' along the abscissa in the inverse hodograph plane (Figure 1c). Since the slope at the inflection point is a maximum, kk'' should be maximum.

A6. Variation in Seepage Velocity

[25] The distribution of the velocity of seeping water normal to the channel contour can be found by using equation (1) along $c'b'$ ($-\infty < \zeta \le 0$) as

$$\frac{dZ}{dW} = \frac{1}{u-iv} = \left(\frac{c-k'}{ck'}\right)\frac{F_2(\zeta,\beta)}{F_1(\alpha,\beta)} - \frac{i}{c} \quad \text{(A31)}$$

Substituting value of c and k' and then manipulating yields

$$\frac{u+iv}{u^2+v^2} = \frac{2yK\left(\sqrt{\alpha/\beta}\right)}{kd_w\sqrt{\beta}}\frac{F_2(\tau,\beta)}{\int_0^{\infty}F_4(\tau,\alpha,\beta)\,d\tau} - i\frac{T}{2kd_w}\frac{K\left(\sqrt{\alpha/\beta}\right)}{K\left(\sqrt{(\beta-\alpha)/\beta}\right)} \quad \text{(A32)}$$

Separating real and imaginary parts and then squaring and adding give equation (13).

A7. Quantity of Seepage

[26] Equation (2) at the point a'' ($\zeta = \beta$; $W = kd_w + iq_s/2$) generates

$$kd_w + \frac{q_s}{2}i = \frac{kd_w\sqrt{\beta}}{2K\left(\sqrt{\alpha/\beta}\right)}\int_0^{\beta}\frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} = kd_w + \frac{ikd_wK\left(\sqrt{(\beta-\alpha)/\beta}\right)}{K\left(\sqrt{\alpha/\beta}\right)} \quad \text{(A33)}$$

because

$$\int_0^\beta \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} = \int_0^\alpha \frac{dt}{\sqrt{t(\alpha-t)(\beta-t)}} + i\int_\alpha^\beta \frac{dt}{\sqrt{t(t-\alpha)(\beta-t)}} \tag{A34}$$

$$\int_\alpha^\beta \frac{dt}{\sqrt{t(t-\alpha)(\beta-t)}} = \frac{2}{\sqrt{\beta}}K\left(\sqrt{(\beta-\alpha)/\beta}\right) \tag{A35}$$

Equating imaginary parts in equation (A33) leads to equation (15). The real part of equation (2) at the point a' ($\zeta = 1$; $W = kd_i + iq_s/2$) yields equation (10b) wherein

$$\int_0^1 \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} = \int_0^\alpha \frac{dt}{\sqrt{t(\alpha-t)(\beta-t)}} + i\int_\alpha^\beta \frac{dt}{\sqrt{t(t-\alpha)(\beta-t)}} + \int_\beta^1 \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} \tag{A36}$$

$$\int_\beta^1 \frac{dt}{\sqrt{t(t-\alpha)(t-\beta)}} = \frac{2}{\sqrt{\beta}}F\left(\sin^{-1}\sqrt{(1-\beta)/(1-\alpha)}, \sqrt{\alpha/\beta}\right) \tag{A37}$$

A8. Special Case: Condition A′

[27] When the water table is below the top of the drainage layer ($d_w \geq d$) then there will be no inflection point on the phreatic line (Figure 2). The point a'' vanishes after approaching to a' in various planes, which result to $\beta = 1$. With $\beta = 1$, the mappings in various planes onto ζ plane convert into equations (19)–(21). Along the channel contour $c'b'$ ($-\infty < \zeta \leq 0$), equation (21) becomes

$$Z = \frac{kd}{2K(\sqrt{\alpha})}\left(\frac{1}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\int_0^{-\zeta}\int_0^t \frac{d\tau}{\sqrt{\tau(\tau+1)}}\frac{-i\,dt}{\sqrt{t(t+\alpha)(t+1)}} - \frac{i}{c}\int_0^{-\zeta}\frac{i\,dt}{\sqrt{t(t+\alpha)(t+1)}}\right) - iy \tag{A38}$$

Sorting out real and imaginary parts gives

$$X = \frac{kd}{2cK(\sqrt{\alpha})}\int_0^{-\zeta}\frac{dt}{\sqrt{t(t+\alpha)(t+1)}} = \frac{kd}{cK(\sqrt{\alpha})}F\left(\sin^{-1}\sqrt{\frac{-\zeta}{-\zeta+\alpha}}, \sqrt{1-\alpha}\right) \tag{A39}$$

$$Y = \frac{kd}{2\pi K(\sqrt{\alpha})}\left(\frac{1}{k}-\frac{1}{c}\right)\int_0^{-\zeta}\int_0^t \frac{d\tau}{\sqrt{\tau(\tau+1)}}\frac{dt}{\sqrt{t(t+\alpha)(t+1)}} - y \tag{A40}$$

where

$$\int_0^{-\zeta}\frac{dt}{\sqrt{t(t+\alpha)(t+1)}} = 2F\left(\sin^{-1}\sqrt{\frac{-\zeta}{-\zeta+\alpha}}, \sqrt{1-\alpha}\right) \tag{A41}$$

Equations (A39) and (A40) at the point b' ($\zeta = -\infty$; $Z = T/2$) give equations (22a), (22b), and (23), where

$$\int_0^\infty \frac{dt}{\sqrt{t(t+\alpha)(t+1)}} = 2K\left(\sqrt{1-\alpha}\right) \tag{A42}$$

$$\int_0^\infty\int_0^t \frac{d\tau}{\sqrt{\tau(\tau+1)}}\frac{dt}{\sqrt{t(t+\alpha)(t+1)}} = 4\int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}} \tag{A43}$$

Equations (22a) and (A38) produce

$$Z = \frac{T}{4K\left(\sqrt{1-\alpha}\right)}\int_0^{-\zeta}\frac{dt}{\sqrt{t(t+\alpha)(t+1)}} + i\frac{y}{2}\int_0^{-\zeta}\frac{\sinh^{-1}\sqrt{t}\,dt}{\sqrt{t(t+\alpha)(t+1)}}\Big/\int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}} - iy \tag{A44}$$

Separating real and imaginary parts and simplifying give equations (24a) and (24b).

[28] An alternate relation for the accessory parameter can be obtained by using physical plane equation along the centerline $c'f'$ ($0 \leq \zeta \leq \alpha$) as

$$Z = \frac{kd}{2K(\sqrt{\alpha})}\left(\frac{-2i}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\int_0^\zeta \frac{\sin^{-1}\sqrt{t}\,dt}{\sqrt{t(\alpha-t)(1-t)}} - \frac{i}{c}\int_0^\zeta \frac{dt}{\sqrt{t(\alpha-t)(1-t)}}\right) - iy \tag{A45}$$

Combining equations (22a) and (22b) in equation (A45) at the point f' ($\zeta = \alpha$; $Z = -id$) yields

$$\frac{2}{K(\sqrt{\alpha})}\frac{d}{y} - \frac{1}{K\left(\sqrt{1-\alpha}\right)}\frac{T}{y} = \frac{2}{K(\sqrt{\alpha})}\left(1 + \int_0^{\sin^{-1}\sqrt{\alpha}}\frac{\tau\,d\tau}{\sqrt{\alpha-\sin^2\tau}}\Big/\int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}}\right) \tag{A46}$$

In lieu of equation (23), equation (A46) can be used to find the accessory parameter α for specified d/y and T/y.

Comparing equations (23) and (A46) leads to the following identity:

$$\int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}} + \int_0^{\sin^{-1}\sqrt{\alpha}} \frac{\tau\,d\tau}{\sqrt{\alpha-\sin^2\tau}} = K\left(\sqrt{\alpha}\right)\frac{\pi}{2} \qquad \text{(A47)}$$

For the phreatic line segment $a'b'$ ($1 \le \zeta < \infty$), the Z plane mapping equation (21) changes to

$$Z = \frac{kd}{2K(\sqrt{\alpha})}\int_\infty^\zeta \left(\frac{2}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\cosh^{-1}\sqrt{t} - \frac{i}{k}\right)\frac{dt}{\sqrt{t(t-\alpha)(t-1)}} + \frac{T}{2} \qquad \text{(A48)}$$

Using equation (22a) in equation (A48) and manipulating capitulate

$$Z = y\int_{\cosh^{-1}\sqrt{\zeta}}^\infty \frac{\tau\,d\tau}{\sqrt{\cosh^2\tau-\alpha}} \Bigg/ \int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}} - \frac{i\,d}{2K(\sqrt{\alpha})}\int_\zeta^\infty \frac{dt}{\sqrt{t(t-\alpha)(t-1)}} + \frac{T}{2} \qquad \text{(A49)}$$

Equating real and imaginary parts gives equations (25a) and (25b) where

$$\int_\zeta^\infty \frac{dt}{\sqrt{t(t-\alpha)(t-1)}} = 2F\left(\sin^{-1}\sqrt{1/\zeta},\,\sqrt{\alpha}\right) \qquad \text{(A50)}$$

Another relation for the seepage width B (m) at the drainage layer can be obtained using physical plane mapping along the drainage layer $f'a'$ ($\alpha \le \zeta \le 1$) as

$$Z = \frac{kd}{2K(\sqrt{\alpha})}\left(\frac{2}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\int_\alpha^\zeta \frac{\sin^{-1}\sqrt{t}\,dt}{\sqrt{t(t-\alpha)(1-t)}} + \frac{1}{c}\int_\alpha^\zeta \frac{dt}{\sqrt{t(t-\alpha)(1-t)}}\right) - i\,d \qquad \text{(A51)}$$

and then at point a' ($\zeta = 1$; $Z = -id + B/2$) as

$$\frac{B}{y} = \frac{T}{y} + 2\int_{\sin^{-1}\sqrt{\alpha}}^{\pi/2} \frac{\tau\,d\tau}{\sqrt{\sin^2\tau-\alpha}} \Bigg/ \int_0^\infty \frac{\tau\,d\tau}{\sqrt{\alpha+\sinh^2\tau}} \qquad \text{(A52)}$$

Rewriting equation (19) for the velocity of seeping water normal to the channel contour $c'b'$ ($-\infty < \zeta \le 0$) as

$$\frac{1}{u-iv} = \frac{2}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\sinh^{-1}\sqrt{-\zeta} - \frac{i}{c} \qquad \text{(A53)}$$

Substituting value of c and manipulating gives equation (27).

A9. Special Case: Condition A′ With Infinite Depth of Drainage Layer and Water Table

[29] If the porous medium is homogeneous and isotropic of infinite extent then both the depth of the drainage layer and the position of water table lie at very large depth ($d \to \infty$ and $d_w \to \infty$). For $d \to \infty$ and $d_w \to \infty$; $\alpha = \beta = 1$ and the mappings onto ζ plane result in [*Chahar*, 2006]

$$\frac{dZ}{dW} = \frac{1}{\pi}\left(\frac{1}{k}-\frac{1}{c}\right)\int_0^\zeta \frac{dt}{\sqrt{t(t-1)}} - \frac{i}{c} \qquad \text{(A54)}$$

$$W = \frac{q_s}{2\pi}\int_0^\zeta \frac{dt}{(t-1)\sqrt{t}} = \frac{q_s}{2\pi}\ln\left|\frac{\sqrt{\zeta}-1}{\sqrt{\zeta}+1}\right| \qquad \text{(A55)}$$

$$Z = \frac{q_s}{\pi c}\tan^{-1}\sqrt{-\zeta} - i\left(y - \frac{2q_s}{\pi^2}\left(\frac{1}{k}-\frac{1}{c}\right)\int_0^{\sinh^{-1}\sqrt{-\zeta}} \frac{\tau\,d\tau}{\cosh\tau}\right) \qquad \text{(A56)}$$

Using equation (A54) at the point b' ($\zeta = -\infty$; $Z = T/2$) and simplifying generate

$$c = \frac{k}{T}\left(T + \frac{\pi^2}{4G}y\right) \qquad \text{(A57)}$$

$$q_s = ky\left(\frac{\pi^2}{4G} + \frac{T}{y}\right) \qquad \text{(A58)}$$

where $G = 0.915965594\ldots =$ Catalan's constant. The equation for the shape of the channel contour is

$$\frac{Y}{y} = \frac{1}{2G}\left(\int_0^{\sinh^{-1}\tan(\pi X/T)} \frac{\tau\,d\tau}{\cosh\tau} - 2G\right) \qquad \text{(A59)}$$

The parametric equations for the phreatic line are

$$X = \frac{T}{2} + \frac{y}{2G}\int_{\cosh^{-1}\sqrt{\zeta}}^\infty \frac{\tau\,d\tau}{\sinh\tau} \qquad \text{(A60)}$$

$$Y = \left(T + \frac{\pi^2}{4G}y\right)\ln\tanh\left(\frac{\cosh^{-1}\sqrt{\zeta}}{2}\right) \qquad \text{(A61)}$$

The width of seepage flow at infinity B (m) comes out

$$B = T + \frac{\pi^2}{4G}y \qquad \text{(A62)}$$

Finally the seepage velocity in this condition is given by

$$V = \frac{k(T/y + \pi^2/4G)}{\sqrt{\left(\left(\pi \sinh^{-1} \tan(\pi X/T)\right)/2G\right)^2 + (T/y)^2}} \tag{A63}$$

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