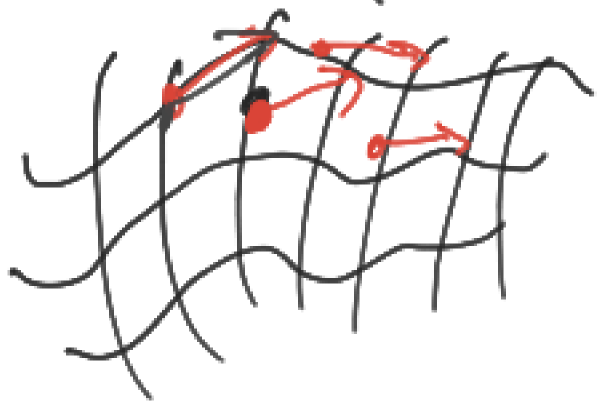
Manifolds

$$h(x, y, z) = 0$$

(x, y, z)



$$x^2 + y^2 + z^2 - 1 = 0$$

$$h_1(x_1, x_2, x_3, \ldots, x_n) = 0$$

$$\vdots$$

$$h_m(x_1, x_2, x_3, \ldots, x_m) = 0$$

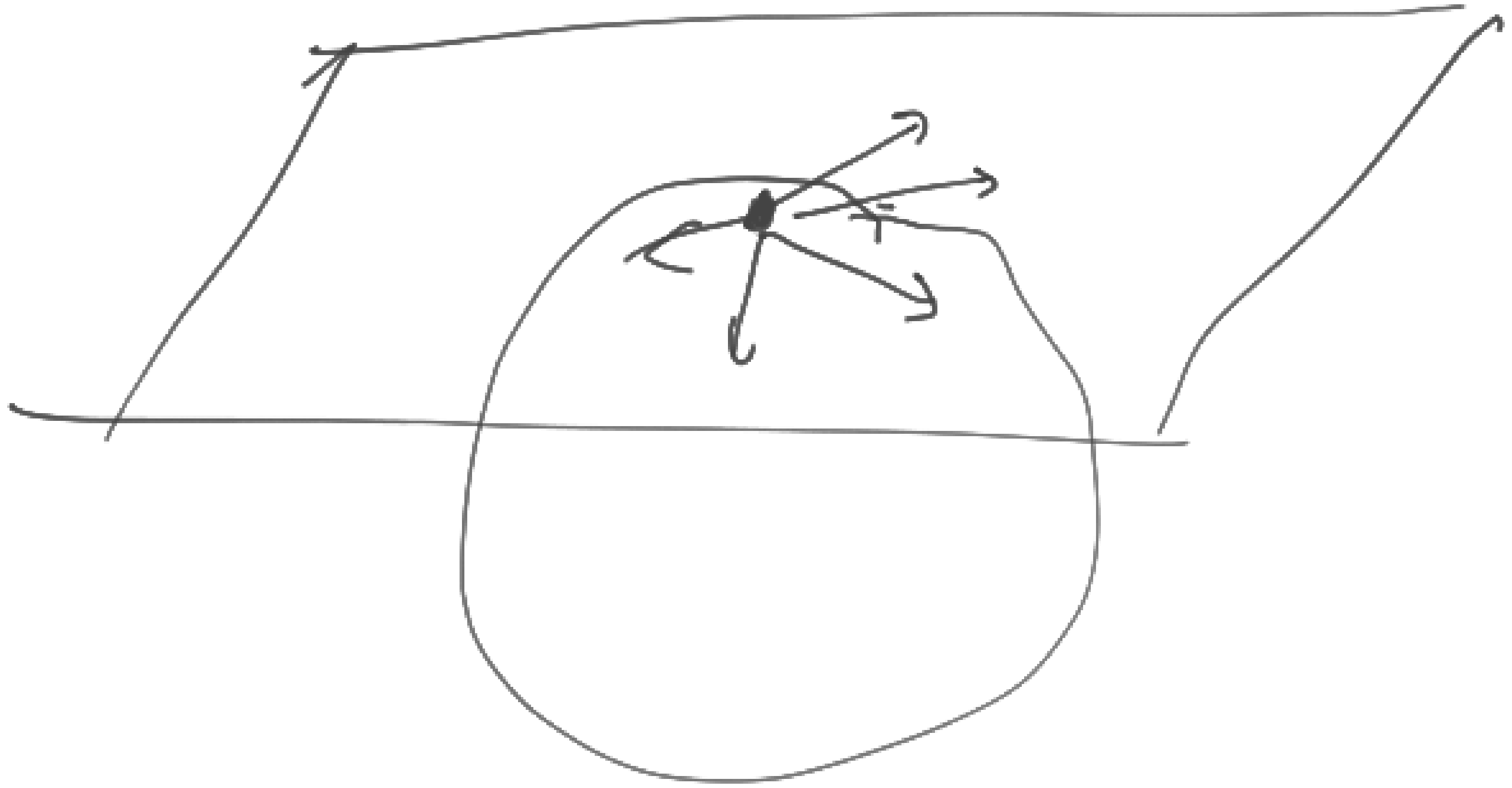
Vector field.

$$f : M \longrightarrow TM$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

$T_x M \rightarrow$ tangent Space at x.

$$\underbrace{\bigcup_{x \in M} T_x M = TM}_{\text{Tangent Bundle}}$$



Distribution.

$$\Delta : \overset{\text{span}}{=} \{ \underline{f_1(x)}, \ \underline{f_2(x)}, \ \dots, \ \underline{f_m(x)} \}$$

$$\dot{x} = \underline{f(x)} + \underline{g(x)}\,u$$

$f(x)$, $\underline{h(x)}$

$$\mathcal{L}_f h(x) = \frac{\partial h}{\partial x} f(x)$$

(Lie Derivative of $h(x)$ w.r.t Vector field $f(x)$)

$$\frac{dh}{dt} = \frac{\partial h}{\partial x} \dot{x} = \frac{\partial h}{\partial x} f(x)$$

$$\dot{x} = f(x)$$

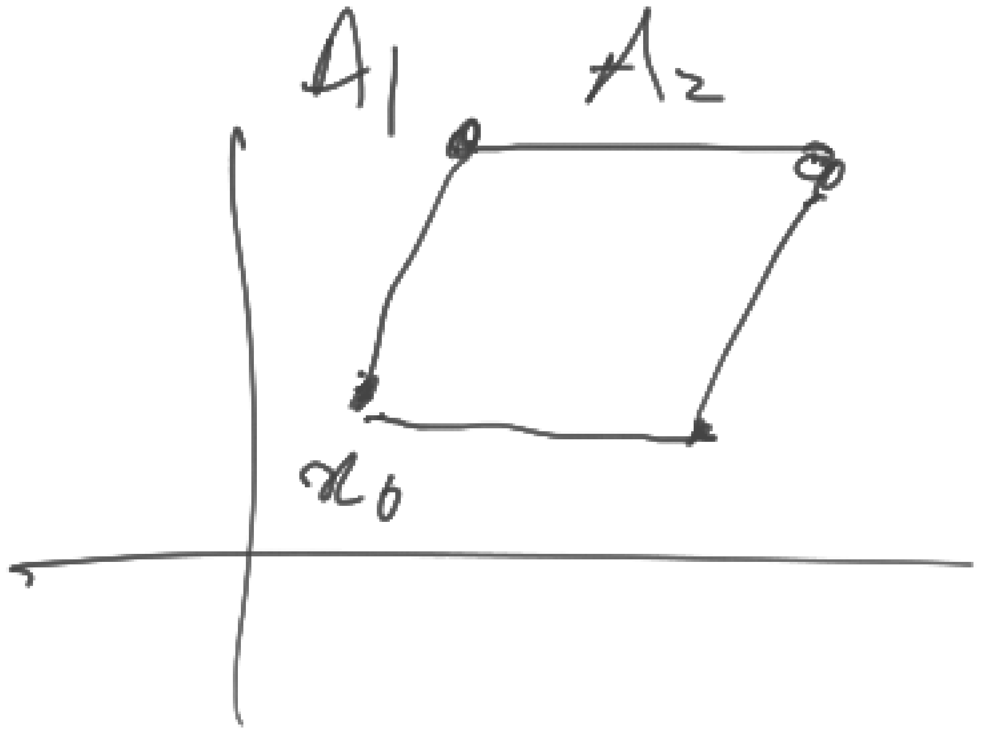
Lie Bracket. $\qquad \dot{x} = \underbrace{(A_1 x)}_{f} u_1 + \underbrace{(A_2 x)}_{g} u_2$

$f, \quad g$

$$\underbrace{[f, g]} := \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

$$= A_2 A_1 x - A_1 A_2 x$$

$$= [A_2 A_1 - A_1 A_2] x$$



$$\Delta := \{ f_1(x,y,z) \quad f_2(x,y,z) \}$$

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2$$

$$\boxed{z = \phi(x,y)}$$

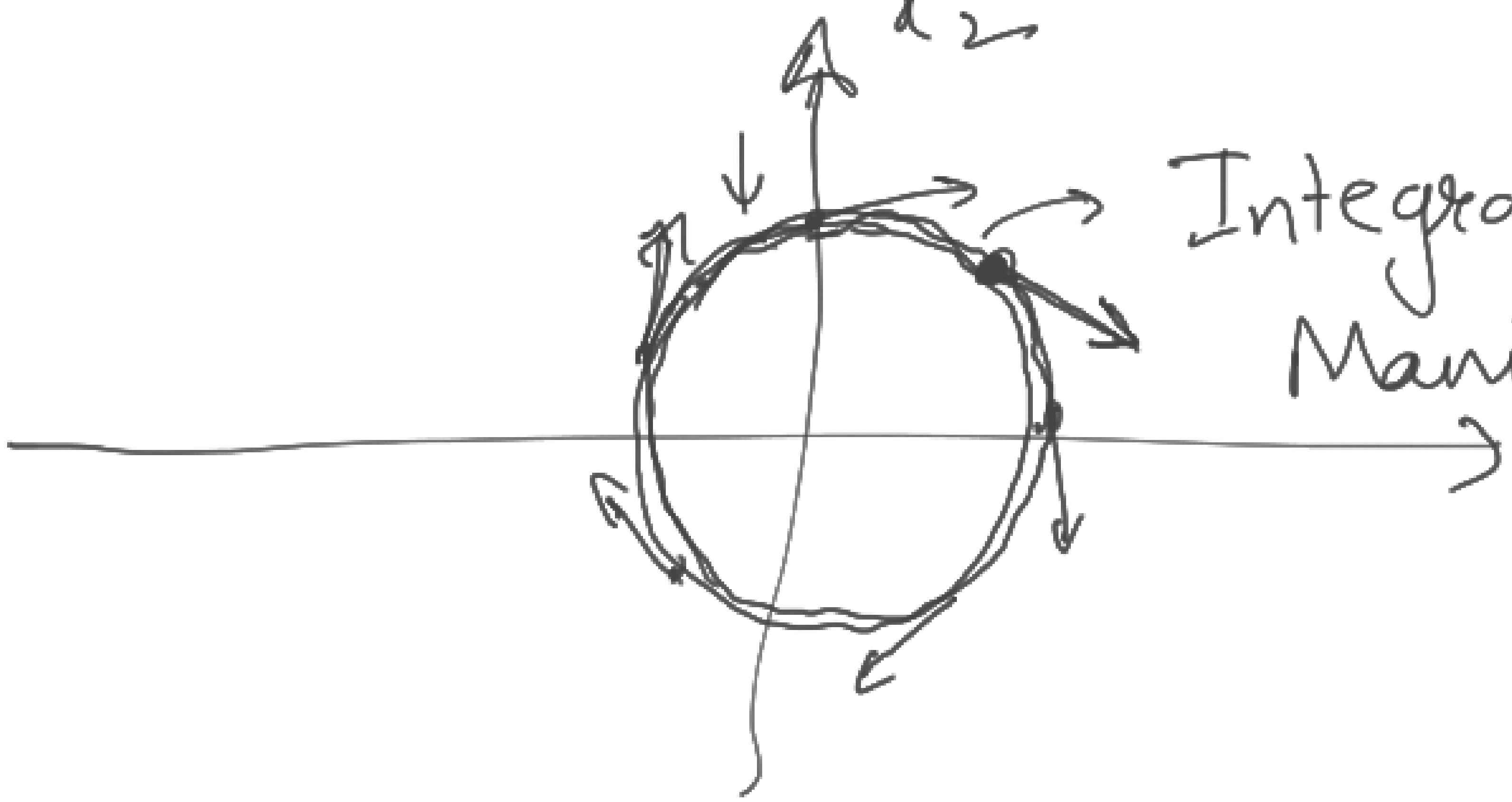
$$\frac{\partial z}{\partial x} = f_1($$

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = +x_1$$

$$\left\{ \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \right\}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Integral Manifold

$$\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Thm: A distribution Δ is integrable if and only if

$$\Delta := \text{span}\{f_1(x), f_2(x), \ldots, f_m(x)\}.$$

$$[f_i, f_j] \in \Delta \text{ for all } i, j \in \{1, 2, \ldots, m\}$$

"Involutive Distribution"

$$z = \phi(x, y)$$

A, B $\quad [f, g] = \underline{ad_f}(g)$

$$[f, [f, g]] = \underline{ad_f^2 (g)}$$

$\vdots$

$$\underbrace{[f, [f, \ldots, [f, g]]]}_{K\text{-times}} = \underline{ad_f^K} (g)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 0 \\ f(x, y, z) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ g(x, y, z) \end{bmatrix} \right\}$$

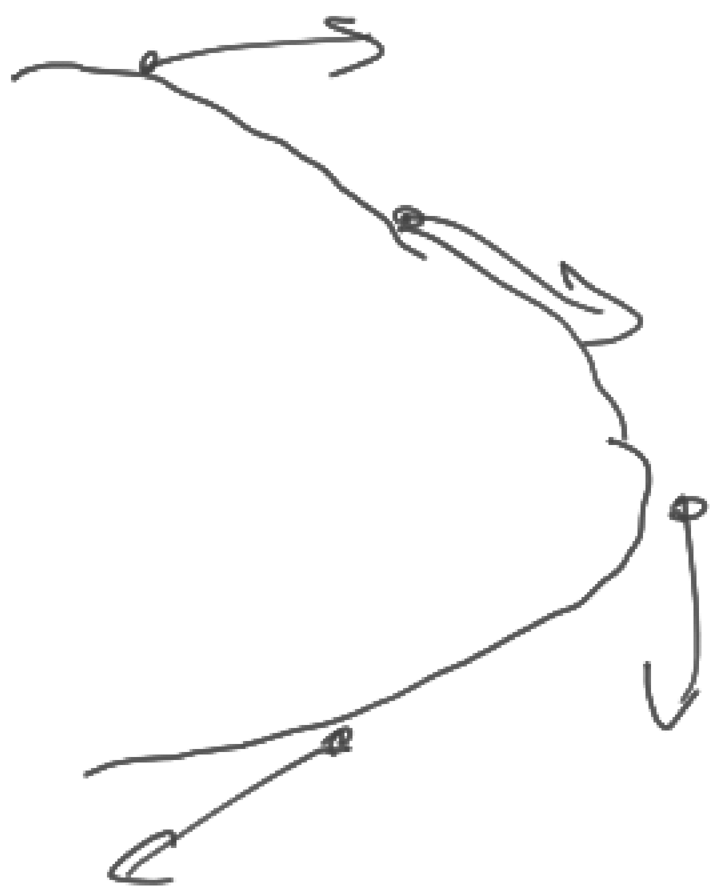
$$\underline{z = \phi(x, y)}$$

$$\begin{bmatrix} c_1 \\ 0 \\ c_1 f \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ c_2 g \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_1 f + c_2 g \end{bmatrix}$$

$$\dot{z} = \frac{\partial \phi}{\partial x} c_1 + \frac{\partial \phi}{\partial y} c_2 = f(x, y, z) c_1 + g(x, y, z) c_2$$

$$\downarrow$$

$$\boxed{\frac{\partial \phi}{\partial x} = f(x, y, \phi), \quad \frac{\partial \phi}{\partial y} = g(x, y, \phi)}$$



Theorem: $\dot{x} = \overset{Ax}{f(x)} + \overset{bu}{g(x)\,u}$, $x \in \mathbb{R}^n$

is feedback linearizable if and only if there exists an open region $U \subset \mathbb{R}^n$

$0 \in U$ s.t.

1.) $\left\{ \overset{b}{g}, \overset{Ab}{ad_f(g)}, \overset{A^2 b}{ad_f^2(g)}, \overset{\cdots}{\ldots}, \overset{A^{n-1} b}{ad_f^{n-1}(g)} \right\}$

are linearly independent in U.

2. The distribution $\Delta = \text{Span}\left\{ \underline{g, ad_f(g), \ldots, ad_f^{n-2}(g)} \right\}$

is involutive.

$$\dot{x} = f(x) + g(x)u$$

$$z = T(x), \qquad u = \alpha(x) + \beta(x)v$$

$$\dot{z} = Az + Bv$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$z = T(x) = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} \quad \dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x} f(x) + \frac{\partial T}{\partial x} g(x)\, u$$

$$= \underline{A}\, z + \underline{B} v = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ v \end{bmatrix}$$

$$L_f T_1 = \frac{\partial T_1}{\partial x} f \qquad L_g T_1 = \frac{\partial T_1}{\partial x} g$$

$$L_f T_1 + \cancel{L_g T_1} u = T_2$$

$$L_f T_2 + \cancel{L_g T_2} u = T_3$$

$$\vdots$$

$$L_f T_n + L_g T_n u = v$$

$$L_f T_1 = T_2$$
$$L_f T_2 = T_3$$
$$\vdots$$
$$L_f T_{n-1} = T_n.$$
$$L_f T_n + L_g T_n u = v$$

$$\left.\begin{aligned} L_g T_1 &= 0 \\ L_g T_2 &= 0 \\ &\vdots \\ L_g T_{n-1} &= 0 \end{aligned}\right\}$$

$$L_g T_1 = 0$$
$$\underline{L_g(L_f T_1) = 0}$$
$$\Downarrow$$
$$L_{[f,g]} T_1 = 0$$

$\Downarrow$ can be shown.

$$L_{[f,[f,[f,\cdots[f,g]]]} T_1 = 0$$

$$L_g T_n = L_g L_f T_{n-1}$$
$$= L_{[f,g]} T_{n-1} - L_{[f,g]} L_f T_{n-2}$$
$$= \cdots$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{MgL}{J_1}\sin x_1 - \frac{K}{J_1}(x_1 - x_3)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{K}{J_2}(x_1 - x_3) + \frac{1}{J_2}u$$

$$= \underbrace{\begin{bmatrix} x_2 \\ -\frac{MgL}{J_1}\sin x_1 - \frac{K}{J_1}(x_1 - x_2) \\ x_4 \\ \frac{K}{J_2}(x_1 - x_3) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J_2 \end{bmatrix}}_{g(x)} u$$

$$\text{rank}\{g,\ ad_f(g),\ ad_f^2(g),\ ad_f^3(g)\} = 4$$

$$\Delta = \text{Span}\{g,\ ad_f(g),\ ad_f^2(g)\}$$

is involutive.

$$L_g T_1 = 0 \qquad L_f T_1 = T_2$$

$$L_g T_2 = 0 \qquad L_f T_2 = T_3$$

$$\vdots$$

$$L_g T_3 = 0 \qquad L_f T_3 = T_4$$

$ad_f\, g = [f, g]$

$ad_f^2\, g = [f, [f, g]]$

$ad_f^3\, g = [f, [f, [f, g]]]$

all are constant vector fields hence.

$$[ad_f^l\, g,\ ad_f^m(g)] = 0$$

Select $z_1 = T_1(x_1, x_2, x_3, x_4] = x_1$

$$\dot{z}_1 = \mathcal{L}_f T_1 = x_2 = T_2 = z_2$$

$$\dot{z}_2 = \mathcal{L}_f T_2 = \dot{x}_2 = -\frac{MgL}{J_1} \sin x_1 - \frac{K}{J_1}(x_1 - x_3) = T_3 = z_3$$

$$\dot{z}_3 = \mathcal{L}_f T_3 = -\frac{MgL}{J_1} \dot{x}_1 \cos x_1 - \frac{K}{J_1}(\dot{x}_1 - \dot{x}_3)$$

$$= -\frac{MgL}{J_1} x_2 \cos x_1 - \frac{K}{J_1}(x_2 - x_4) = T_4 = z_4$$

$$\dot{z}_4 = -\frac{MgL}{J_1}(\dot{x}_2 \cos x_1 - x_2 \dot{x}_1 \sin x_1) - \frac{K}{J_1}(\dot{x}_2 - \dot{x}_4)$$

$$= \alpha(x) + \beta(x) v$$