Lecture 4: Limit Cycles / Periodic orbits

$\underline{\dot{x} = f(x)}$, $\quad x(0) = x_0$

$x(t, x_0) \longrightarrow$ State trajectory / orbit

<u>Defn</u>: [Periodic orbit]

$x(t)$ to $\dot{x} = f(x)$ is periodic if $\exists\, T > 0$

s.t $x(t+T) = x(t) \quad \forall\ t \geq 0$

$T \rightarrow$ Time period.

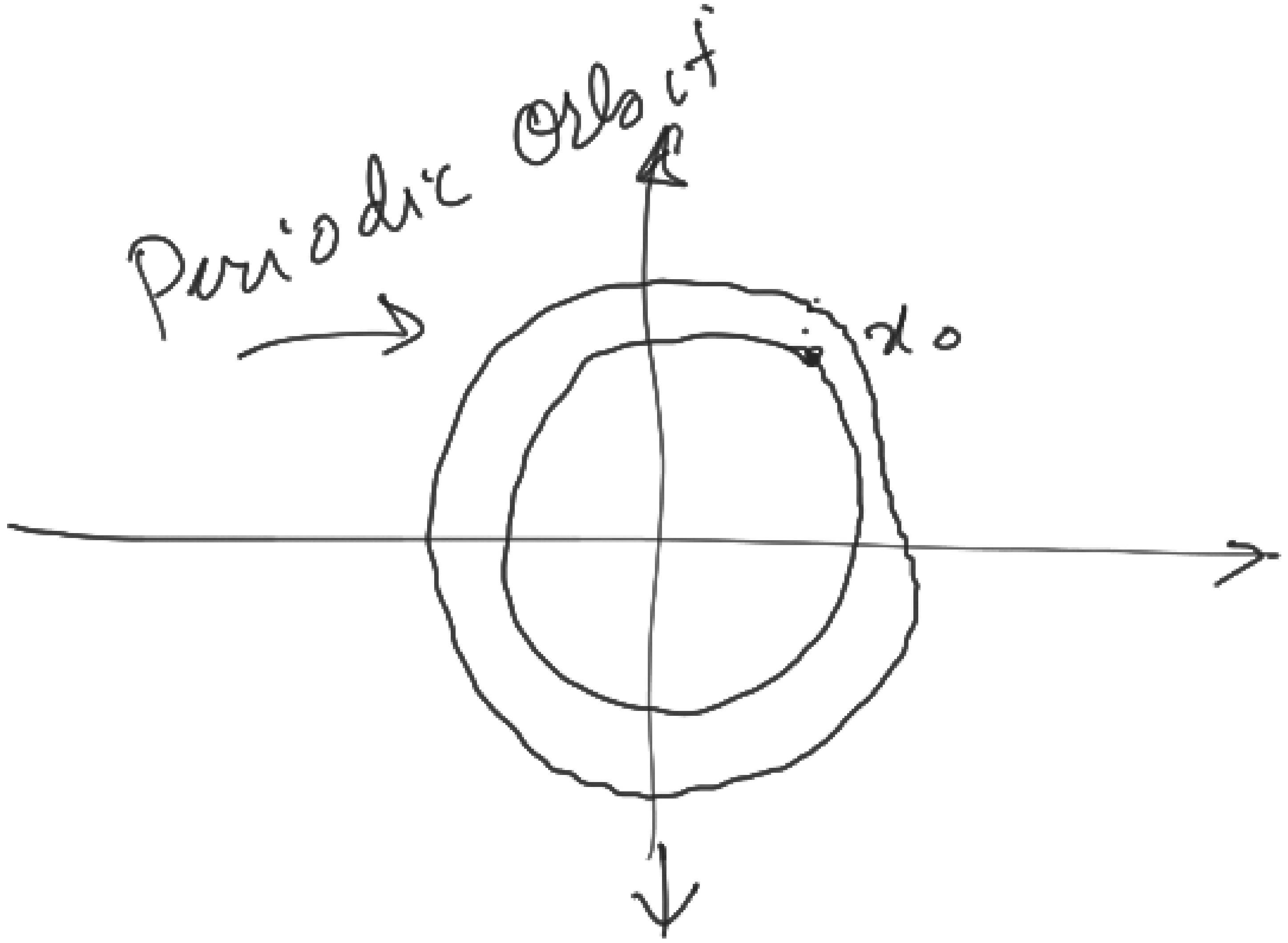
min $T > 0$ s.t $x(t+T) = x(t)$ is "fundamental" period.

Example-

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 \end{aligned} \Bigg\} \longrightarrow \begin{aligned} x_1(t) &= r_0 \cos(t + \phi_0) \\ x_2(t) &= r_0 \sin(t + \phi_0) \end{aligned}$$

$$r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\phi_0 = \tan^{-1}\left(\frac{y_0}{x_0}\right)$$



$$\dot{x}_1 = x_2 + \alpha x_1 \left(\beta^2 - x_1^2 - x_2^2\right)$$

$$\dot{x}_2 = -x_1 + \alpha x_2 \left(\beta^2 - x_1^2 - x_2^2\right)$$

$$r^2 = x_1^2 + x_2^2 ,$$

$$\phi = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\dot{\phi} = -1$$

$$\cancel{2}\cancel{r}\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= \alpha\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)$$

$$= \alpha r^{\cancel{2}} \left(\beta^2 - r^2\right)$$

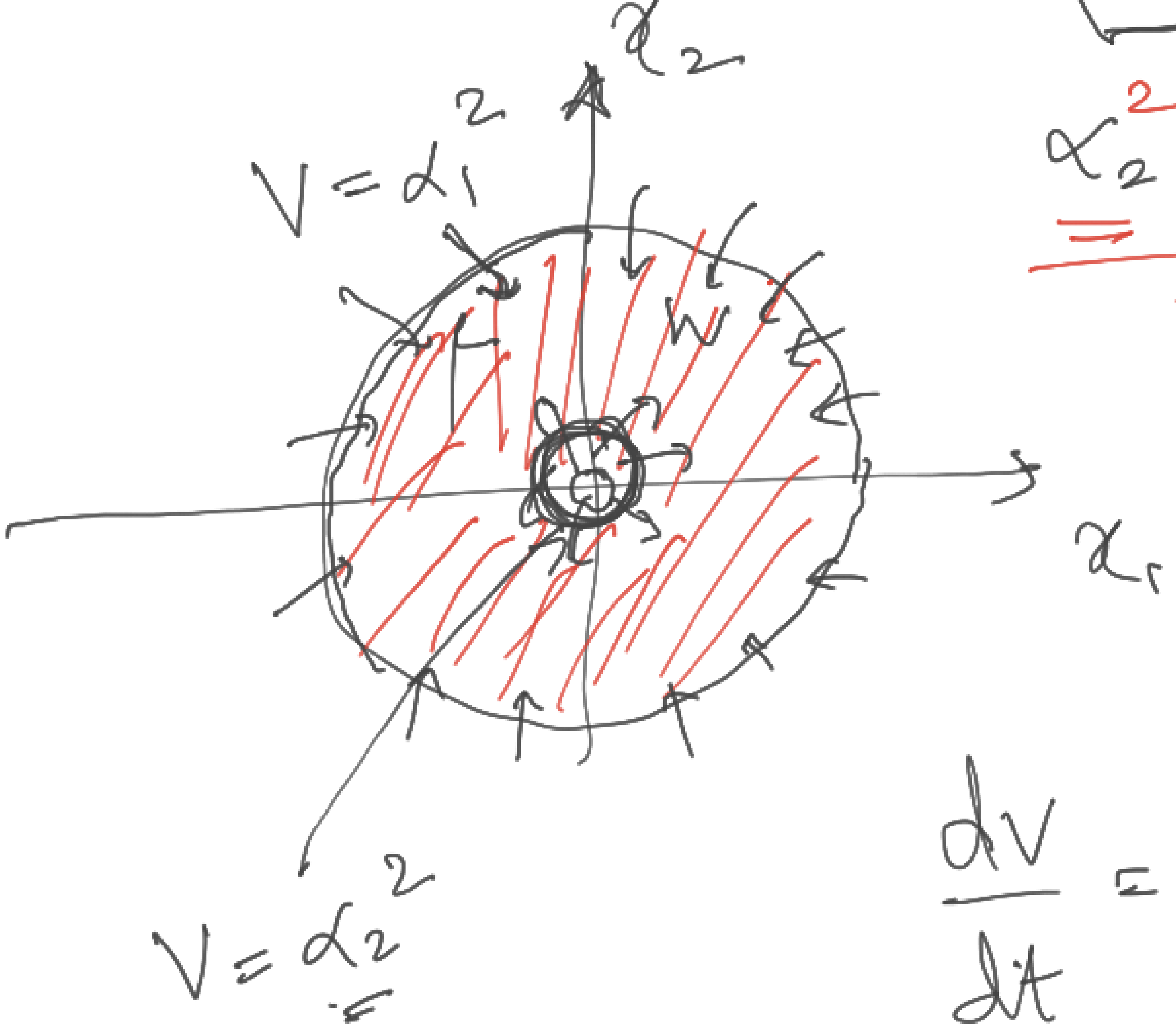
$$\boxed{\dot{r} = \alpha r \left(\beta^2 - r^2\right)}$$

$$\boxed{\alpha_2^2 < \beta^2 < \alpha_1^2}$$

$$\alpha_2^2 < \underline{x_1^2 + x_2^2} < \alpha_1^2$$

$$\Updownarrow$$

$$V$$

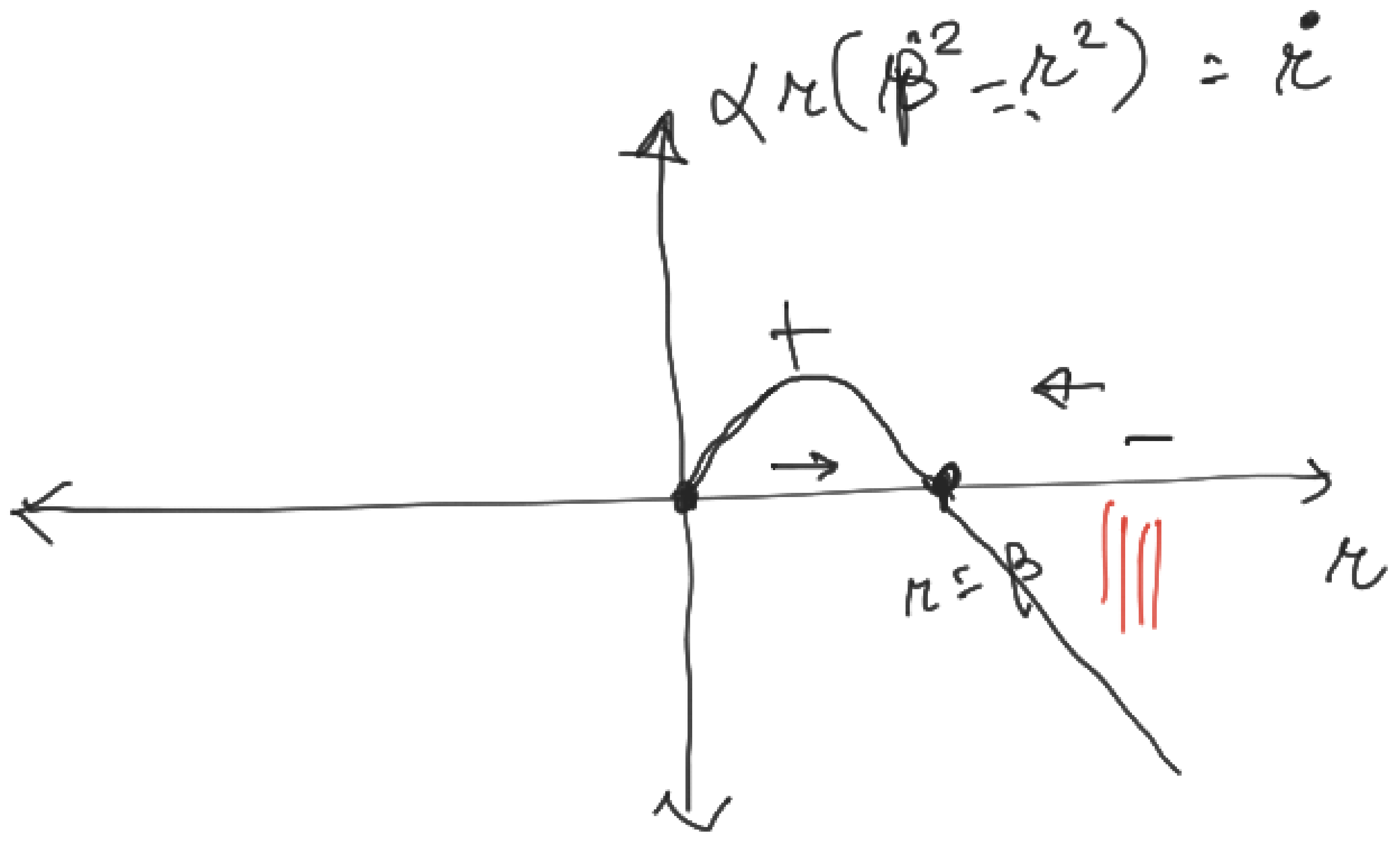


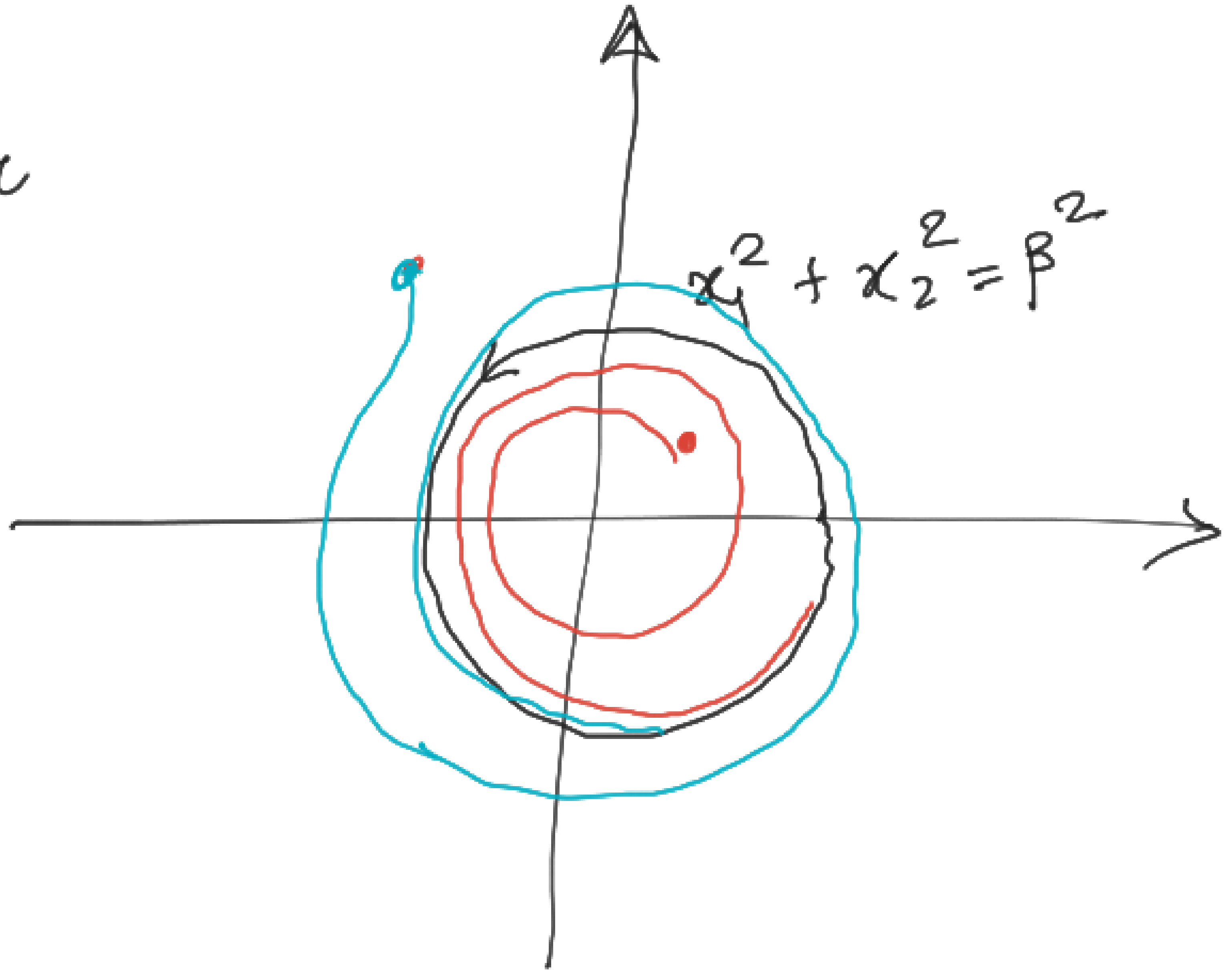
$$\frac{dV}{dt} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2\cancel{\alpha} V(\beta^2 - V)$$

$V > \beta^2$, $\underline{\dot{V} < 0}$

$\underline{V < \beta^2 , \dot{V} > 0}$


$\alpha r(\beta^2 - r^2) = \dot{r}$
+
−
$r = \beta$
r


$x_1^2 + x_2^2 = \beta^2$

Theorem [Poincare - Bendixson Theorem]

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\} \longrightarrow (1)$$

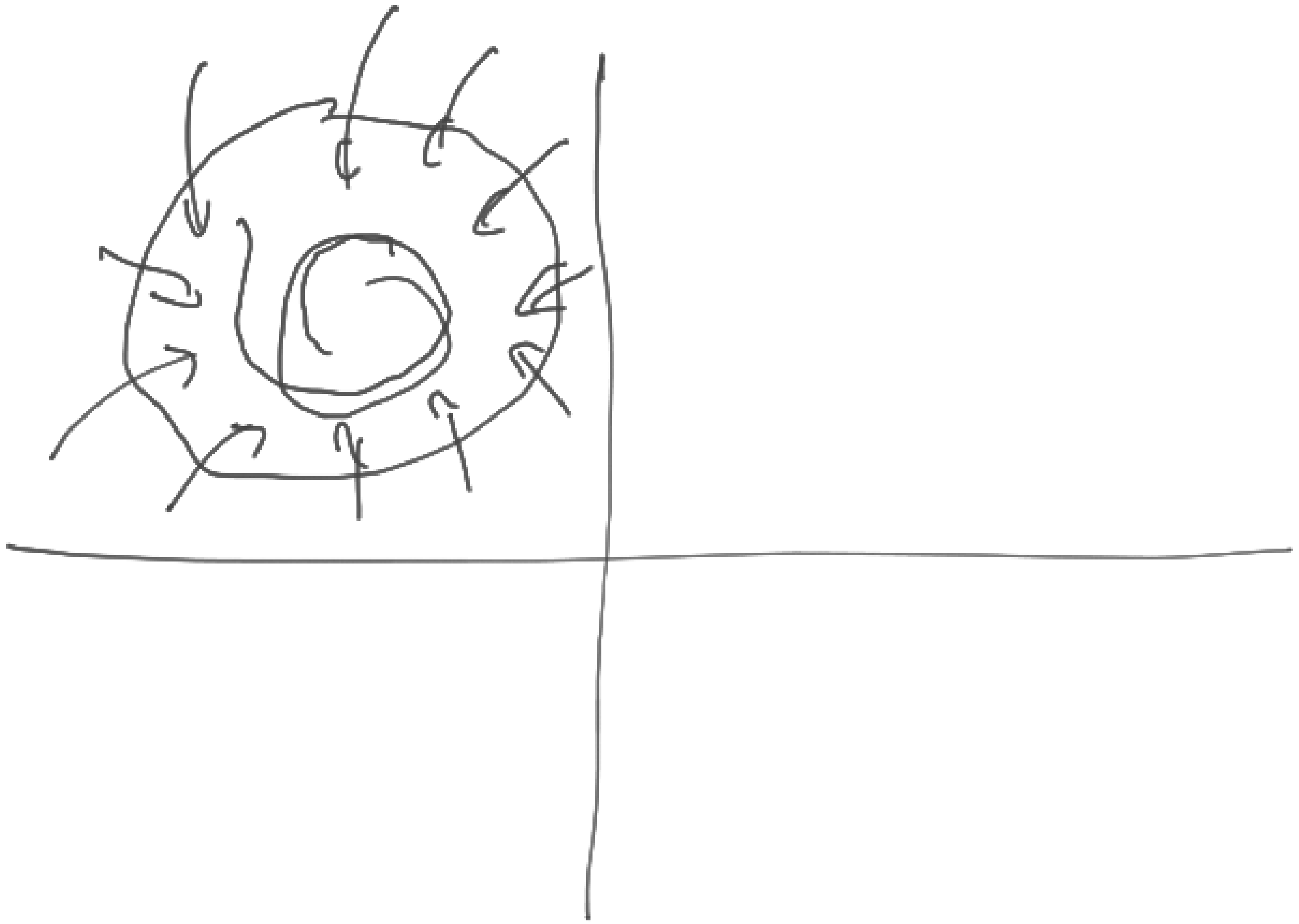
Let $W \subset \mathbb{R}^2$ Closed & bounded. (compact)

s.t. (i) W Contains no equilibrium points of (1)

(ii) $\forall\ x(t)$ of (1) with $x(0) \in W$, we have.

$x(t) \in W \quad \forall\ t \geq 0$

then set W Contains a Periodic orbit



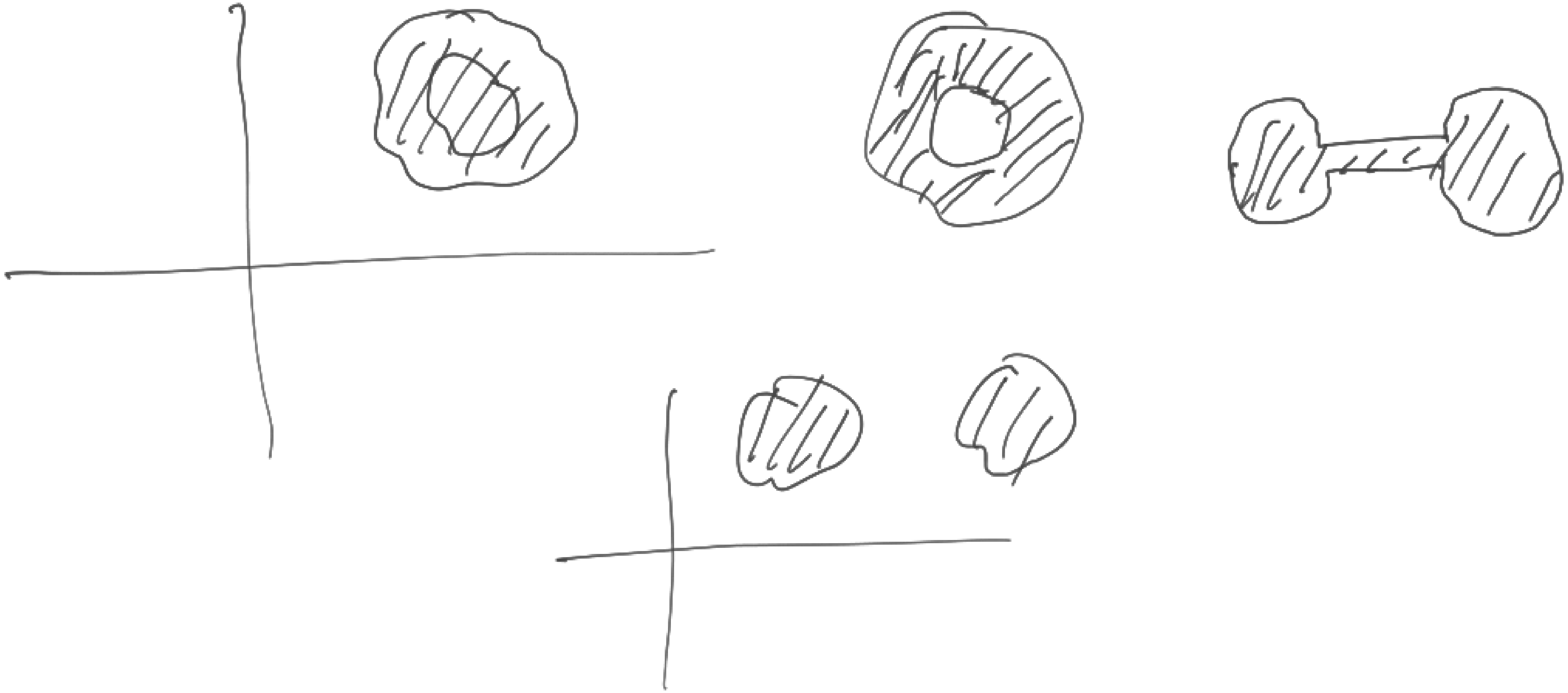
Theorem (Bendixson's Theorem)

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2) \quad \text{--- (i)}$$

Let D be a simply connected region in $\mathbb{R}^2$ s.t. there are no eq. pts of (i) in D.

If (i) $\exists\, x \in D$ s.t. $\underbrace{\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}}$ $\neq 0$

(ii) $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ does not change its sign on D. then Periodic orbit does not exists in D.



Examples

(i) $\ddot{x} + \underline{\underline{\alpha \dot{x}}} + g(x) = 0$, $\quad g(0) = 0$, $\quad \alpha > 0$

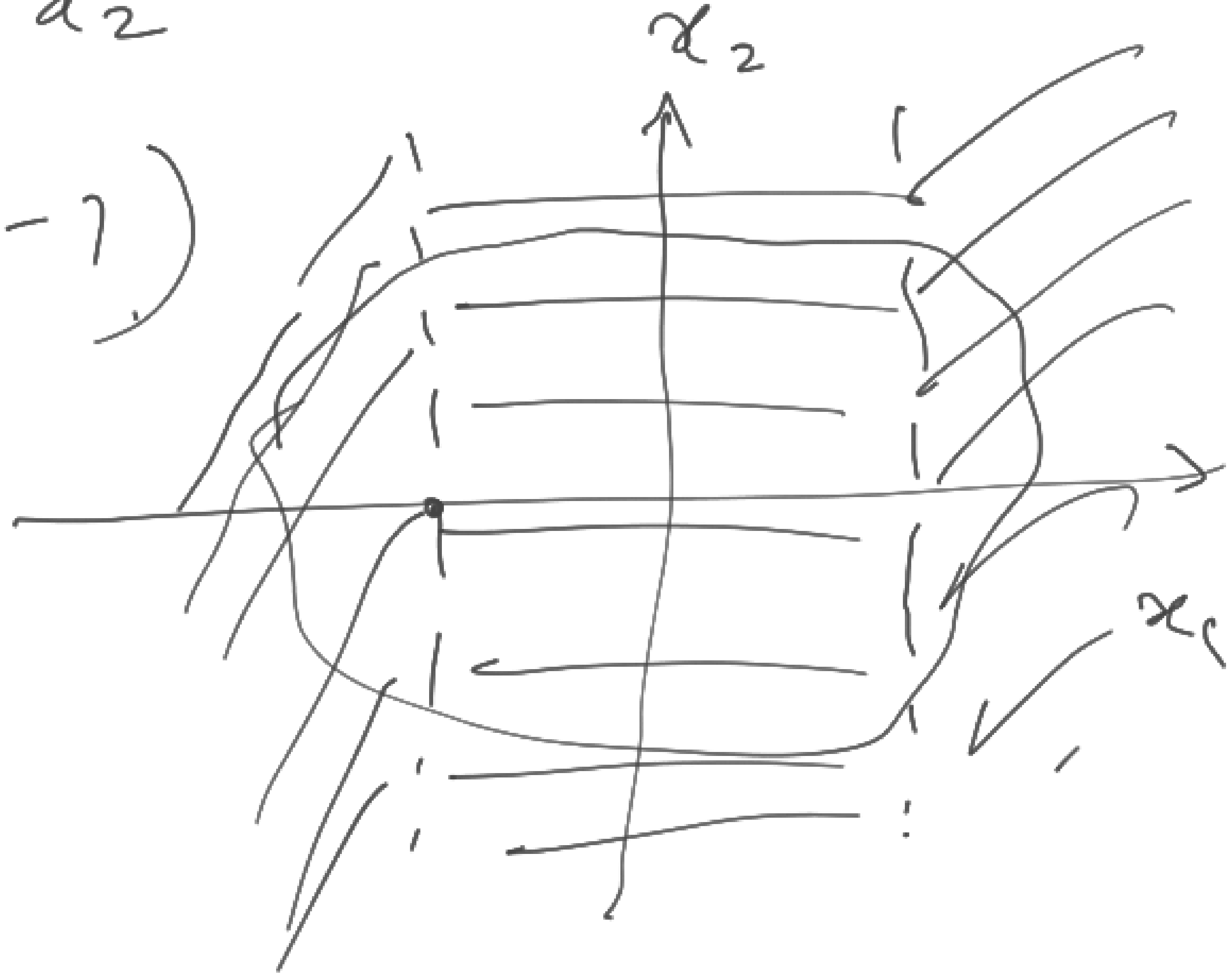
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha x_2 - g(x_1)$$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \alpha \neq 0$$

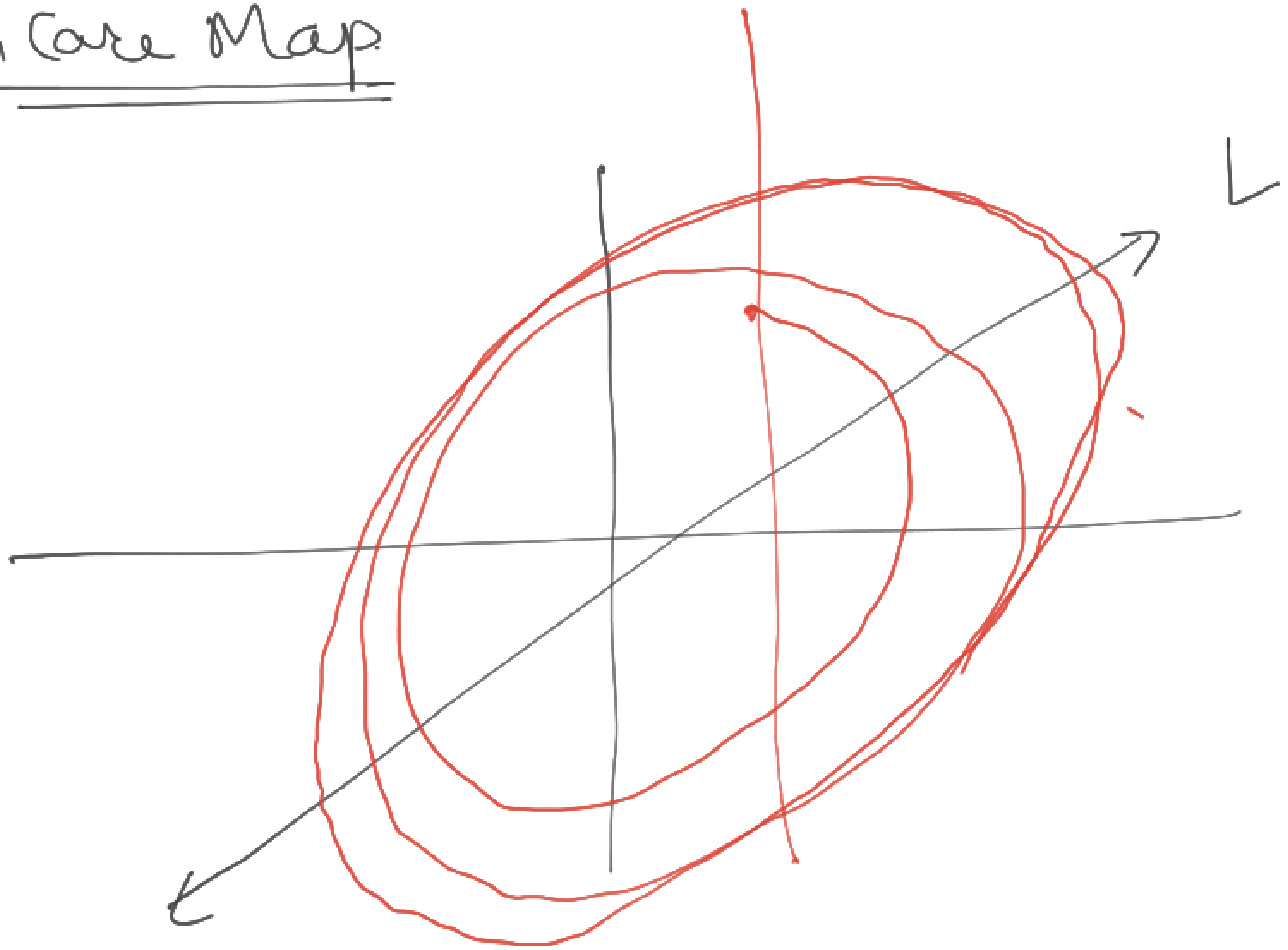
e.g. $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, $\mu > 0$

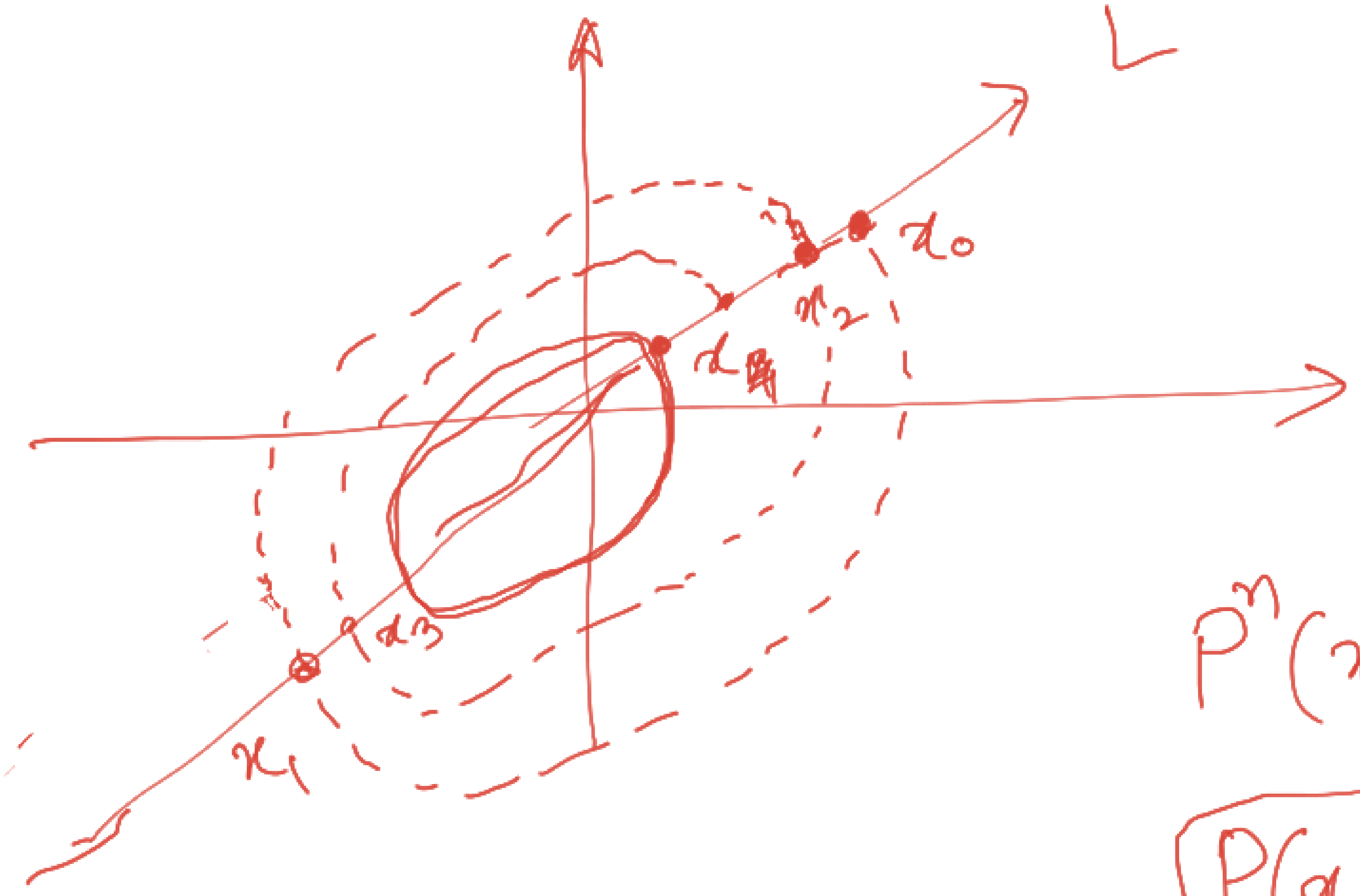
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \mu(x_1^2 - 1)x_2$$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -\mu(\underline{\underline{x_1^2 - 1}})$$



Poincare Map





$\rightarrow P : L \rightarrow L$

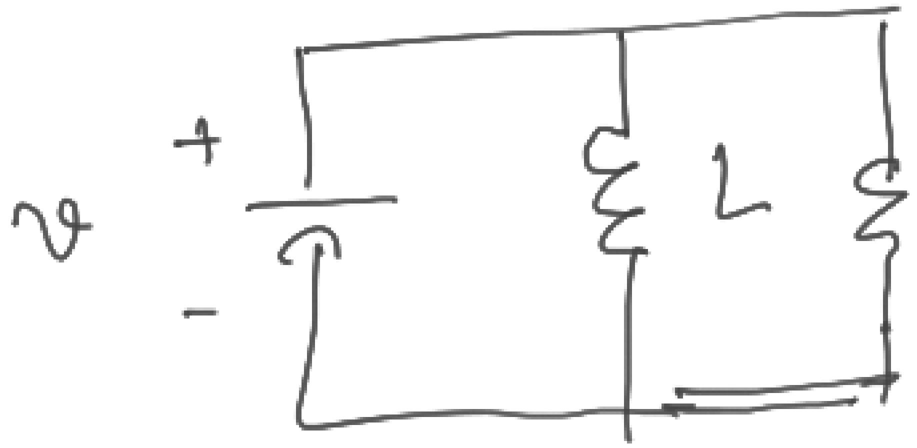
$x_0 \rightarrow P(x_0) = x_2$

$P(x_2) = x_4$

$P^n(x_0) \longrightarrow x^*$

$$\boxed{P(x^*) = x^*}$$

Van der pol Oscillator.



$$i = \underline{h(v)} = -v + v^3.$$

$$\cancel{C}\frac{dv}{dt} + \frac{\cancel{1}}{\cancel{L}}\int v\,dt + (-v + v^3) = 0$$

$$\ddot{v} + v + (-\dot{v} + 3v^2\dot{v}) = 0$$

$$\ddot{v} + \underbrace{(-1 + 3v^2)}_{\frac{dh}{dv}}\dot{v} + v = 0$$

$$\dot{x} = y - h(x)$$
$$\dot{y} = -x$$

$$x = v$$
$$y = \dot{v} + (-v + v^3)$$
$$= \underline{\dot{v} + h(v)}$$

$$\dot{x} = y - h(v)$$
$$\dot{y} = \ddot{v} + \frac{dh}{dv}\dot{v}$$
$$= -x$$