Lecture 7: Existence & Uniqueness of solutions to ODE's.

$\boxed{\dot{x} = f(x)}$, $x(0) = x_0$

$\boxed{x(t) \text{ is a solution}}$

Defn: [Normed linear Space]

A normed linear space consists of a vector space X with a norm $\|\bullet\|$ defined on it.

It is denoted by $(X, \|\cdot\|)$.

The norm $\|\cdot\|$ is a real valued function

$(X, \|\cdot\|)$

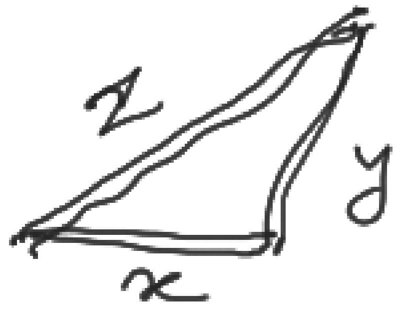
$$\|\cdot\| : X \to \mathbb{R}$$
$$x \mapsto \|x\| \in \mathbb{R}$$

that Satisfy following Properties

(i) $\|x\| \geq 0$ for all $x \in X$ and (Positivity).
$\|x\| = 0$ if & only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$, $\alpha \in \mathbb{R}$.
(Homogeneity)

(iii) $\|x + y\| \leq \|x\| + \|y\|$ $\forall$ $x, y \in X$.
(triangle inequality)



$|x+y| \leq |x| + |y|$

Examples i) $(\mathbb{R}^n, \|\cdot\|_2) \longrightarrow$ Normed linear space.

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

(ii) $(\mathbb{R}^n, \|\cdot\|_\infty)$

$$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

(iii) $(\mathbb{R}^n, \|\cdot\|_1)$, $\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|$

(iv) $(\mathbb{R}^n, \|\cdot\|_p)$, $\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$.

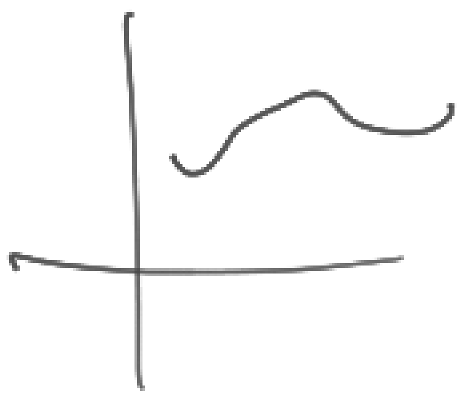
$p \geq 1$, $p < 1$ (not a norm)

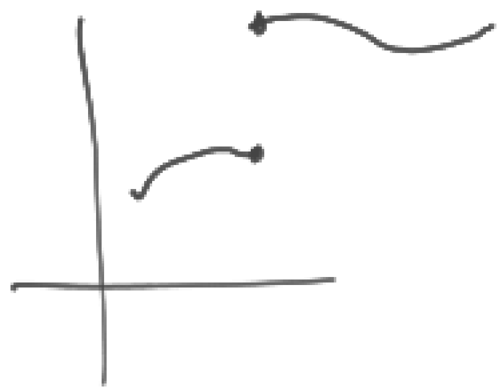
(v) $\underline{C[t_0, t_1]}$ is a Space of continuous functions on $[t_0, t_1]$

$$C[t_0, t_1] := \left\{ f : [t_0, t_1] \to \mathbb{R}^n,\quad t \mapsto \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \;\middle|\; f_i \text{ are continuous} \right\}$$

<u>Exercise</u>

Verify $C[t_0, t_1]$ is a vector space.





$\left(C[t_0, t_1] , \|\cdot\|_p \right)$ Normed linear Space $\mathcal{L}_p$.

for $p \geq 1$

$$\|f(t)\|_p := \left(\int_{t_0}^{t_1} \sum_{i=1}^{n} |f_i(t)|^p \, d\tau \right)^{1/p}$$

$f(t)$

$\left(C(t_0, t_1), \|\cdot\|_\infty \right)$

$p = 1$

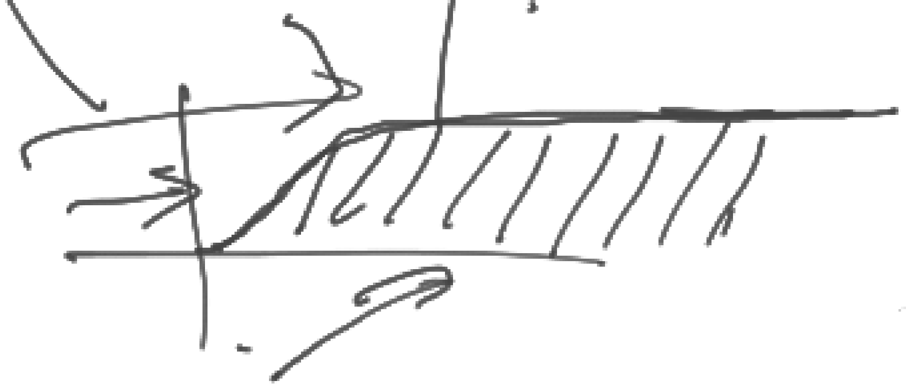
$$\|f(t)\|_1 := \left(\int_{t_0}^{t_1} |f_i(t)| \, d\tau \right)$$

X

$\left(C(t_0, t_1), \|\cdot\|_1 \right)$

$p = 2$

$$\|f(t)\|_2 := \left(\int_{t_0}^{t_1} |f_i(t)|^2 \, d\tau \right)^{1/2}$$



$p = \infty$

$$\|f(t)\|_\infty := \sup_{t \in [t_0, t_1]} \{ |f_i(t)| \}$$

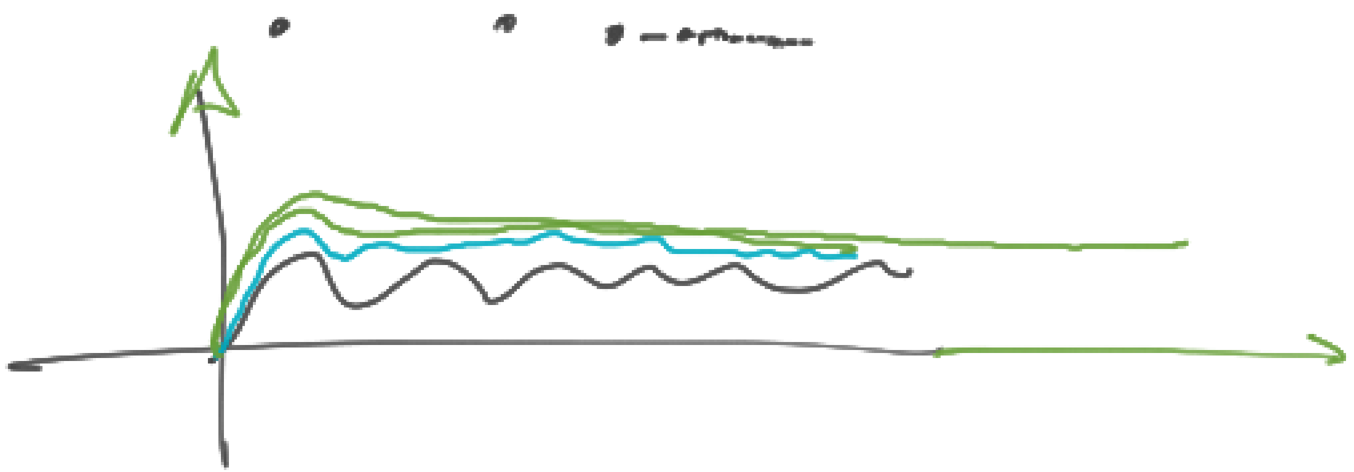
Sequences in $(X, \|\cdot\|)$

$$(x_0, x_1, \ldots\ldots, x_n, \ldots\ldots) = \{x_n\}_{n=0}^{\infty}$$

Defn: [Cauchy Sequence]

A sequence $\{x_n\}_{n=0}^{\infty}$ in a normed linear Space $(X, \|\cdot\|)$ is said to be a "Cauchy" sequence if for all $\epsilon > 0$, there is $N = N(\epsilon) > 0$ s.t. $\|x_i - x_j\| < \epsilon \quad \forall\, i, j > N$

Example $(\mathbb{Q}, |\cdot|)$



$$x_n = \left(\frac{1}{2}\right)^n$$

$\epsilon = 0.1$, N afterwhich $\left|\left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^j\right| < 0.1$

$\epsilon = 0.01$ N

$\forall\ i, j > N.$

$\epsilon = 0.1$, $N = 4$

$\epsilon = 0.01$, $N = 10$

$$\left|\left(\frac{1}{2}\right)^5 - \left(\frac{1}{2}\right)^6\right|$$

$$\left|\frac{1}{32} - \frac{1}{64}\right| < 0.1$$

Defn: [Sequence converging to x in $(X, \|\cdot\|)$

A sequence $\{x_n\}_{n=0}^{\infty}$ converges to an element

"x" if & only if

$\rightarrow$ [$\forall \epsilon > 0 \quad \exists \, N \geq 0$.
s.t. $\|x - x_m\| < \epsilon \quad \forall \, m \geq N$]

In otherwords.

$\rightarrow \lim_{m \to \infty} \|x - x_m\| = 0$, or $\lim_{m \to \infty} x_m = x$.

or. $x_m \to x$ as $m \to \infty$.

Example $\boxed{(\mathbb{Q}, |\cdot|)}$, $\sqrt{2} = 1.4142135623 7\ldots$

$1, \ 1.4, \ 1.41, \ 1.414, \ 1.4142, \ 1.41421, \ \ldots$

$\ldots\ldots\ldots \quad \sqrt{2} \notin \mathbb{Q}$

This Sequence is a Cauchy sequence in $(\mathbb{Q}, |\cdot|)$ which does not converge to an element in $\mathbb{Q}$.

$(X, \|\cdot\|)$ → Sequences

Cauchy

Non-cauchy

if all Converge to element in X

if $\exists$ a Cauchy Sequence not converging to an element in X.

⇓

↓

→ "Complete" Normed Linear Space

⇒ Not Complete.

→ $(\mathbb{Q}, |\cdot|)$ is not complete.

→ $(C[t_0, t_1], \|\cdot\|_\infty)$ is complete.

Def^n: [Completeness of a Normed Linear Space]

A normed linear space $(X, \|\cdot\|)$ is said to be "Complete" normed linear space ("Banach" Space) if every Cauchy sequence in X converges to an element in X.

$\rightarrow (\mathbb{Q}, |\cdot|)$ is "not" a Banach space.

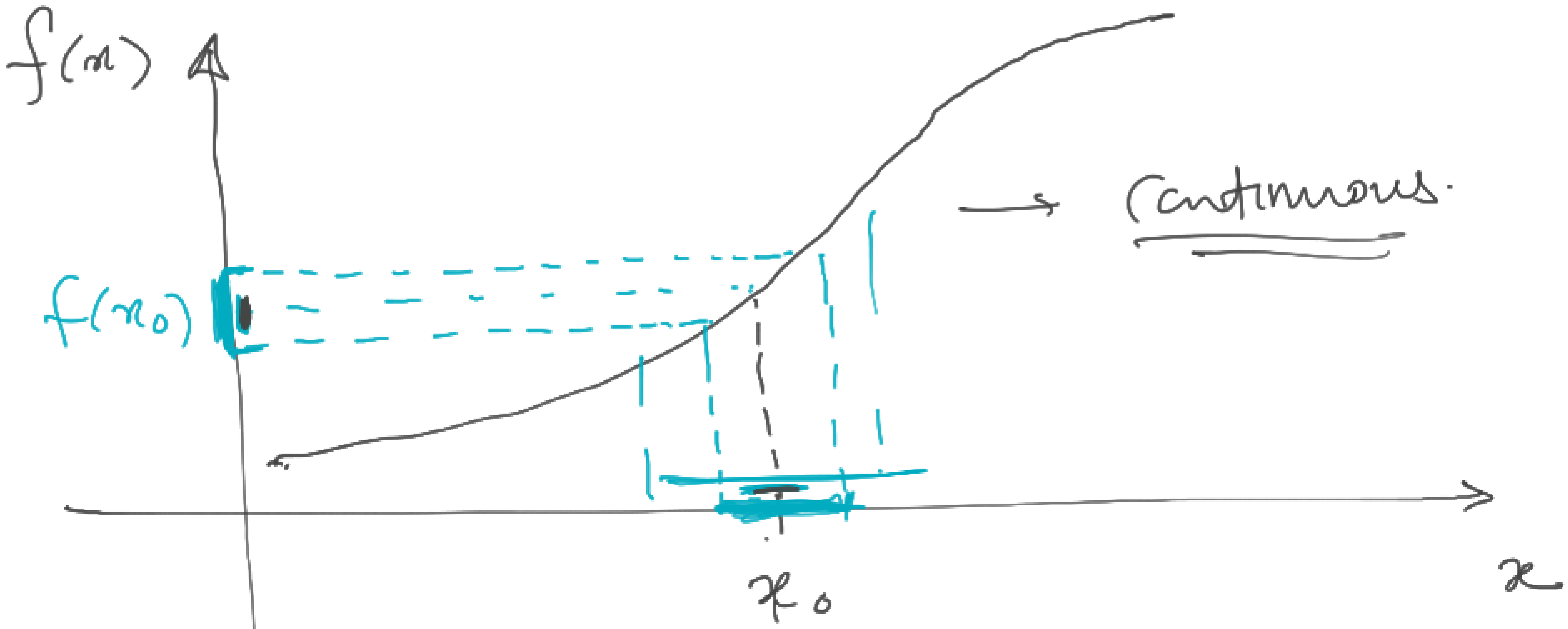
$\rightarrow (C[t_0, t_1], \|\cdot\|_\infty)$ is a Banach space.

Def^{n}. [Continuous function] $f(x) = x^2$ at $x = 1$

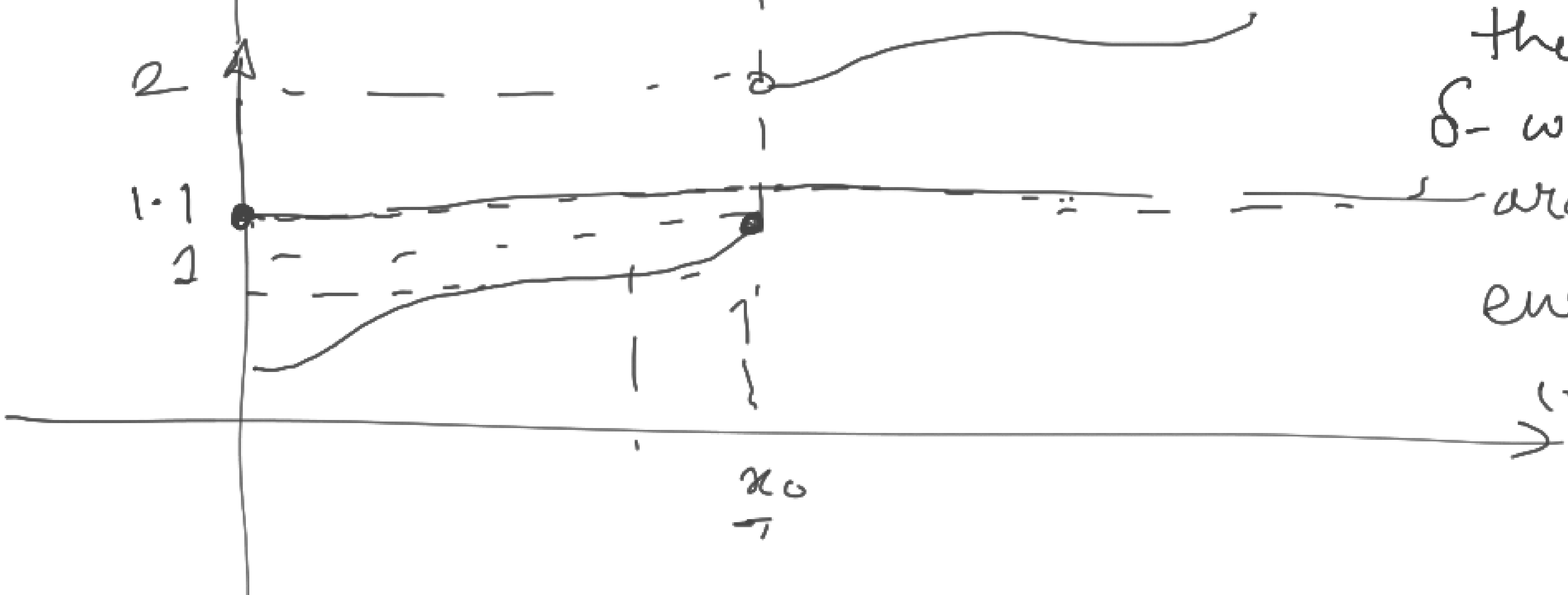
A function $f : \mathbb{R}^n \to \mathbb{R}$ is Continuous at $x = x_0$ if $t \mapsto f(t)$

$$\left[\text{for all } \epsilon > 0, \text{ there is } \delta = \delta(\epsilon, x_0) > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon. \right]$$

If δ does not depend on x_0, then f is uniformly Continuous.



→ Continuous.



there is no δ-width interval around x_0 to ensure f(x) is in ε-width interval.

Defⁿ: [continuous function between normed linear space]

Let $(\underline{X}, \|\cdot\|_\alpha)$ and $(\underline{Y}, \|\cdot\|_\beta)$. det

$f: X \longrightarrow Y$ be a map.

then f is a continuous map at $x = x_0 \in X$

if $\forall\, \underline{\epsilon} > 0$, $\exists\, \delta = \delta(\epsilon, x_0)$ s.t.

$$\|\underline{x - x_0}\|_\alpha < \delta \Rightarrow \|f(\underline{x}) - f(x_0)\|_\beta < \epsilon$$

If δ is independent of x_0 then f is uniformly continuous

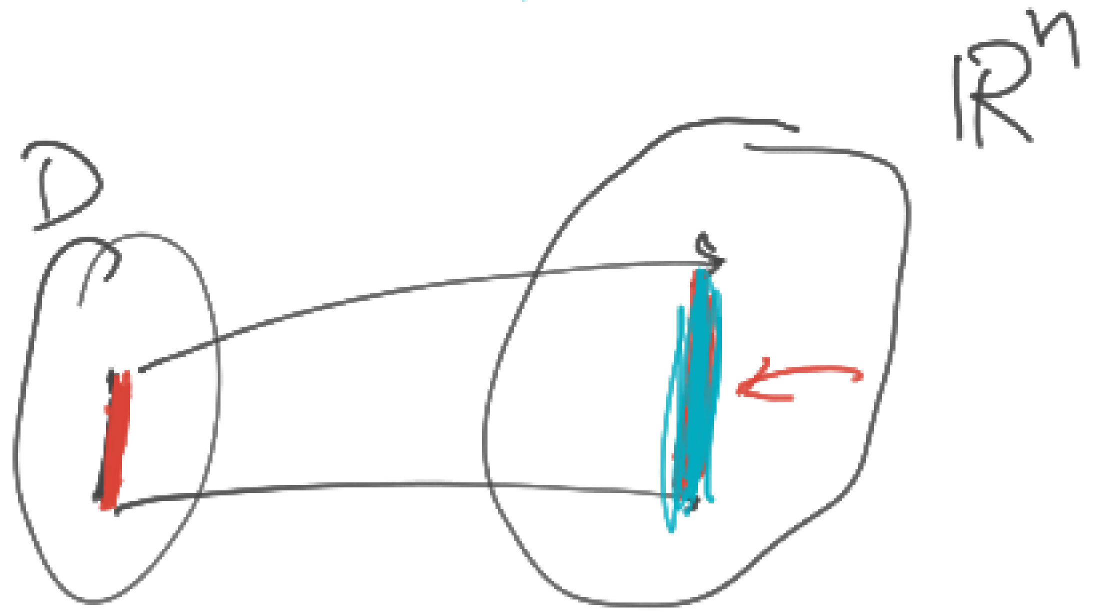
Def^n: [Lipschitz Continuous function]

$f : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is "Lipschitz"

at $x_0 \in D$ if there is $L > 0$ and

$N_{x_0} \subset D$ (Neighbourhood of x_0) s.t.

$$\| f(x) - f(y) \|_{\$} \leq L \, \| x - y \|_{\$} \quad \forall x, y \in N_{x_0}$$



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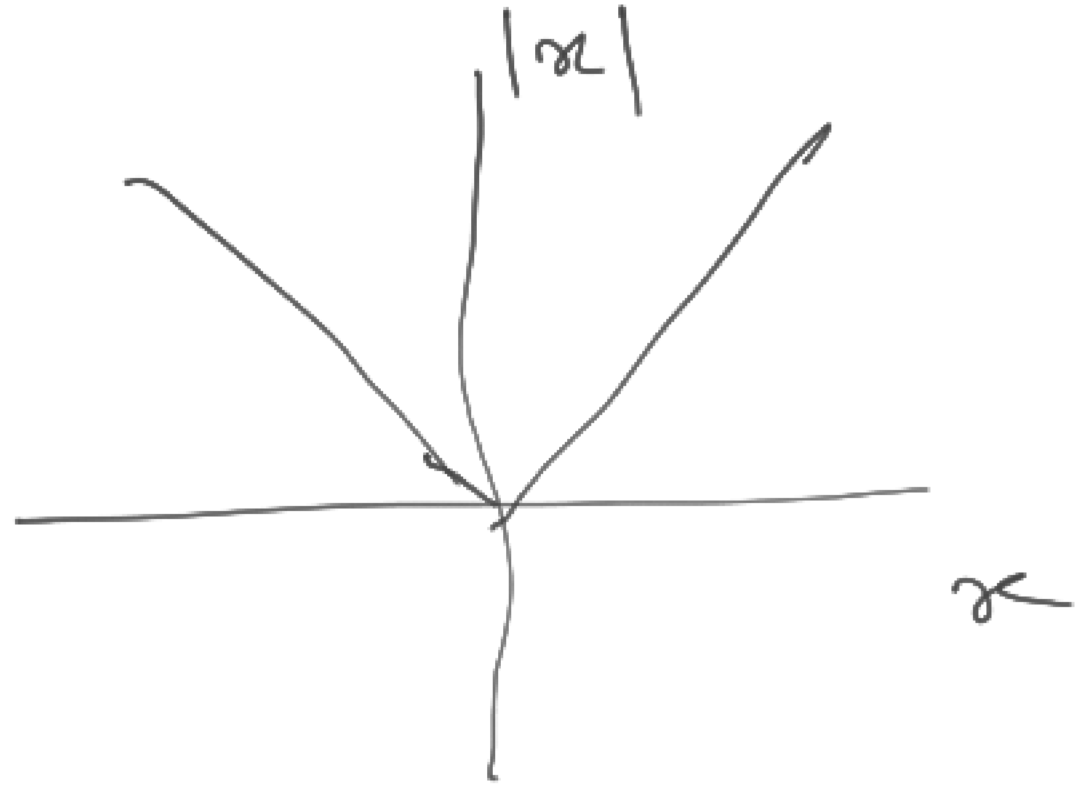
Example $f(x) = |x|$

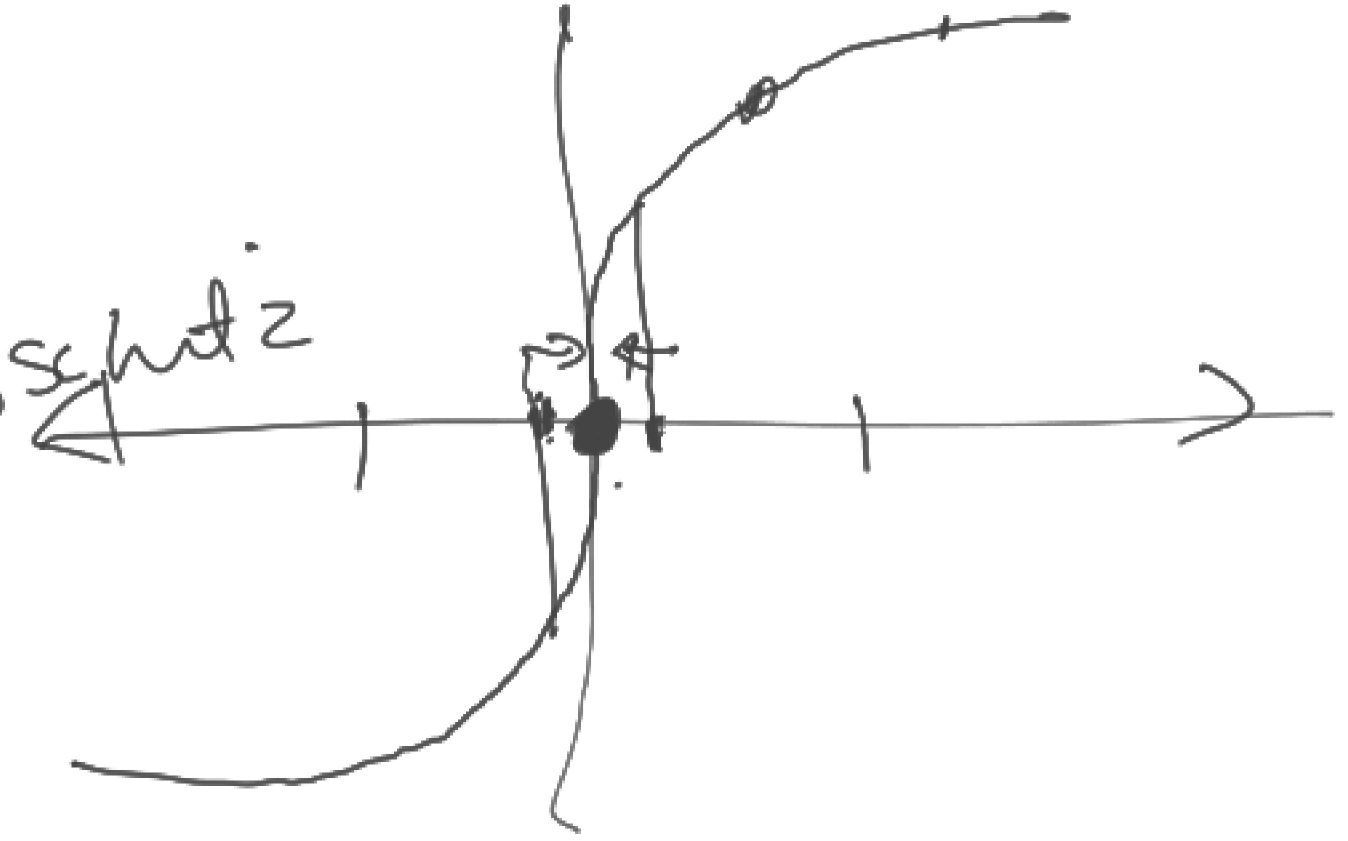


$$|f(x) - f(y)| = \big||x| - |y|\big|$$

$$\leq 1|x - y|$$

$L = 1$

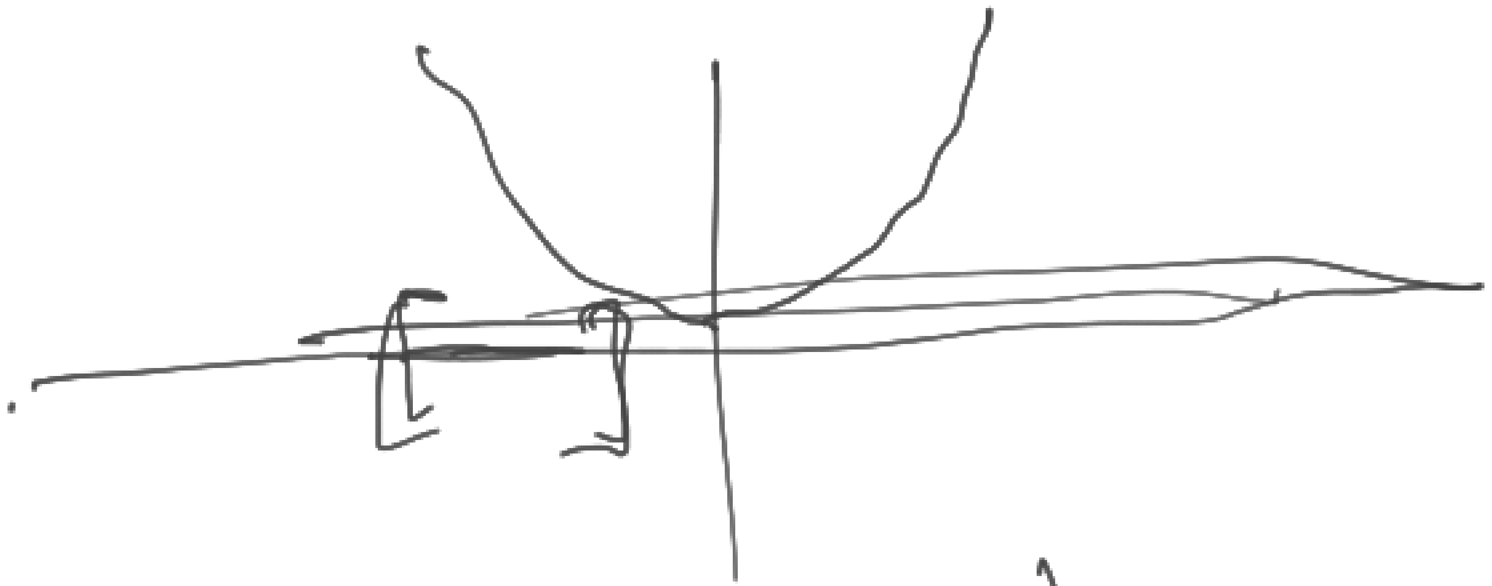
$f(x) = x^{1/3} \to$ is not Lipschitz at $x = 0$



$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \text{ is unbounded.}$$

for $x \to 0^+$, $y \to 0^-$

$$f(x) = x^2$$



$$|f(x) - f(y)|$$

$$= |x^2 - y^2| \le \underline{|x+y|}\,|x-y|$$

$$L = \max_{(x,y)\in N_{x_0}} |x+y|$$

Lipschitz continuous at each pt.

But it is not uniformly Lipschitz contn

$$f(x_0 + \Delta x) \approx f(x_0) + \left[\frac{\partial f}{\partial x}\right]^T \Big|_{x = x_0} \Delta x \;\cdots$$

$$\| f(x) - f(x_0) \| \leq \left\| \frac{\partial f}{\partial x} \right\|^T_{x = x_0} \| x - x_0 \|$$

$\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$

Induced matrix norms

f is differentiable $\xrightarrow{\text{"Locally"}}$ Lipschitz Continuous
$\Longrightarrow$

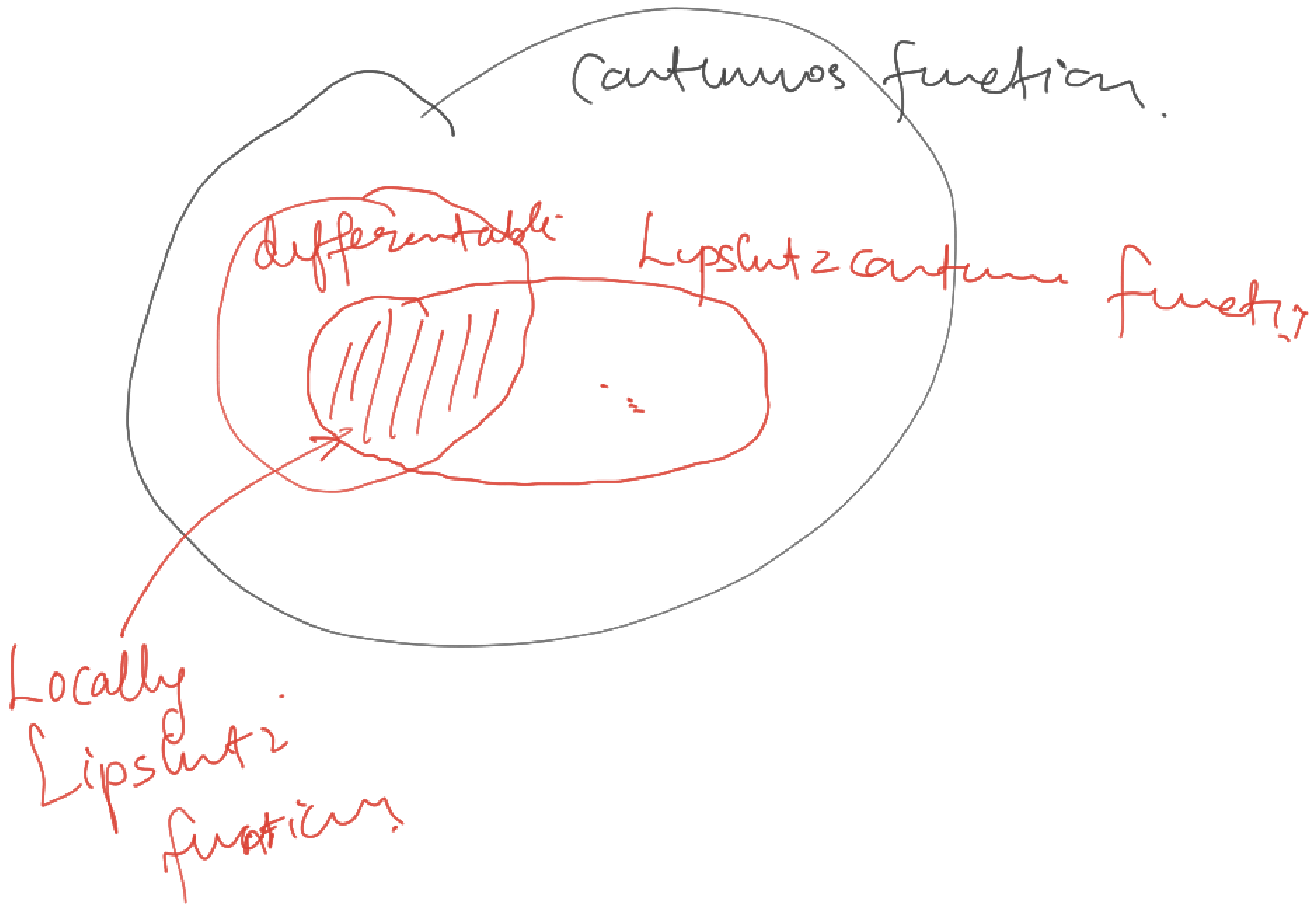
$\not\Leftarrow$, $\underline{x^2}$
global Lipschitz continuity

f is Lipschitz $\overset{?}{\not\Longrightarrow}$ f is differentiable

$|x|$

f is Continuous $\not\Longrightarrow$ f Lipschitz

$x^{1/3}$


continuous function.
differentiable
Lipschitz continuous function
Locally Lipschitz function.