Thm: [Existence & Uniqueness of Solution to ODE]

Consider $\dot{x} = f(x)$, $x(t_0) = x_0 \in \mathbb{R}^n$ — (I)

If f is Lipschitz continuous at x_0

then there exists $\delta > 0$ s.t. [I] has a unique continuous solution on interval $[t_0, t_0+\delta]$

Proof: 1. $(C[t_0, t_1], \|\cdot\|_\infty) = X$ is a Banach Space.

2.

$$x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}\, dt = \int_{t_0}^{t} f(x)\, dt$$

$$x(t) = \boxed{x(t_0) + \int_{t_0}^{t} f(x(\tau))\, d\tau.}$$

$$P : X \to X$$

$$x(t) \longmapsto x(t_0) + \int_{t_0}^{t} f(x(\tau))\, d\tau = P\{x(t)\}$$

Picard's iteration

$$\boxed{x_{n+1}(t) = \underset{\sim}{P}\{x_n(t)\}}, \quad x_0(t) \equiv x_0$$

$x_0(t), x_1(t), \ldots \ldots$

$$\boxed{x^*(t) = P(x^*(t))} = x^*(t_0) + \int_{t_0}^{t} f(x^*(\tau))\,d\tau$$

Defn: [Contraction Map]

Consider a ~~A~~ map $T : X \to X$, $x \mapsto T(x)$ where $(X, \|\cdot\|)$ is a normed linear space. Then T is contraction map on $S \subseteq X$ if there is $\rho \in [0, 1)$ s.t.

$$\forall x, y \in S, \quad \|T(x) - T(y)\| \le \rho \|x - y\|$$

Theorem: [Banach Fixed Point theorem]

$S \subseteq X$ be a closed subset of $(X, \|\cdot\|)$ which is Complete. Let $T : S \to S$ be a contraction map on S i.e. $\exists\, 0 \leq \delta < 1$ s.t.

$$\|T(x) - T(y)\| \leq \delta \|x - y\| \quad \forall\, x, y \in S$$

Then (i) $\exists$ a "unique" $x^* \in S$ s.t. $x^* = T(x^*)$

(ii) $\{x_n\}_{n=}^{\infty}$ generated by $x_{n+1} = T(x_n)$ Converges to x^* as $n \to \infty$ for any $x_0 \in S$

Proof: $x_0, x_1, x_2, \cdots, x_i, \cdots, x_j$

To show.

$\left[\forall\, \epsilon > 0,\ \exists\, N_\epsilon \text{ s.t. } \|x_i - x_j\| < \epsilon,\ \forall\, i, j \geq N_\epsilon\right]$

$i > j$, $x_i = T(x_{i-1})$

$x_j = T(x_{j-1})$

$$\|x_i - x_j\| = \|x_i - x_{i-1} + x_{i-1} - x_{i-2} + x_{i-2} + \cdots + x_{j+1} - x_j\|$$

$$\leq \|x_i - x_{i-1}\| + \|x_{i-1} - x_{i-2}\| + \cdots + \|x_{j+1} - x_j\|$$

$$\leq \rho^i \|x_1 - x_0\| + \rho^{i-1} \|x_1 - x_0\| + \cdots + \rho^{j+1} \|x_1 - x_0\|$$

$$\|x_i - x_j\| \leq \left(\rho^i + \rho^{i-1} + \dots + \rho^{j+1}\right) \|x_1 - x_0\|$$

$$\leq \rho^{j+1}\left(1 + \rho + \rho^2 + \dots + \rho^{i-j-1}\right) \|x_1 - x_0\|$$

$$\leq \rho^{j+1}\left(\sum_{k=0}^{\infty} \rho^k\right) \|x_1 - x_0\|$$

$$= \rho^{j+1} \frac{1}{1-\rho} \|x_1 - x_0\| < \epsilon \quad \forall\, i,j \geq N_\epsilon$$

Since $\rho < 1$, $\exists N_\epsilon$ s.t. $\dfrac{\rho^{N_\epsilon}}{1-\rho}\|x_1 - x_0\| < \epsilon$

Hence $\{x_n\}_{n=0}^{\infty}$, $x_{n+1} = T(x_n)$ is a Cauchy sequence

By Completeness & closedness $x_n \to x^* \in S$

Suppose $x^* \neq y^*$, s.t $T(x^*) = x^*$

$T(y^*) = y^*$.

$\|x^* - y^*\| = \|\underline{T(x^*) - T}(y^*)\| \leq \delta \|x^* - y^*\|$

$\Rightarrow \delta \geq 1$ Contradiction! $x^* = y^*$

Uniqueness!

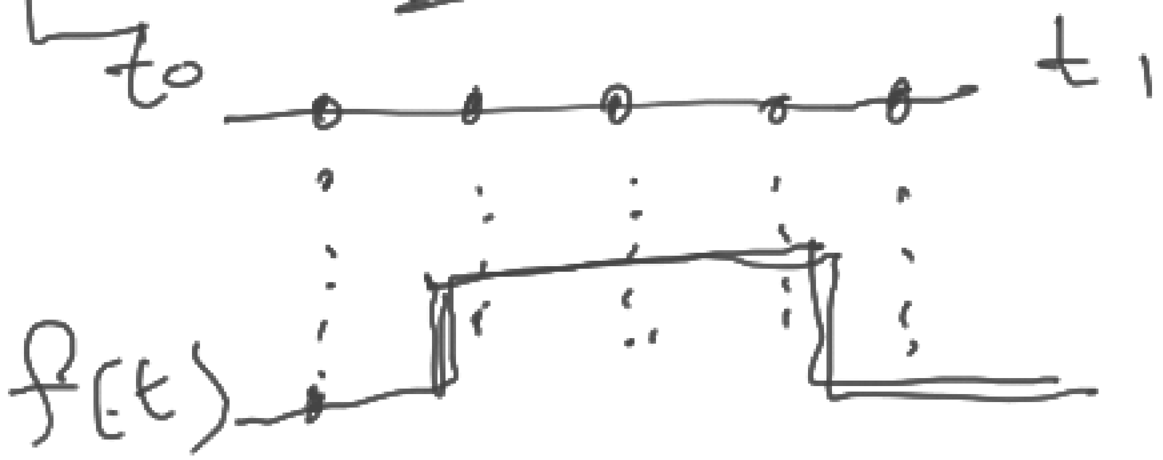
Theorem: $\left(C[t_0, t_1], \|\cdot\|_\infty\right)$ is a Banach Space

or "complete" Normed linear space.

Proof: Let $f_n(t)$ be an arbitrary Cauchy sequence in $C[t_0, t_1]$

$\left[\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ s.t } \|f_n(t) - f_m(t)\|_\infty \leq \epsilon \quad \forall n, m \geq N_\epsilon\right]$

[fix $t \in [t_0, t_1]$



at time t

$\forall \epsilon > 0, \exists M_\epsilon > 0$ s.t

$\|f_n(t) - f(t)\| < \epsilon \quad \forall n > M_\epsilon$

To show: $f(t)$ is continuous at every pt $w \in [t_0, t_1]$

To show:

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|t - w\| < \delta \Rightarrow \|f(t) - f(w)\| < \epsilon$

$$\|f(t) - f(w)\| = \|f(t) - f_N(t) + f_N(t) - f_N(w) + f_N(w) - f(w)\|$$

$$\le \|f(t) - f_N(t)\| + \|f_N(t) - f_N(w)\| + \|f_N(w) - f(w)\|$$

for fixed t, $\forall \epsilon/3 > 0$, $\exists N_{\epsilon/3}$ s.t $\|f(t) - f_N(t)\| < \epsilon/3$ $\forall n > N_{\epsilon/3}$

Next p.

$$\leq \frac{\varepsilon}{3} + \underline{\|f_N(t) - f_N(w)\|} + \varepsilon/3$$

$\forall \ \varepsilon/3 > 0, \ \exists \ \delta > 0$ s.t $\|t - w\| \leq \delta$

$$\Downarrow$$

$$\|f_N(t) - f_N(w)\| < \varepsilon/3$$

$$\leq \varepsilon$$

$\Rightarrow f$ is continuous everywhere on $[t_0, t_1]$

$\Rightarrow (C[t_0, t_1], \|\cdot\|_\infty)$ is complete.

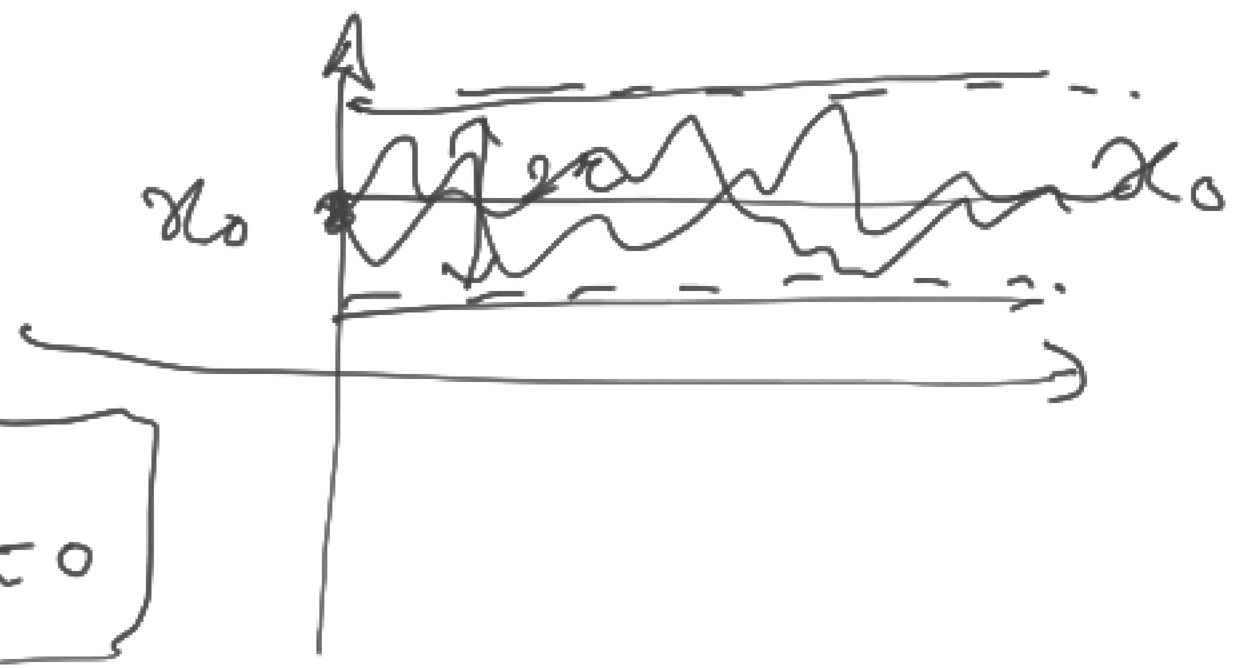
Proof of Existence & Uniqueness theorem.

Show $P\{x(t)\} = x(t_0) + \int_{t_0}^{t} f(x(\tau))\,d\tau$

is a contraction map on $S \subset X$

$$\underline{S} := \left\{ x(t) \in X \;\middle|\; \begin{array}{l} x(t_0) = x_0 \\ \|x(t) - x(t_0)\|_\infty \le r \end{array} \right\}$$

$$X = \left(C[t_0, t_1], \|\cdot\|_\infty \right)$$



$t_1 \geq t_0 + \delta$, $\boxed{\delta \le t_1 - t_0}$

$$\| P\{x(t)\} - x_0 \| = \| \cancel{x_0} + \int_{t_0}^{t} f(x(\tau))\, d\tau - \cancel{x_0} \|$$

$$= \| \int_{t_0}^{t} f(x(\tau))\, d\tau \|$$

$$= \| \int_{t_0}^{t} \big(f(x(\tau)) - f(x_0) + f(x_0) \big)\, d\tau \|$$

$$\leq \int_{t_0}^{t} \big(\| f(x(\tau)) - f(x_0) \| + f(x_0) \big)\, d\tau$$

$$\leq \int_{t_0}^{t} \big(L \| x(\tau) - x_0 \| + f(x_0) \big)\, d\tau \leq \big(L r + f(x_0) \big)(t - t_0)$$

$$\leq \left(Lr + f(x_0)\right)(t - t_0) \leq \left(Lr + f(x_0)\right)\delta \leq r$$

$$\left(Lr + f(x_0)\right)\delta \leq r$$

$$\boxed{\delta \leq \frac{r}{Lr + f(x_0)}} \Rightarrow \boxed{P : S \to S}$$

$$\delta \leq t_1 - t_0$$

To ensure P is contraction map

$$\| \underline{P}\{\underline{\underline{x(t)}}\} - P\{\underline{\underline{y(t)}}\}\|_\infty$$

$$= \left\| \int_{t_0}^{t} f(x(\tau))\,d\tau - \int_{t_0}^{t} f(y(\tau))\,d\tau \right\|_\infty$$

$$\leq \int_{t_0}^{t} \| \underline{f(x(\tau)) - f(y(\tau))} \|_\infty \, d\tau$$

$$\leq \int_{t_0}^{t} L \, \| \underline{\underline{x(\tau) - y(\tau)}} \|_\infty \, d\tau \leq \int_{t_0}^{t} LM \, d\tau$$

$$\leq LM(t - t_0)$$

$$\leq LM\delta$$

$$\leq L\delta \, \|x(\tau) - y(\tau)\|_\infty \leq \rho \, \|x(\tau) - y(\tau)\|_\infty$$

$$\underline{L\delta \leq \rho < 1}$$

$$\delta \leq \frac{\rho}{L} \Rightarrow P \text{ is Contraction.}$$

$$\text{If } \quad \delta \leq \min \left\{ \frac{\rho}{L}, \; \frac{r}{Lr + f(x_0)}, \; t_1 - t_0 \right\}$$

then (i) $P : S \to S$

(ii) P is contraction map

Using Banach fixed point theorem.

(i) $\exists$ unique $x^* \in S$ s.t $P(x^*) = x^*$

(ii) $x_{n+1} = P(x_n)$, $x_0(t) \equiv x_0$

Converges to x^* as $n \to \infty$

$\square$

$$\dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = \sin x_1 = f_2(x_1, x_2)$$

$$\dot{x} = Ax$$

$$x_{n+1}(t) = P\{x_n(t)\} = x_0 + \int_{t_0}^{t} f(x_n(\tau))\,d\tau$$

$$= x_0 + A\int_{t_0}^{t} x_n(\tau)\,d\tau$$

$$x_0(t) \equiv x_0 \qquad \int_{t_0}^{t} A x_0\,d\tau$$

$$x_1(t) = \underline{x_0 + A(t - t_0)}\,x_0$$

$$x_2(t) = x_0 + A\int_{t_0}^{t} \left(A(t - t_0)x_0 + x_0\right)d\tau$$

$$x_2(t) = x_0 + A(t-t_0)x_0 + \frac{A^2}{2}(t-t_0)^2$$

$$\vdots$$

$$x_n(t) = x_0 + A(t-t_0)x_0 + \frac{A^2}{2!}(t-t_0)^2 x_0 + \cdots + \frac{A^n}{n!}(t-t_0)^n x_0$$

$$\vdots$$

$$x(t) = e^{At}x_0 \qquad \dot{x} = Ax$$

Theorem [Continuous dependence on initial Conditions]

Consider $\dot{x} = f(x)$ and $f : D \to \mathbb{R}^n$ is uniform Lipschitz continuity on D.

Let $x(t)$ that starts from $x_0 \in D$.

& $y(t)$ that starts from $y_0 \in D$.

then $\forall\ \varepsilon > 0$ & $t \in [t_0, t_1]$, $\exists\ \delta = \delta(\varepsilon, t_1 - t_0) > 0$

s.t. $\|x_0 - y_0\| < \delta \Rightarrow \|x(t) - y(t)\| \leq \epsilon$