"State - Space" Methods for Control.

states.

$$\dot{x} = Ax + Bu \quad , \quad x(t) \in \mathbb{R}^n \text{ at all time } t$$

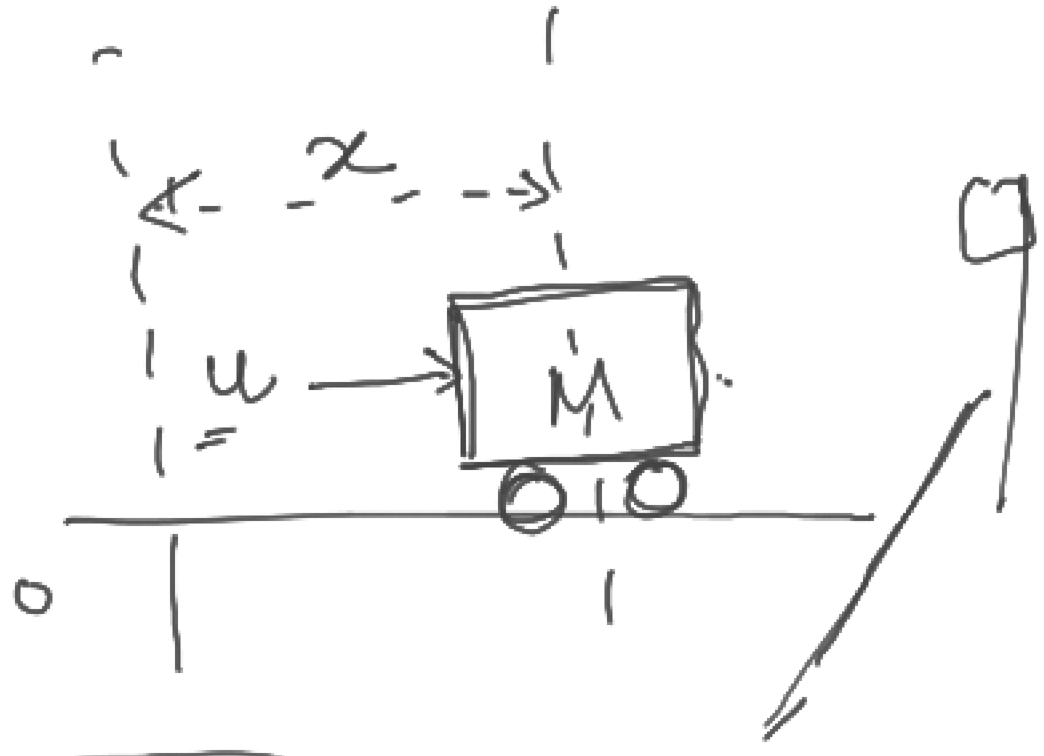
$$y = Cx + \cancel{Du}^{\,0} \quad u(t) \in \mathbb{R}^m \text{ inputs.}$$

$$y(t) \in \mathbb{R}^p$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

$C \in \mathbb{R}^{p\times n}$

example



$$\underline{M\ddot{x} = u(t)}$$

$$\left.\begin{matrix} x(0) \\ \dot{x}(0) \end{matrix}\right] \rightarrow \text{State Variables}$$

two state
one output
one input.

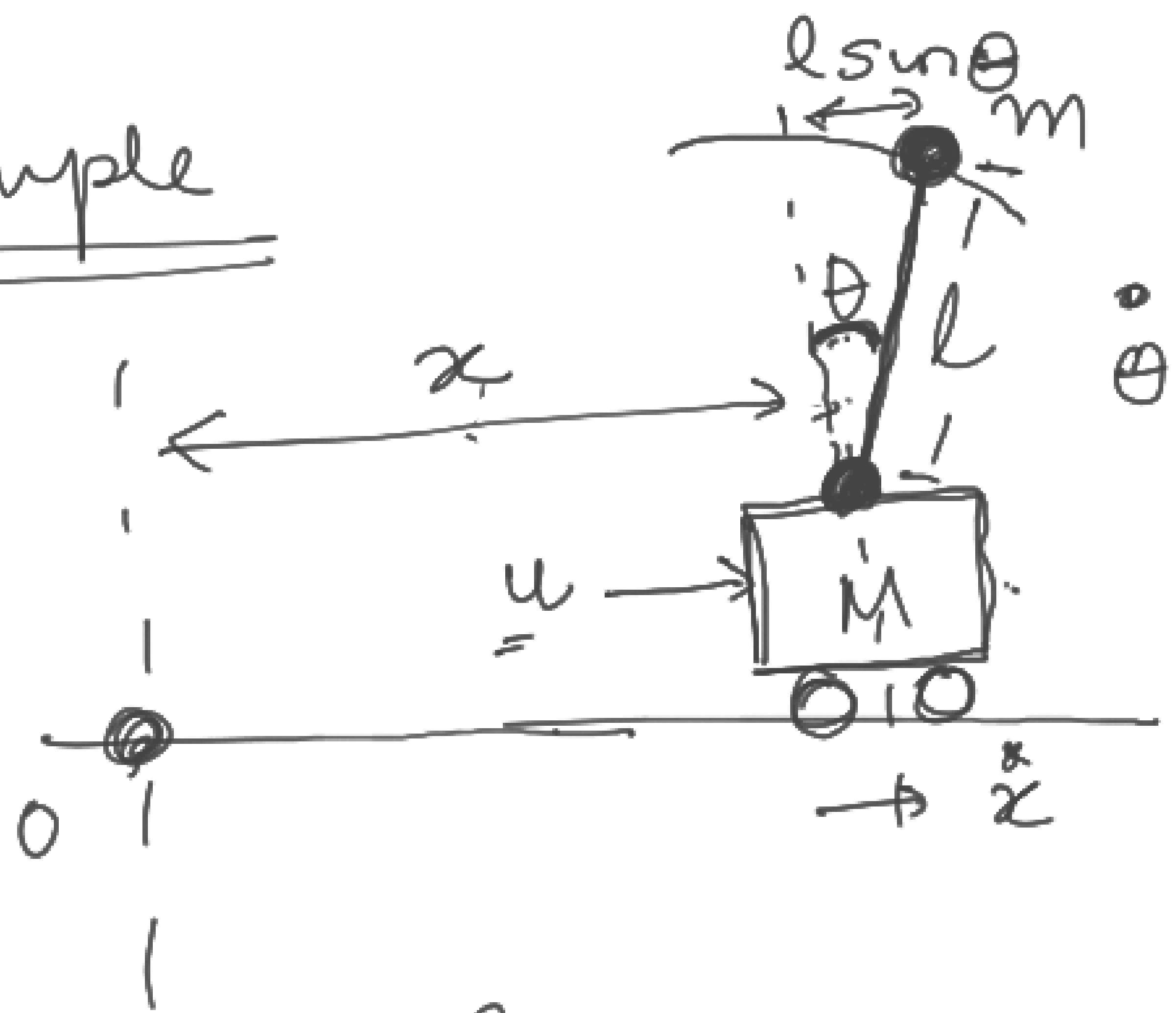
$$x_1 = x \quad , \quad \dot{x}_1 = x_2$$

$$x_2 = \dot{x} \quad \quad \dot{x}_2 = \frac{1}{M} u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{B} u(t)$$

$$y = x_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Example



Segway Robots

$\theta = 0$

$\dot{\theta} = 0$

$x = 0$

$\dot{x} = 0$

$$m \frac{d^2}{dt^2}(x + l\sin\theta) + M \frac{d^2 x}{dt^2} + u(t) = 0$$

$$ml^2\ddot{\theta} - mgl\sin\theta + ml\ddot{x}\cos\theta = 0$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} ml\cos\theta & M+m \\ ml & m\cos\theta \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} ml\dot{\theta}^2 \sin\theta + u(t) \\ mg\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} ml\cos\theta & M+m \\ ml & m\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} ml\dot{\theta}^2 \sin\theta + u(t) \\ mg\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3, x_4, u) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, x_4, u) = \cdots \\ \dot{x}_3 &= f_3(x_1, x_2, x_3, x_4, u) \\ \dot{x}_4 &= f_4(x_1, x_2, x_3, x_4, u) \\ y &= h(x, u) \end{aligned}$$

$$\left.\begin{aligned} \dot{\mathbf{x}} &= f(\underline{\mathbf{x}}, \mathbf{u}) \\ y &= h(x, u) \end{aligned}\right] \rightarrow \quad \bar{x} = \underline{0},$$

$$\dot{x} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=0}} x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{u=0}} u$$

$$\dot{x} = A x + B u.$$

$$y = \left.\frac{\partial h}{\partial x}\right|_{x=0} x + \left.\frac{\partial h}{\partial u}\right|_{u=0} u = Cx + Du.$$

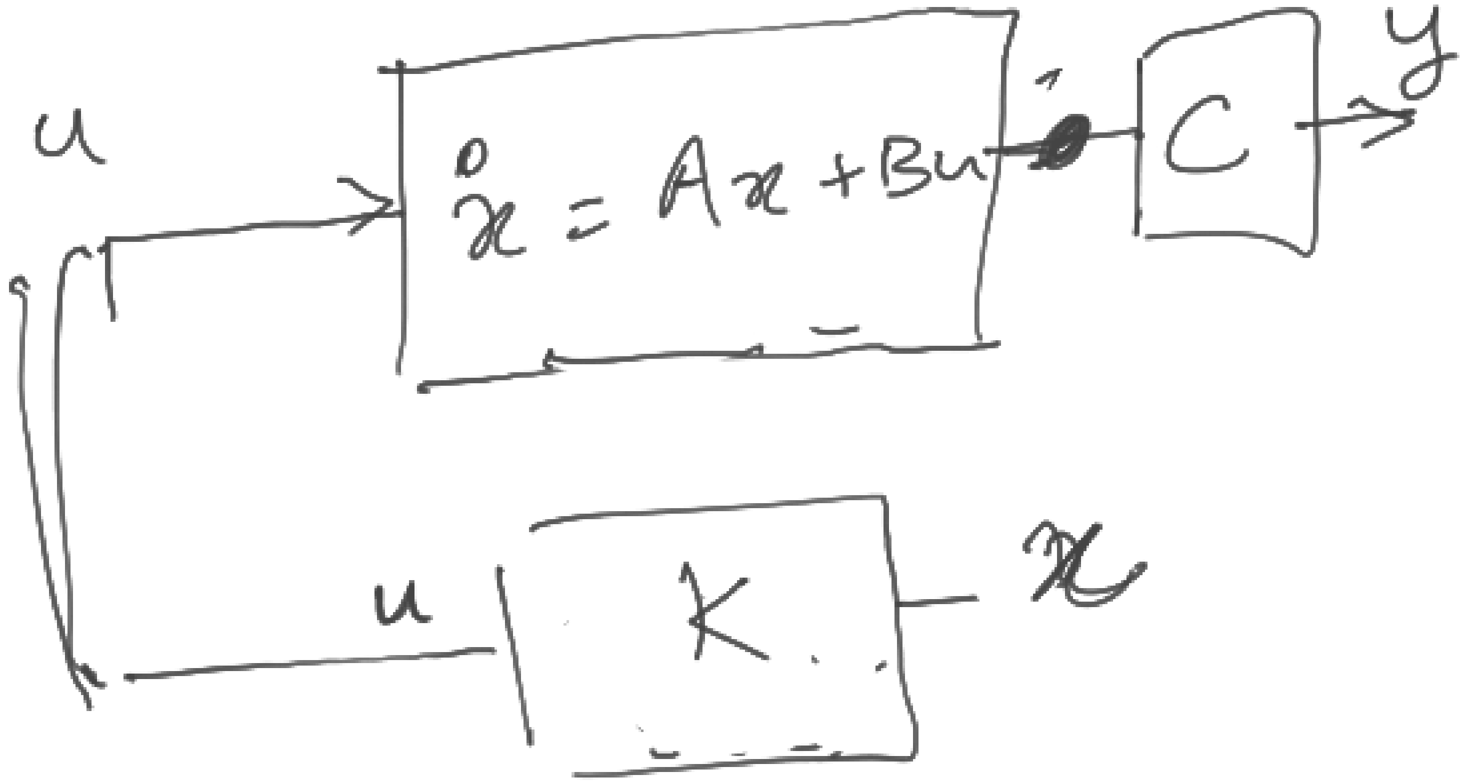
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right]$$

$x \to 0$ as $t \to \infty$

Can we reconstruct x?

$\Uparrow$



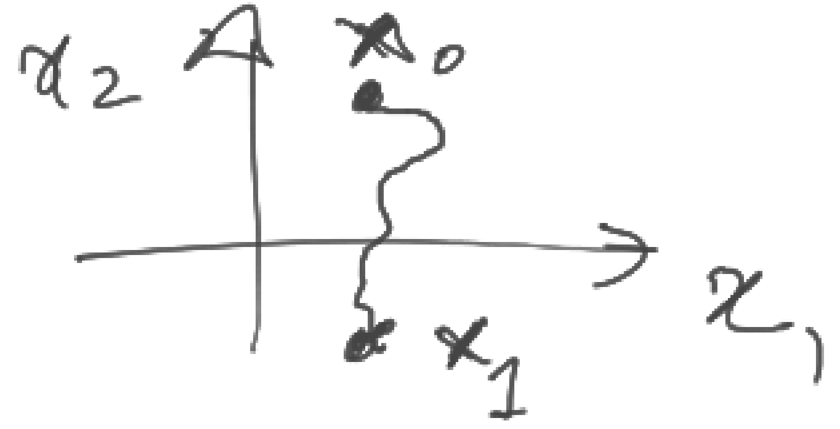
$$\boxed{\dot{x} = Ax + Bf(x)}$$

$x = 0$ to be a asymptotically stable eq. pt.

$$\dot{x} = (A + BK)x$$

Can we choose K s.t. $\Lambda(A + BK) \subset \mathbb{C}^-$?

- Notion of Controllability



$$\dot{x} = Ax + Bu \qquad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}$$

The (A, B) is controllable if

$\forall\, x_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$.

$\exists\, u(t) \in$ set of piecewise continuous functions

s.t. $x(0) = x_0$ and $x(t) = x_1$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$x_1 = \boxed{e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}.$$

Let x_0, x_1 be specified arbitrary.

$$u(\tau) = B^T e^{-A^T\tau} \left[\int_0^t e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau \right]^{-1} \times \left(e^{-At} x_1 - x_0 \right)$$

works if $\int_0^t e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau$ is invertible.

Thm: The following statement are equivalent.

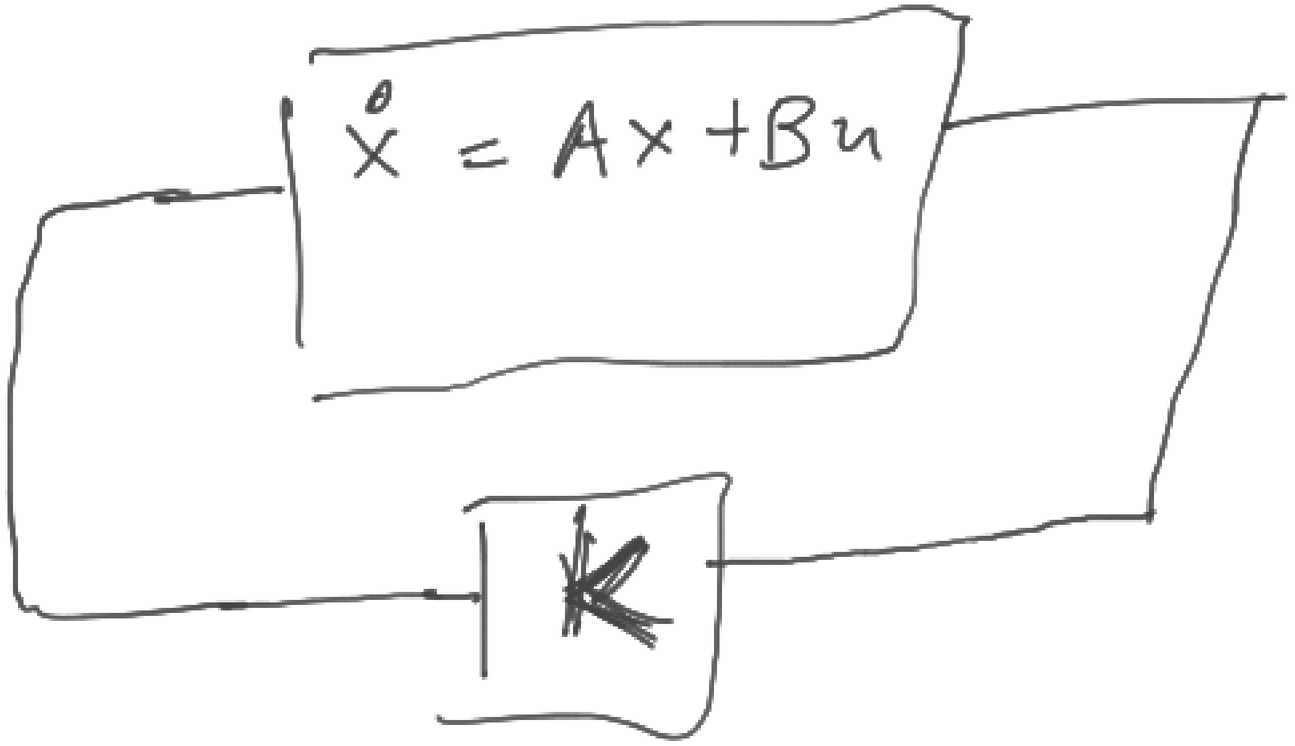
(i) (A, B) Controllable

(ii) $\int_0^t e^{-A\tau} B B^T e^{-A^T\tau} \, d\tau$ is invertible for all $t > 0$.

(iii) $\text{Rank}\left[B \;\; AB \;\; A^2B \; \dots \; A^{n-1}B\right] = n$

(iv) $\text{Rank}\left[\lambda I - A, \;\; B\right] = n \quad \forall \lambda \in \Lambda(A)$

(PBH test) Popov-Belevitch-Hautus test controllability



Can we select $\underline{K}$ s.t. eigenvalues of $\underline{A+BK}$ can be placed arbitrarily?

$$\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = \det(\lambda I - A)$$

Thm: Let (A, B) be a controllable pair

and $\underline{p(\lambda)} = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0$.

$\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + p_0$

$$\Downarrow$$

$\exists K$ s.t. $\chi_{\underline{(A+BK)}}(\lambda) = p(\lambda)$

proof: $B \in \mathbb{R}^{n \times 1}$ (single input case)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad (\underline{A}, \underline{B})$$

$$\operatorname{rank}\begin{bmatrix} B, & AB, & A^2B, & A^3B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -a_3 & * & * \end{bmatrix} = 4$$

$$\underline{\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0}$$

$$\dot{x} = Ax + Bu \qquad \xrightarrow{T} \qquad \dot{z} = \bar{A}z + \bar{B}u$$

$$y = Cx \qquad\qquad\qquad\qquad y = \bar{C}z.$$

$$z = \underline{T}x$$

$$\dot{z} = T\dot{x} = TAx + TBu.$$

$$\dot{z} = \underbrace{TAT^{-1}}_{\bar{A}}z + \overline{TB}u \qquad TB = \bar{B}$$

$$y = \underbrace{CT^{-1}}_{\bar{C}}z. \qquad \bar{A} = TAT^{-1}$$

$$\bar{A} = TAT^{-1} \qquad A = T^{-1}\bar{A}T$$

$$(A, B)$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0+\bar{k}_1 & -a_1+\bar{k}_2 & \cdots & & -a_{n-1}+\bar{k}_n \end{bmatrix} \;\;\text{"}$$

$$\bar{A} + \bar{B}\bar{k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ \bar{k}_1 & \bar{k}_2 & \cdots & & \bar{k}_n \end{bmatrix}$$

$$\underline{k = \bar{k}\, T^{-1}}$$

$$\bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{B}\bar{K} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n \end{bmatrix}$$

$$K = \bar{K} T$$

$$\underline{(A + BK)} = \left(T^{-1}\bar{A}T + T^{-1}\bar{B}\bar{K}T\right)$$

$$= T^{-1}\left(\underline{\bar{A} + \bar{B}\bar{K}}\right)T$$

$$\chi_{(\bar{A}+\bar{B}\bar{K})}(\lambda) = \chi_{(A+BK)}(\lambda)$$

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}_{3\times 1}, \quad \chi_A(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3)$$

$$\text{Rank} \begin{bmatrix} 1/2 & -1/2 & +1/2 \\ 1/2 & -1 & +2 \\ 1/2 & -3/2 & +9/2 \end{bmatrix} = 3$$

$\to C$

$$\text{Rank} \begin{bmatrix} \lambda+1 & 0 & 0 & 1/2 \\ 0 & \lambda+2 & 0 & 1/2 \\ 0 & 0 & \lambda+3 & 1/2 \end{bmatrix} = 3 \quad \text{for all } \lambda \in \{-1, -2, -3\}$$

$\Rightarrow (A, B)$ controllable.

Choose $u = \underline{K} x$ s.t $\chi_{(A+BK)}(\lambda)$

$$= (\lambda + 100)(\lambda^2 + 8\lambda + 32)$$

$$= \lambda^3 + 108\lambda^2 + 832\lambda + 3200.$$

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix}_{3\times 3}$$

$$T = C C^{-1}$$

$$\bar{A} = T^{-1} A T$$

$$\bar{B} = T^{-1} B$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{A} + \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6+\bar{K}_1 & -11+\bar{K}_2 & -6+\bar{K}_3 \end{bmatrix}, \quad \chi_{(\bar{A}+\bar{B}\bar{K})}(\lambda)$$

$$= \lambda^3 + (\bar{K}_1 - 6)\lambda^2 + (\bar{K}_2 - 11)\lambda + \bar{K}_3 - 6$$

$\bar{K}_1 = 114.$

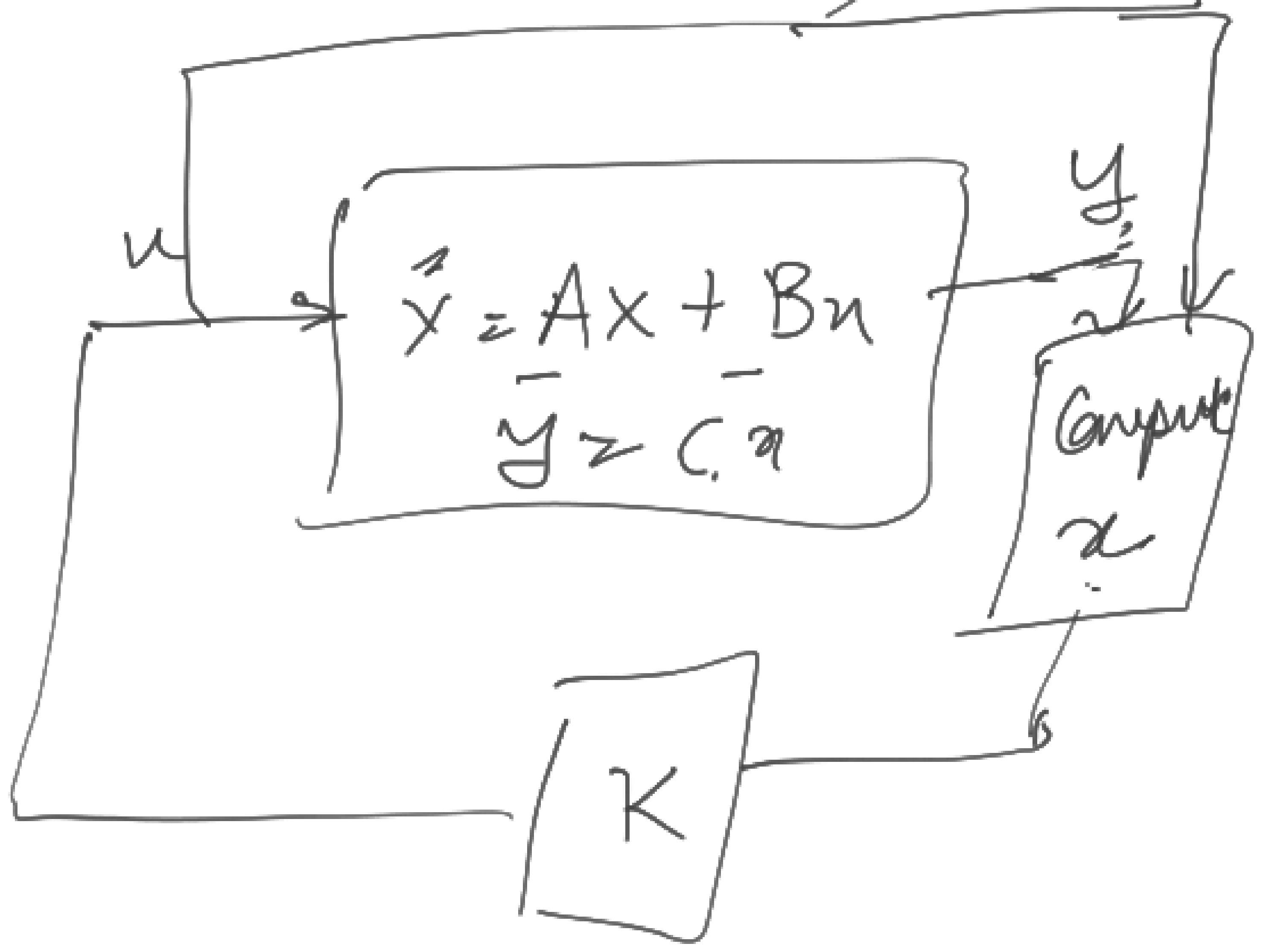
$\bar{K}_2 = 843$, $\bar{K} = [3194 \quad 821 \quad 102]$

$\bar{K}_3 = 3206$

$\bar{A} + \bar{B}\bar{K}$ will have desired Characteristic Polynomial.

$K = \bar{K}T^{-1}$

place $(A, B, p(\lambda)) \rightarrow K$



given y measurements

$\downarrow$ $A, B, C, u(t)$

Can we compute $x(t)$?

(observability!)

Next class.

Pole Placement based State feedback.