Distributed Computation of Minimum Time Consensus for Multi-Agent Systems

Ameer Mulla, Deepak Patil, Debraj Chakraborty
IIT Bombay, India

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Minimum time consensus

There are N double integrator agents with bounded inputs.

- **Goal**: Achieve consensus in minimum time.
- **Strategy**: Computation (in advance) of the consensus state.
- **Key result**: Computation can be distributed (By Helly's Theorem).
- **Tool**: Attainability set.

Outline

Problem Statement

Assume N identical agents with double integrator dynamics

$$\dot{x}_i(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x_i(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u_i(t) \qquad i = 1, \ldots, N$$

with $x_i(t) = \begin{bmatrix} r_i(t) \\ v_i(t) \end{bmatrix}$, $x_i(0) = x_{i0} = \begin{bmatrix} r_{i0} \\ v_{i0} \end{bmatrix}$ and $|u_i(t)| \leq 1$.

Problem

Find $\bar{x}$ and min $\bar{t}$ such that, for all i, j

- $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \bar{t}$
- $x_i(\bar{t}) = x_j(\bar{t}) = \bar{x}$ and $x_i(t) = x_j(t)$ for all $t \geq \bar{t}$

Definitions

Attainable Set from point p at time t

The set of all points that an agent can reach from initial condition p at time t using admissible input i.e. $|u_i(t)| \leq 1$

$$\mathscr{A}_p(t) = \left\{ x : x = e^{At} p + \int_0^t e^{A(t-\tau)} b u(\tau) d\tau, \ \forall u(t) : |u(t)| \leq 1 \right\}$$

- For double integrator $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
- Boundary of attainability set

$$x(t) = e^{At} p \pm \int_0^{t_s} e^{A(t-\tau)} B d\tau \mp \int_{t_s}^{t} e^{A(t-\tau)} B d\tau \ ; \ 0 \leq t_s \leq t < \infty$$

- Quadratic in t and t_s

Definitions

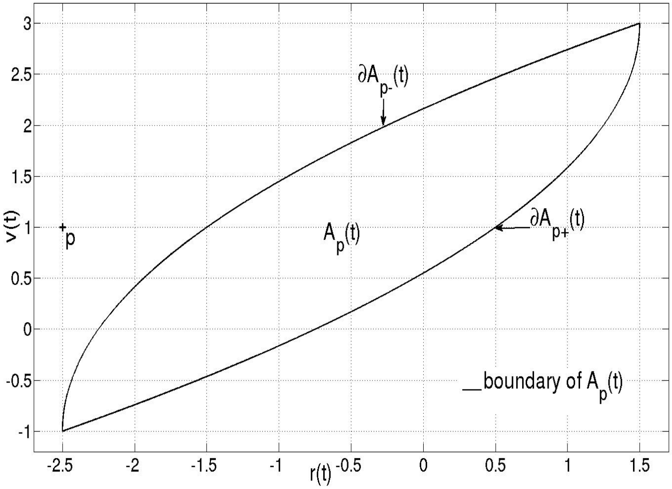

Figure: $\mathscr{A}_p(t)$

Outline

Way to Consensus

- Possible if attainable sets of all agents at time t intersect
- Let $\bar{t}$ be minimum t such that $\bigcap_{1 \leq i \leq N} \mathscr{A}_i(t) \neq \phi$ $(\mathscr{A}_i(t) := \mathscr{A}_{x_{i0}}(t))$
- Requires solution of coupled polynomial equations and inequalities
- Computation cannot be distributed

Main tool:Helly's Theorem

Helly's Theorem

F a finite family of convex sets in $\mathbb{R}^n$
Every $n+1$ members of F have a point in common

$$\Updownarrow$$

all the members of F have a point in common

- $\mathscr{A}_i(t) \subseteq \mathbb{R}^2$ is convex for all i and t
- Attainability set of every triplet intersect $\Longleftrightarrow$attainability set of all intersect
- Allows to distribute computation for all $\binom{N}{3}$ triplets evenly among N agents

Way to Consensus

- Let $\bar{t}_{ijk}$: Minimum time to consensus for agents $\{a_i, a_j, a_k\}$
- Then, $\bar{t} \geq \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk}$

Lemma

If $\mathscr{A}_i(t') \cap \mathscr{A}_j(t') \cap \mathscr{A}_k(t') \neq \phi$ for some $t' > 0$, then $\mathscr{A}_i(t) \cap \mathscr{A}_j(t) \cap \mathscr{A}_k(t) \neq \phi$ for all $t \geq t'$

- Thus, $\bar{t} = \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk}$
- Computing closed form expressions for three agents case suffices

Consensus: Three agents

- Let $\{a_p, a_q, a_r\}$ be the triple of agents and $\bar{t}_{pqr} = \max_{1 \le i,j,k \le N} \bar{t}_{ijk}$.

Theorem

The minimum time to consensus $\bar{t} = \bar{t}_{pqr}$
The corresponding consensus point $\bar{x} = \bar{x}_{pqr}$

Theorem

For two different $\{a_p, a_q, a_r\}$ *and* $\{a_{p'}, a_{q'}, a_{r'}\}$
If $\bar{t}_{pqr} = \bar{t}_{p'q'r'} = \max_{1 \le i,j,k \le N} \bar{t}_{ijk}$ *then* $\bar{x}_{pqr} = \bar{x}_{p'q'r'}$

Consensus: Three agents

- Two ways towards consensus of three agents
- For a triple $\{a_i, a_j, a_k\}$, $\bar{t}_{ijk} \geq \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$
  - WLOG let $\bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$.
  - Two cases:
    - **Case 1:** $\bar{t}_{ijk} = \bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ i.e $\bar{x}_{ij} \in \mathscr{A}_k(\bar{t}_{ij})$
    - **Case 2:** $\bar{t}_{ijk} > \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ i.e. $\bar{x}_{ij} \notin \mathscr{A}_k(\bar{t}_{ij})$

Consensus: Three agents

Case 1: $\bar{t}_{ijk} = \bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ i.e $\bar{x}_{ij} \in \mathscr{A}_k(\bar{t}_{ij})$

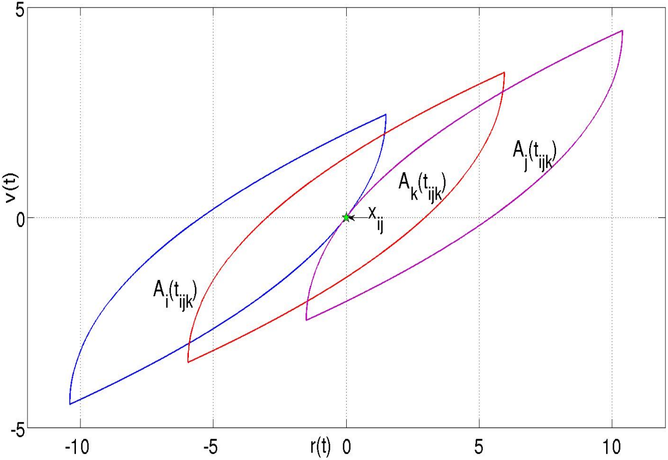

Figure: Case 1

Consensus: Three agents

Case 2: $\bar{t}_{ijk} > \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ i.e. $\bar{x}_{ij} \notin \mathscr{A}_k(\bar{t}_{ij})$

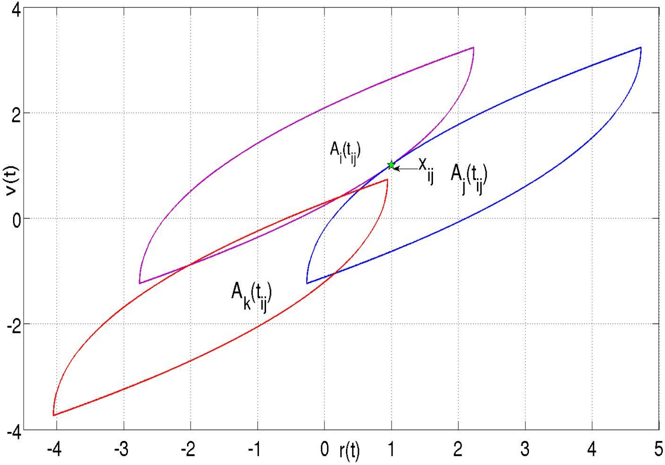

Figure: Case 2: $\bar{x}_{ij} \notin \mathscr{A}_k(\bar{t}_{ij})$

Consensus: Three agents

Case 2: $\bar{t}_{ijk} > \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ i.e. $\bar{x}_{ij} \notin \mathscr{A}_k(\bar{t}_{ij})$

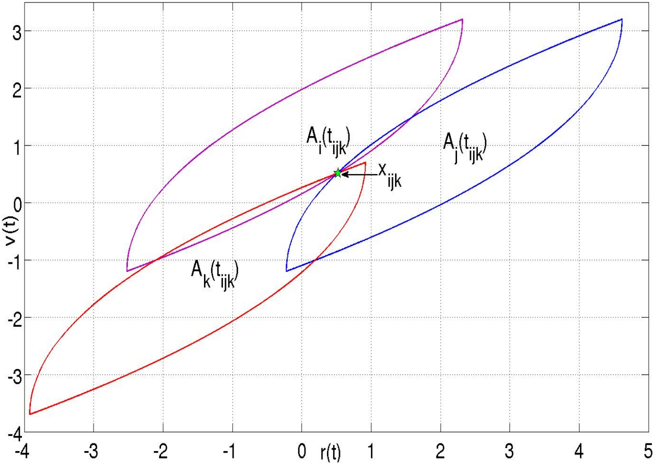

Figure: Case 2: Consensus of three agents

Role of Agents

- All N agents do not play a role in computation of $\bar{t}$ and $\bar{x}$
- Triplet$\{a_i, a_j, a_k\}$ for which $\bar{t} = \bar{t}_{ijk}$ determine $\bar{x}$ and $\bar{t}$
- For these agents, $\bar{x} \in \partial \mathscr{A}_i(\bar{t})$
- Must use minimum-time control law to reach $\bar{x}$ at $\bar{t}$
  In other words, $x_{i0}, x_{j0}, x_{k0} \in \partial R_{\bar{x}}(\bar{t})$
- For all other agents (say a_q), $\bar{x} \in \text{int}\,\mathscr{A}_q(\bar{t})$,
- Must use a control law that allow them to reach $\bar{x}$ at $\bar{t}$
  May use scaled down version of the input bounds

Six Agents Consensus

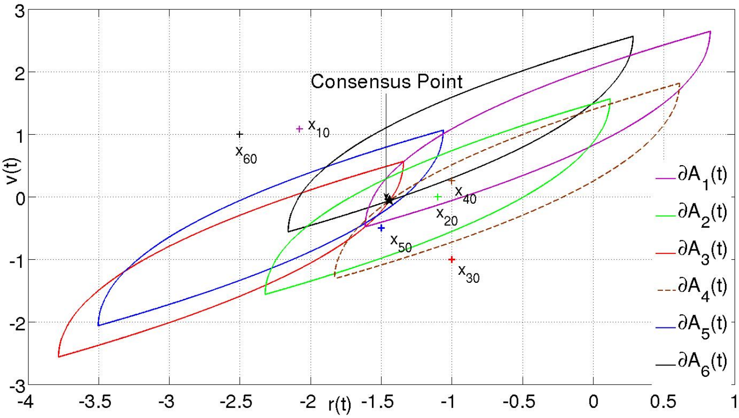

Figure: Attainable Sets of Agents at $\bar{t}$

Six Agents Consensus

Definition

The set of all initial conditions from which an agent can reach p at time t using admissible input i.e. $|u_i(t)| \leq 1$

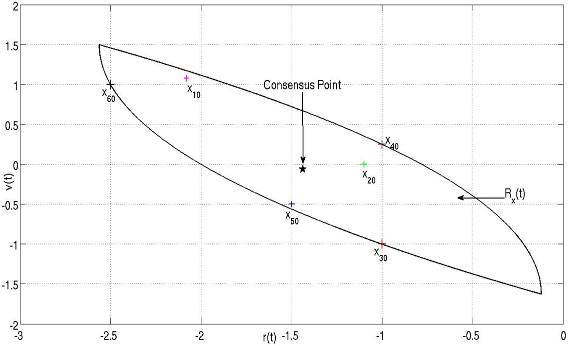

Figure: Reachable Set to $\bar{x}$ at $\bar{t}$

Minimum time consensus: Single integrator

- N agents with single integrator dynamics $\dot{x}_i = u_i$, $x_i \in \mathbb{R}$ and $|u_i| \leq 1$
- Attainable set of agents are intervals in $\mathbb{R}$ given by $x_i(t) = x(0) + \int_0^t u_i(t)dt$
- Initial condition p then at time t attainable set is $[p-t, p+t]$
- If every pair of the intervals intersect, then all the intervals intersect
- The minimum time is $\bar{t}_{ab} = \max_{ij} \bar{t}_{ij} = \frac{\max_{ij} |x_{i0} - x_{j0}|}{2}$ for some agents a and b (endpoints)
- That is $\bar{t} = \frac{\max_{ij} |x_{i0} - x_{j0}|}{2}$ and $\bar{x} = \frac{\max_{ij} (x_{i0} + x_{j0})}{2}$.
- For all other agents, any input that drives $x_i(t)$ to $\bar{x}$ in time $t \leq \bar{t}_{ab}$
- For example, $u_i(t) = sign(\bar{x} - x_i)$

More can be said

- Single integrators interacting over a connected undirected graph $G = (V, E)$
- Prior computation of the $\bar{t}$ and corresponding $\bar{x}$ is not required
- Control law $u_i(t) = sign(\sum_{(i,j) \in E}(x_j - x_i))$ ensures proper input
- Agreement dynamics: $\dot{p}(t) = -sign(Lp(t))$, where L is Laplacian matrix
- Consensus occurs at $\bar{x} = \frac{\max_{ij}(x_{i0} + x_{j0})}{2}$

Conclusions

- An algorithm for computing minimum time consensus
- Computation is to be done for each possible triplet.
- Can be distributed evenly among each agent
- Final values of $\bar{x}$ and $\bar{t}$ are broad-casted to all
- Agreement becomes easier for single integrator agents