Switching Surfaces and Null-controllable Region of a class of LTI systems using Gröbner basis

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***Abstract*— The problem of time optimal *feedback* control of a single input, continuous time, linear time invariant (LTI) system is considered. The control input is constrained to obey $|u(t)| \leq 1$. It is known that the solution to this problem is bang-bang with the input switching between the extreme values ± 1 according to some "switching surfaces" in the state space. It is shown that for a class of LTI systems, these switching surfaces can be represented as semi-algebraic sets and a method to construct these switching surfaces using Gröbner basis, is proposed. Numerical simulations demonstrate the effectiveness of this feedback control strategy.**

I. INTRODUCTION

The time optimal control problem for linear time invariant (LTI) systems involves finding a control (from a constrained set of inputs e.g. $|u| < 1$) which transfers a given initial state to the origin of the state space in the minimum possible time. This important problem is usually solved by the Pontryagin's maximum principle (PMP) [1], which shows that the optimal solution is bang-bang and provides necessary conditions characterizing the optimal input. These necessary conditions, when solved (either analytically or by numerical ordinary differential equation solvers) usually provide the optimal open loop control input. However, a feedback solution is highly desirable mainly for the following reasons: (a) A feedback controller would transfer the system states to the origin from any initial condition (in the admissible initial condition set) in the minimum time possible. Hence such a controller would be easier to implement in any real time system since it would not involve recalculation of the entire input signal for each distinct initial condition (b) A feedback solution would be robust to uncertainties and disturbances and drive the system to origin repeatedly even if the system is forced away from the optimal trajectory due to external disturbances. Several attempts have been made to analyze this problem and the feedback solution structure was discovered by [2] and [3]. However, a general method for constructing the feedback solution seems to be unavailable. In this article, we provide a partial solution to this synthesis problem, by providing an algorithmic method for constructing the time-optimal feedback function for controllable linear systems with rational positive eigenvalues.

Consider a n^{th} order linear time invariant system:

$$\dot{x(t)} = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{1}$$

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where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state vector and the input $u(t)$ is constrained within a normalized interval $|u(t)| \leq 1$. Denote the set of admissible inputs $U = \{u(t) : |u(t)| < 1\}$. We assume that the initial condition x_0 of system (1) is in the set $X_0 \subset \mathbb{R}^n$. The objective of the control law is to force the state trajectory $x(t)$ from x_0 to origin in the minimum possible time. The PMP predicts that the optimal input must switch between the maximum and minimum admissible values ± 1 at not more than $n-1$ specified time instants called switching instants (see e.g. [1] and [4]), which in turn completely characterizes the optimal input. Alternatively, the optimal switching can also be computed according to the so called "switching surfaces" in the state space. The optimal input switches between the extreme admissible values, when the state trajectory intersects these switching surfaces. Hence, if the switching surfaces are known, a time optimal feedback strategy can be constructed just by checking the location of the current states with respect to these switching surfaces.

Switching surfaces are mentioned in [1], and constructed explicitly for some examples of second-order systems. Analysis of such surfaces is done by [2] in great details. The existence of time optimal feedback control for scalar-input systems is shown and the structure and the properties of the switching surfaces are described. In [3], these results are generalized for multiple-input systems. There are several articles considering synthesis of a these surfaces for limited cases e.g. for second-order LTI systems [1][4][5][6] and n-integrator chains [7]. But in general there is no method to synthesize such surfaces for n^{th}-order system. In this article, for controllable systems with rational positive eigenvalues, we show that the resulting switching surfaces can be described by semi-algebraic sets. Moreover, using Gröbner basis based implicitization algorithm [8], we derive explicit expressions in terms of polynomials and rational functions describing the switching surfaces. Using these expressions, a feedback logic for optimal switching is synthesized.

It is also well known [1], that for unstable systems only those initials conditions belonging to a particular set (often called the null-controllable set) can be steered to the origin. As a natural extension to the proposed method we show that the null-controllable region is also a semi-algebraic set and an algorithm for the computation of this set is provided.

Formally, the primary research problems studied in this work can be framed as follows:

Problem 1: Find a feedback function $f : X_0 \to [-1, 1]$ which solves the following problem: $\min_f t_f$ such that the solution to (1) satisfies $x(t_f) = 0$ for some $t_f \in [0, \infty)$.
The class of admissible feedback functions can be assumed

to be piecewise continuous with no more than $n-1$ discontinuities, since it is well known that the optimal solution is of this class [1].

Problem 2: Find the maximal set X_0 from which there exists a feedback function f such that $x(t_f)=0$ at some $t_f \in [0,\infty)$.

The remaining article is arranged as follows. In section II the parametric formulation of the switching surface and null-controllable region is discussed. In section III we apply Gröbner basis based elimination technique to eliminate parameters and obtain formulations of the switching surfaces and the null-controllable region, solely in terms of the state-variables. Using the switching surface a switching algorithm is proposed, which will achieve time-optimal state transfer to origin. The proposed method is demonstrated by an example of third order unstable system.

II. PRELIMINARIES

The theory of terminal manifolds and switching locus [2] is reviewed briefly in this section. Consider an LTI system described by (1). We assume A to be anti-stable with rational eigenvalues ($\lambda(A) \in \mathbb{Q}$) and the pair $\{A,B\}$ to be controllable. The solution to (1) is:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

The states should reach the origin eventually. Hence, (2) becomes

$$x_0 = -\int_0^t e^{-A\tau}Bu(\tau)d\tau \tag{3}$$

If the input $u(t)$ is allowed to vary over the set of admissible inputs U, then (3) defines all initial conditions which can be steered to origin in time t. We will call such initial conditions as null-controllable states in time t and they are characterized by following set:

$$R(t) = \{x : x = \int_0^t e^{-A\tau}Bu(\tau)d\tau,\ \forall u(t) \in U\} \tag{4}$$

Thus, the set of all null-controllable states is,

$$X_0 = \bigcup_{t\in[0,\infty)} R(t)$$

The set X_0 can be written in a parametric form by using the following theorem [2].

Theorem 1: For any $k = 1,\dots,n$ and a sequence $0 < t_1 < \dots < t_k$, consider the control u on $[0,t_k]$ with values ± 1 alternating in intervals $[0,t_1),\dots,[t_{k-1},t_k)$. Then u is an optimal control, so that the point,

$$x = \pm\left(-\int_0^{t_1} + \int_{t_1}^{t_2} - \dots + (-1)^{k-1}\int_{t_{k-1}}^{t_k}\right)e^{-A\tau}Bd\tau$$

has t_k as the least time required to reach origin. Conversely, every optimal control on any interval $(0,\theta)$ where $0 < \theta < \infty$ is of the described form.

To characterize all states which can be steered to origin in time t_k using a bang-bang input with k-switches, we define following functions:

$$\left.\begin{aligned} F_k^+(t_1,\dots,t_k) &= \Big(-\textstyle\int_0^{t_1} + \int_{t_1}^{t_2} - \\ &\quad \dots + (-1)^{k-1}\textstyle\int_{t_{k-1}}^{t_k}\Big)e^{-A\tau}Bd\tau \\ F_k^-(t_1,\dots,t_k) &= -F_k^+(t_1,\dots,t_k) \end{aligned}\right\} \tag{5}$$

Observe that there is a physical binding on t_i, $i = 1,2,\dots,k$ i.e $0 \le t_1 \le t_2 \le \dots \le t_k$. Let, $V_k = \{(t_1,t_2,\dots,t_k) : 0 \le t_1 \le t_2 \le \dots \le t_k < \infty\}$ for $k = 1,\dots,n$.

Definition 1: The set of states, which can be steered to origin in less than k-switches with $u=+1$ to begin with is denoted by M_k^+ and can be defined as follows:

$$M_k^+ = \{x : x = F_k^+(v), \forall v \in V_k\}$$

Observe that $M_k^- = -M_k^+$.

The following is a consequence of Theorem 1:

Corollary 1: For a system defined by (1), a bang-bang control on an interval $[0,\infty)$ is time-optimal iff it has atmost $n-1$ discontinuities.

Thus the set of all states which can be steered to origin in $n-1$ switches can be defined as follows:

$$M_n = \{x : x = F_n^{\pm}(v),\ \forall v \in V_n\} \tag{6}$$

Lemma 1: $M_n = X_0$.

Proof: Clearly $M_n \subset X_0$. Now, it is known that, for a system described by (1), if there exists at least one control which transfers the state x_0 to origin, there also exists an optimal control which transfers the state x_0 to origin [1]. Hence $X_0 \subset M_n$ implying $M_n = X_0$. ■

Lemma (1) is extremely useful for our purposes, since it states that the null-controllable region with only bang-bang inputs (with at most $n-1$ switches) is actually the entire null controllable region. Hence any state in X_0 can be driven to the origin with such a bang-bang control input. Moreover, such an input is always time optimal.

It is further shown in [2], that the set X_0 is divided symmetrically into two disjoint parts M_n^+ and M_n^- by M_{n-1}, which itself is divided into two disjoint parts M_{n-1}^+ and M_{n-1}^- by M_{n-2} and so on. In general we can write,

$$M_k = M_k^+ \cup M_k^- \forall k = 1,\dots,n$$

and

$$X_0 = M_n^+ \cup M_n^-$$

Clearly, $M_0 \subset M_1 \subset \dots \subset M_n$. We illustrate the structure of M_k for a simple 2-state system:

Example 1: For a second order LTI system $\dot{x} = Ax + Bu$ with $A = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, $B = \begin{bmatrix}1\\1\end{bmatrix}$ and $|u| \le 1$, the corresponding structure of X_0, which is divided in two parts namely M_2^+ and M_2^- by $M_1^+ \cup M_1^-$ is as shown in the figure 1.

Based on this structure of X_0, an iterative switching logic can be defined as follows that drives any state in X_0 to the origin in minimum time. For any initial condition $x_0 \in X_0$, time-optimal switching should ideally induce the following

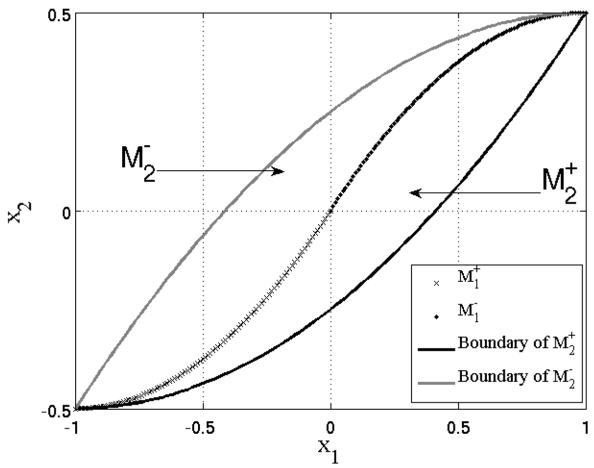

Fig. 1. $X_0 = M_2^+ \cup M_2^-$ for case of example 1

set of events. We assume $x_0 \in M_n^+$ for illustration. A similar sequence would be valid for $x_0 \in M_n^-$ with opposite signs:

- Input $u = 1$ pushes x from M_n^+ to the manifold M_{n-1}^-.
- As soon as $x \in M_{n-1}^-$, switch 2 pushes x to M_{n-2}^+

$\vdots$

- Input at $(n-1)^{th}$ switch pushes x from M_1^+(if n is odd) or M_1^-(if n is even) to the origin.

On inspection of this logic, it is evident that we need to compute M_k^+ and M_k^- to implement feedback based switching. Before we give an algorithm for this computation in the next section, we note the following:

Note 1: Recall that A was assumed to have rational positive eigenvalues. Moreover, for the construction of the switching surfaces M_k $(k = 1,...,n-1)$ and for using the switching logic defined above, we can assume A to be diagonal without loss of generality. This is because, for any real similarity transformation $\widehat{x} = Tx$ on a system, the corresponding set $\widehat{M}_k^+ = \{\widehat{x} = Tx : \ x \in M_k^+\}$ and similarly $\widehat{M}_k^- = \{\widehat{x} = Tx : x \in M_k^-\}$ [2]. Thus, computation of M_k^+ and M_k^- for the diagonalized system is enough to compute the corresponding M_k^+ and M_k^- for all similar systems.

III. Construction of the Region of Null-controllability(X_0) and Switching Surface

Using a simple substitution of variables, we show first that the switching surfaces M_k , $k = 1,...,n-1$ and the null-controllable set $X_0 = M_n$ can be represented parametrically as polynomials. These polynomials are then implicitized using Gröbner basis techniques to get polynomials equalities and inequalities involving only the state variables. These semi-algebraic sets define both the switching surfaces and the null-controllable region.

A. Polynomial Representation

All $x \in M_k$ are characterized by function F_k^+ or F_k^- defined in (5), which takes $t_1,\ldots,t_k$ as arguments. Since we can assume A to be diagonal without loss of generality (see Section II: Note 1 above), we can alternatively write each component of state x $(x_i,\ i = 1,..,n)$ as some other function, denoted as $f_{ki}^{\pm}$, which takes arguments $e^{-\lambda_i t_1},\ldots,e^{-\lambda_i t_k}$ for all $i = 1,\ldots,n$. Here f_{ki}^+ corresponds to F_k^+ and f_{ki}^- to F_k^-. Thus,

$$x_i = f_{ki}^{\pm}(e^{-\lambda_i t_1},\ldots,e^{-\lambda_i t_k})\ \forall i = 1,\ldots,n \text{ and } \lambda_i \in \lambda(A).$$

Now recall that the eigenvalues of A are assumed to be rational. Let us denote the denominators of λ_i by d_i, and $l = lcm(d_1,\ldots,d_n)$ and by substituting $z_i = e^{-\frac{t_i}{l}}\ \forall i = 1,\ldots,k$ we again write

$$x_i = f_{ki}^+(z_1^{p_i},\ldots,z_k^{p_i})$$

where $p_i \in \mathbb{Z}_+$ $(i = 1,...,n)$ and f_{ki}^+ is a polynomial in z_j, $(j = 1,\ldots,k)$. The inequality $0 \le t_1 \le t_2 \le \ldots \le t_k < \infty$ in terms of $z_1,\ldots,z_k$ becomes $0 < z_k \le z_{k-1} \le \ldots \le z_1 \le 1$. Now we have a polynomial representation for M_k^+ and M_k^- which is as follows:

$$\begin{aligned} M_k^+ =& \{(x_1,\ldots,x_n) : x_i = f_{ki}^+(z_1^{p_i},\ldots,z_k^{p_i}) \\ & \forall i = 1,\ldots,n, 0 < z_k \le z_{k-1} \le \ldots \le z_1 \le 1\} \\ M_k^- =& -M_k^+ \end{aligned} \tag{7}$$

The above expressions for M_k^+ and M_k^- involves $z_1...z_k$ and hence cannot be used for state based switching. It would be convenient to eliminate the variables z_1, z_2,..., z_k from the representation of M_k^+ and M_k^- and find an alternate representation of M_k^+ and M_k^- in terms of only the state variables $x_1,\ldots,x_n$. Such a representation can be used directly for state-feedback based switching of the input values between ± 1. For this purpose, we will use an implicitization method based on construction of Gröbner bases to eliminate $z_1\ldots,z_k$ and form an implicit representation for the set $M_k^{\pm}$. Detailed treatment of Gröbner basis based implicitization can be found in [8].

B. Switching Surface

Recall that the $n-1$ dimensional switching surface is described by $M_{n-1}^+ \cup M_{n-1}^-$. Using (7), for $k = n-1$,

$$x_i = f_{n-1,i}^+(z_1^{p_i},\ldots,z_{n-1}^{p_i})\ \forall i = 1,\ldots,n$$

along with the inequality

$$0 \le z_{n-1} \le z_{n-2} \le \ldots \le z_1 \le 1 \tag{8}$$

parametrically describes the switching surface. Thus,

$$\begin{aligned} M_{n-1}^+ =& \{(x_1,\ldots,x_n) : x_i = f_{n-1,i}^+(z_1^{p_i},\ldots,z_{n-1}^{p_i}) \\ & \forall i = 1,\ldots,n, 0 \le z_{n-1} \le z_{n-2} \le \ldots \le z_1 \le 1\} \end{aligned} \tag{9}$$

To eliminate z_k, $\quad k = 1,\ldots,n-1$ from (9) we will follow the implicitization procedure

1) Form an ideal $J_{n-1}^+ = \langle x_1 - f_{n-1,1}^+,\ldots,x_n - f_{n-1,n}^+\rangle$.
2) Compute Gröbner basis G_{n-1}^+ of J_{n-1}^+ w.r.t. lexicographic ordering as $z_1 \succ z_2 \succ \ldots \succ z_{n-1} \succ x_1 \succ \ldots \succ x_n$.
3) The element $g_{n-1}^+ \in G_{n-1}^+ \cap \mathbb{Q}[x_1,\ldots,x_n]$ defines the smallest variety containing the parametric representation $x_i = f_{n-1,i}^+$.

In general, the variety defined by g^+_{n-1} may be larger than the parameterized surface defined by M^+_{n-1}. However, by uniqueness of time-optimal control, it follows that for given $x \in X_0$ there are unique switching instants $t_1,\ldots,t_{n-1}$ and hence $z_1,\ldots,z_{n-1}$. Thus all variables $x_1,\ldots,x_n \in X_0$ satisfying $g^+_{n-1}(x_1,\ldots,x_n) = 0$ are always extendable to entire variety. To describe M^+_{n-1} completely and ensure that $x_1,\ldots,x_n \in X_0$, we need to impose condition (8). For that we need to rewrite inequality (8) in terms of $x_1,x_2,\ldots,x_n$. This is accomplished by solving each z_k in terms of $x_1,x_2,\ldots,x_n$ and then imposing inequality (8). The technique described below for this purpose, works identically for each z_k, $k = 1,...,n-1$.

Let $G^+_{z_k}$ be the Gröbner basis obtained by using ordering $.. \succ .. \succ \ldots \succ z_k \succ x_1 \succ \ldots \succ x_n$. Consider the set $G^+_{z_k} \cap \mathbb{Q}[z_k,x_1,\ldots,x_n]$: let there be m elements in this set and let us denote the elements by $g^+_1,g^+_2,...,g^+_m$. Clearly, each of these elements is a polynomial in the variables $(z_k,x_1,\ldots,x_n)$ and the equations $g^+_i = 0\ \forall i = 1,\ldots,m$ can be solved for z_k in terms of $(x_1,\ldots,x_n)$. For M^+_{n-1} we denote the respective z_k by z^+_k for $k = 1,\ldots,n-1$ and similarly for M^-_{n-1}. Thus, M^+_{n-1} can now be defined as follows:

$$M^+_{n-1} = \{(x_1,\ldots,x_n) : g^+_{n-1}(x_1,x_2,\ldots,x_n) = 0$$
$$0 \le z^+_{n-1} \le z^+_{n-2} \le \ldots \le z^+_1 \le 1\}$$

Using this expression, any point on the state space $(x_1,...,x_n)$ can be tested to check whether it belongs to the switching surface M_{n-1}. This is illustrated in Algorithm 1. Since switching takes place only when the state trajectory hits the switching surface, algorithm 1 can be used to synthesize the switching logic described in algorithm 2.

Algorithm 1 Switching Surface

if $z^+_{n-1} \le z^+_{n-2} \le \ldots \le z^+_1 \le 1$ **then**
  $SwitchSurf = g^+_{n-1}$
else
  if $z^-_{n-1} \le z^-_{n-2} \le \ldots \le z^-_1 \le 1$ **then**
    $SwitchSurf = g^-_{n-1}$
  end if
end if

Algorithm 2 Switching logic

Require: current $x(t)$ and past $u(t^-)$
  Execute Algorithm 1
  if Algorithm 1 is executed **then**
    $u = -sign(SwitchSurf)$
  else
    $u = u(t^-)$
  end if

Remark 1: It can be seen from algorithm 2, that the switching takes place only on the basis of the $n-1$ dimensional surface M_{n-1}, whereas the sequence of desired switching events described at the end of section II, seem to require all the lower dimensional switching surfaces M_{n-k} $(k > 1)$. This can be justified as follows.

Recall the set of switching events from section II. If $x_0 \in M^+_n$ then $u = +1$ will push it to M^-_{n-1}. Ideally *SwitchSurf* for M^-_{n-1} should instantaneously change sign when state trajectory intersects M^-_{n-1}. But, in practice or in simulation, instantaneous change of sign is not possible. Thus, the state trajectory would overshoot by a small amount say, $\delta \ge 0$ at first switch. The state-trajectory, which ideally should have stayed in M^-_{n-1} after hitting M^-_{n-1}, overshoots and enters M^-_n slightly (by δ) before the first switch occurs. Since the state is now in M_n the switching surface M_{n-1} can be used again for the second switch. This situation will repeat for all the $n-1$ switches and it can be shown that the generated trajectory remains arbitrarily close to the corresponding time-optimal trajectory. Moreover, by continuity of the vector field, it can be shown that the final value of the trajectory after $n-1$ switches will reach ε close to the origin where the magnitude of ε depends only on the overshoot δ. A complete proof of this result is omitted from this article due to lack of space. This fact is however verified through simulations below.

Remark 2: As mentioned above, the lower dimensional switching surfaces i.e. M_{n-k} for $k > 1$ are not required in practice for switching. However, one might compute all the switching surfaces easily from M_{n-1} itself. Then it is easy to alter algorithm 2 to incorporate switching based in the lower dimensional surfaces exactly as described in section II, The lower dimensional surfaces can be computed by substituting $z_k = 1$ in the inequality (8). This will force $z_i = 1$, $\forall i \le k-1$. Thus,

$$M^+_{n-k} = \{(x_1,\ldots,x_n) : g^+_{n-1}(x_1,x_2,\ldots,x_n) = 0$$
$$0 \le z^+_{n-1} \le z^+_{n-2} \le \cdots \le z^+_k = \cdots = z^+_1 = 1\}$$

Example 2: Consider a system defined as (1) with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The points on switching surface are described by M^+_2 and M^-_2. We have,

$$F^+_2(t_1,t_2) = \begin{bmatrix} 2e^{-t_1} - e^{-t_2} - 1 \\ e^{-2t_1} - \frac{1}{2}e^{-2t_2} - \frac{1}{2} \\ \frac{2}{3}e^{-3t_1} - \frac{1}{3}e^{-3t_2} - \frac{1}{3} \end{bmatrix}$$

We change the equation of F^+_2 by substituting $z_1 = e^{-t_1}, z_2 = e^{-t_2}$ into a polynomial. Thus,

$$F^+_2 = \begin{bmatrix} 2z_1 - z_2 - 1 \\ z_1^2 - \frac{1}{2}z_2^2 - \frac{1}{2} \\ \frac{2}{3}z_1^3 - \frac{1}{3}z_2^3 - \frac{1}{3} \end{bmatrix} \tag{10}$$

The above set of polynomials (10) describes the surface M^+_2. Form an ideal $J^+ = \langle x_1 - 2z_1 + z_2 + 1, x_2 - z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}, x_3 - \frac{2}{3}z_1^3 + \frac{1}{3}z_2^3 + \frac{1}{3}\rangle$. To eliminate z_1 and z_2 and form an implicit equation for the surface M^+_2, we use the elimination procedure described in section III-B. The Gröbner basis G^+

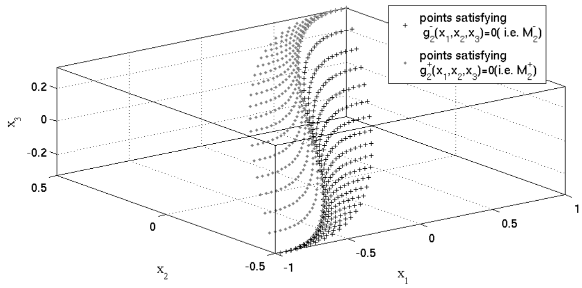

Fig. 2. Points on the Switching Surface

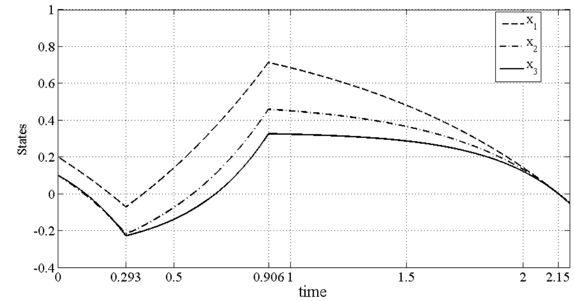

Fig. 3. State against time for initial condition $x_0 = (0.2, 0.1, 0.1)$

with ordering as $z_1 \succ z_2 \succ x_1 \succ x_2 \succ x_3$ for the ideal J^+ is[1],

$$G^+ = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle \quad \text{where} \tag{11}$$

$$\begin{aligned}
s_1 &= x_1^6 + 6x_1^5 - 6x_1^4 x_2 + 12x_1^4 - 24x_1^3 x_2 - 24x_1^3 x_3 + \\
&\quad 36x_1^2 x_2^2 - 72x_1^2 x_3 - 18x_1^2 + 72x_1 x_2^2 + 72x_1 x_2 x_3 \\
&\quad + 72x_1 x_2 - 36x_1 x_3 - 72x_2^3 - 72x_2^2 + 72x_2 x_3 - 18x_3^2 \\
s_2 &= 24z_2 x_2^3 + \ldots - 54x_2 + 63x_3^2 + 36x_3 \\
s_3 &= 9z_2 x_1 x_3 + \ldots + 12x_2 x_3 + 24x_2 - 15x_3 \\
s_4 &= 6z_2 x_1 x_2 + 3z_2 x_1 + 6z_2 x_2 - \ldots - 30x_2 + 21x_3 \\
s_5 &= 3z_2 x_1^2 + 6z_2 x_1 - 6z_2 x_2 + x_1^3 + 3x_1^2 - 6x_1 x_2 - 6x_2 + 6x_3 \\
s_6 &= z_2^2 - 2z_2 x_1 - 2z_2 - x_1^2 - 2x_1 + 4x_2 + 1 \\
s_7 &= 2z_1 - z_2 - x_1 - 1
\end{aligned}$$

Thus $g_2^+(x_1, x_2, x_3) = s_1$. Along with conditions $0 \le z_2 \le z_1 \le 1$ on z_1 and z_2, $g_2^+(x_1, x_2, x_3) = 0$ describes M_2^+. We observe s_5 contains only degree one terms in z_2. Thus, we can write an expression for z_2 as, $z_2 = \frac{-(x_1^3 + 3x_1^2 - 6x_1 x_2 - 6x_2 + 6x_3)}{(3x_1^2 + 6x_1 - 6x_2)}$. Similar expression for z_1 can be found from the Gröbner basis of the ideal J with ordering $z_2 \succ z_1 \succ x_1 \succ x_2 \succ x_3$, which is $z_1 = \frac{-(-x_1^3 - 3x_1^2 - 3x_1 + 3x_3)}{(3x_1^2 + 6x_1 - 6x_2)}$.

Therefore,

$$\begin{aligned}
M_2^+ = &\{(x_1, x_2, x_3) : g_2^+(x_1, x_2, x_3) = 0, \\
&0 < z_2 \le z_1 \le 1\}
\end{aligned} \tag{12}$$

One can obtain M_2^- just by replacing (x_1, x_2, x_3) as $(-x_1, -x_2, -x_3)$. Figure 2 shows points on M_2^+ and M_2^-. The switching surface can be assembled as shown in algorithm 3.

Algorithm 3 Switching Surface

if $z_2^+ \le z_1^+ \le 1$ **then**
  $SwitchSurf = g_2^+$
else
  if $z_2^- \le z_1^- \le 1$ **then**
    $SwitchSurf = g_2^-$
  end if
end if

[1]Gröbner basis computed using a computer algebra package Singular[9]

For an initial condition $x_0 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \end{bmatrix}$ we have $u_0 = -1$. The respective calculated open-loop switching instants are $t_1 = 0.293$, $t_2 = 0.906$, and at $t_3 = 2.15$ the system states are at origin. We implement the above algorithm 2 for these x_0 and u_0 values. As shown in the figure 3, the open-loop switching instants and closed loop switching instants are matching.

C. Region of Null-controllability(X_0)

As per the discussion from section II, $X_0 = M_n^+ \cup M_n^-$. We know,

$$\begin{aligned}
M_n^+ = \{&x_i = f_{ni}^+(z_1^{p_i}, \ldots, z_n^{p_i}) \forall i = 1, \ldots, n \\
&: 0 \le z_n \le z_{n-1} \le \ldots \le z_1 \le 1\}.
\end{aligned}$$

Since each x_i is expressed as a polynomial in z_i, $i = 1, \ldots, n$, we can systematically eliminate z_i's by using Gröbner basis based elimination technique. However, in this case we have $2n$-variables and n-equations defining the variety J_n^+. Therefore, we can eliminate at most $n-1$ variables by using the elimination procedure.

To implicitize

1) Form an ideal $J_n^+ = \langle x_1 - f_{n1}^+, \ldots, x_n - f_{nn}^+ \rangle \subset \mathbb{Q}[z_1, \ldots, z_n, x_1, \ldots, x_n]$.
2) By using lexicographic ordering $z_1 \succ z_2 \succ \ldots \succ z_n \succ x_1 \succ x_2 \succ \ldots \succ x_n$, compute Gröbner basis G_n^+ for J_n^+.
3) Select the element in G_n^+ which contains none of the variables $z_1, z_2, \ldots, z_{n-1}$ i.e. $g_n^+ = G_n^+ \cap \mathbb{Q}[x_1, x_2, \ldots, x_n, z_n]$.

Let us denote that element as $g_n^+(x_1, x_2, \ldots, x_n, z_n)$. By the very nature of parametrization, $t_n \to \infty$ implies $z_n = e^{\frac{-t_n}{l}} \to 0$. Substituting $z_n = 0$ in the equation $g_n^+ = 0$ identifies $(x_1, \ldots, x_n)$ from where it takes infinite time to reach origin. Hence by putting $z_n = 0$, we are selecting boundary of set M_n^+. A similar procedure holds for M_n^-. Hence the sets $\{(x_1, .., x_n) \in \mathbb{R}^n : g_n^+(x_1, \ldots, x_n, 0) = 0\}$ and $\{(x_1, .., x_n) \in \mathbb{R}^n : g_n^-(x_1, \ldots, x_n, 0) = 0\}$ together contains the boundary of null-controllability region. The following theorem 2 gives a sufficient condition to check whether a given point $x \in X_0$:

Theorem 2: If for all $\mu \in [0, 1]$, $g_n^+(\mu x) \neq 0$ and $g_n^-(\mu x) \neq 0$ then $x \in X_0$.

Proof: By lemma 1, $X_0 = M_n$. Also, $0 \in M_n$ and $M_n = M_n^+ \cup M_n^-$. The boundary of M_n^+ is contained in the set $\{(x_1, .., x_n) \in \mathbb{R}^n : g_n^+(x_1, \ldots, x_n, 0) = 0\}$ and similarly boundary of M_n^- is contained in $\{(x_1, .., x_n) \in \mathbb{R}^n :$

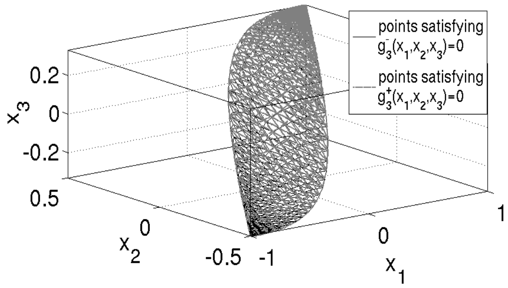

Fig. 4. Boundary of X_0

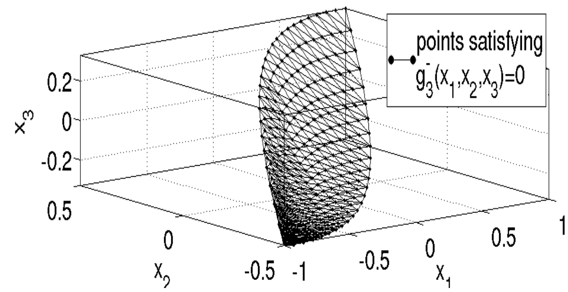

Fig. 5. Points satisfying $g_3^-(x_1,x_2,x_3) = 0$

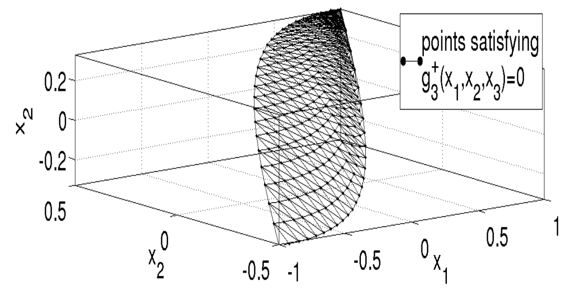

Fig. 6. Points satisfying $g_3^+(x_1,x_2,x_3) = 0$

$g_n^-(x_1,\ldots,x_n,0) = 0\}$. By convexity of X_0, it follows that, for all $x \in X_0$, the line segment joining x and origin should not intersect boundary of X_0. Thus, for all $\mu \in [0,1]$, if $g_n^+(\mu x) \neq 0$ and $g_n^-(\mu x) \neq 0$ then $x \in X_0$. ■
However, this is only a sufficient condition since parts of the surfaces defined by $g_n^+ = 0$ and $g_n^- = 0$ can lie in the interior of X_0. Then either $g_n^+(\mu x) = 0$ or $g_n^-(\mu x) = 0$ can hold for some μ even if a particular $x \in X_0$.

Example 3: We use the system from example 2. The set X_0 is divided into two parts: M_3^+ and M_3^-. We have,

$$F_3^+ = \begin{bmatrix} 2e^{-t_1} - 2e^{-t_2} + e^{-t_3} - 1 \\ e^{-2t_1} - e^{-2t_2} + \frac{1}{2}e^{-2t_3} - \frac{1}{2} \\ \frac{2}{3}e^{-3t_1} - \frac{2}{3}e^{-3t_2} + \frac{1}{3}e^{-3t_3} - \frac{1}{3} \end{bmatrix} \tag{13}$$

We can change the expression for F_3^- by substituting $z_1 = e^{-t_1}$, $z_2 = e^{-t_2}$ and $z_3 = e^{-t_3}$ as follows:

$$F_3^+ = \begin{bmatrix} 2z_1 - 2z_2 + z_3 - 1 \\ z_1^2 - z_2^2 + \frac{1}{2}z_3^2 - \frac{1}{2} \\ \frac{2}{3}z_1^3 - \frac{2}{3}z_2^3 + \frac{1}{3}z_3^3 - \frac{1}{3} \end{bmatrix} \tag{14}$$

We form an ideal $I = \langle 2z_1 - 2z_2 + z_3 - x_1 - 1, z_1^2 - z_2^2 + \frac{1}{2}z_3^2 - x_2 - \frac{1}{2}, \frac{2}{3}z_1^3 - \frac{2}{3}z_2^3 + \frac{1}{3}z_3^3 - x_3 - \frac{1}{3}\rangle$ Gröbner basis G of ideal I w.r.t lexicographic ordering $z_1 \succ z_2 \succ z_3 \succ x_1 \succ x_2 \succ x_3$ is computed. We identify the element in $G \cap \mathbb{Q}[z_3,x_1,x_2,x_3]$, which is

$$\begin{aligned} g_3^+ = {} & z_3^4 - 4z_3^3x_1 - 4z_3^3 - 2z_3^2x_1^2 - 4z_3^2x_1 + \\ & 16z_3^2x_2 + 6z_3^2 + \frac{4}{3}z_3x_1^3 + 4z_3x_1^2 + 4z_3x_1 - \\ & 16z_3x_3 - 4z_3 - \frac{1}{3}x_1^4 - \frac{4}{3}x_1^3 - 2x_1^2 + 16x_1x_3 + \\ & 4x_1 - 16x_2^2 - 16x_2 + 16x_3 + 1 \end{aligned} \tag{15}$$

Thus, for this example the set of points satisfying

$$-\frac{1}{3}x_1^4 - \frac{4}{3}x_1^3 - 2x_1^2 + 16x_1x_3 + 4x_1 - 16x_2^2 - 16x_2 + 16x_3 + 1 = 0$$

and

$$-\frac{1}{3}x_1^4 + \frac{4}{3}x_1^3 - 2x_1^2 + 16x_1x_3 - 4x_1 - 16x_2^2 + 16x_2 - 16x_3 + 1 = 0$$

contains the boundary of X_0. This boundary and the two separate parts defined by the above equations are plotted in figures 4, 5 and 6:

IV. CONCLUSIONS

In this article we use a Gröbner basis based implicitization technique to get a semi-algebraic description of switching surfaces for time-optimal control. It is shown that these surfaces can be combined to synthesize a feedback based switching law that ideally produces time-optimal switching. In implementation, this switching feedback law robustly drives the system to an arbitrarily small neighborhood of origin in finite time. The proposed method is currently limited to controllable diagonalizable systems with positive rational eigenvalues. Relaxing some of these assumptions is the subject of current and future research.

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