Gröbner Basis Computation of Feedback Control for Time Optimal State Transfer

Deepak U. Patil, Ameer K. Mulla, Debraj Chakraborty and Harish Pillai

***Abstract*— The synthesis of time-optimal feedback control of a single input, continuous time, linear time invariant system with bounded inputs is considered. Unlike a recent paper by the authors, the target final state is not necessarily the origin of the state space. Semi-algebraic representations of switching surfaces corresponding to the bang-bang time-optimal control are computed using a Gröbner basis based elimination algorithm. These surfaces are then used to synthesize a nested switching logic for time-optimal feedback control. Non-origin target points introduces unavoidable limit cycles in time optimal trajectories, whose period depends on the target position. This dependence is algebraically characterized and a method to compute the periods of the limit cycles is proposed. As a natural extension, we also provide a semi-algebraic characterization of the set of all points reachable with constrained inputs.**

I. Introduction

The time-optimal control problem of a single input, continuous time, linear time-invariant (LTI) system involves calculating the input (say, from a normalized constrained set $||u(t)|| \leq 1$), which transfers a given initial state to a final target state in the minimum time possible. While the open loop solution to this problem is well known (e.g. Pontryagin's maximum principle (PMP) [1]), general methods for synthesizing the feedback solution seem to be unavailable in the literature. A partial solution to the feedback synthesis problem was provided in [2], where the target point was necessarily the origin of the state space. This paper is an extension of the method proposed in [2] for synthesizing time optimal feedback control which will transfer admissible initial states to arbitrary reachable points (not necessarily origin) in the state space. Although the method proposed earlier extends naturally to the case of non-origin points, the situation is significantly different in practice. While for origin target points it was possible to keep the trajectory arbitrarily close to the target by using the proposed feedback logic, in this case, the trajectory is forced into compulsory limit cycles by the feedback logic after reaching most non-origin targets. Moreover, the period of the limit cycle varies with the location of the target point in the state space. Lastly, the proposed method (which assumed systems to have positive eigenvalues in [2]) is extended to systems with non-zero, distinct and rational eigenvalues in this article.

The current version of the problem is important for many applications. A feedback solution for time optimal state transfer is robust to uncertainties and exogenous disturbances and, unlike the open loop control, would not require recalculation of the optimal input separately for each distinct initial condition. Additionally, many applications require the target point to be different from the origin [3], [4]. For example, systems are often linear over a wide range of operating conditions and it is reasonable to assume the same linear model to track various set points, which would then be different from the origin. Similarly, formation control of multi-agent systems, require target points different from the origin of the combined state space [5], [6]. Lastly, non-linear systems may have to be linearized at points different from the target set points, necessitating the proposed formulation. An example of such a situation is flight control of spacecrafts, where the model is usually linearized about nominal flight conditions whereas actual flight might demand non-zero attitude commands [7], [8].

Consider a n^{th} order linear time invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state vector and the input $u(t)$ is constrained within a normalized interval $|u(t)| \leq 1$. Denote the set of admissible inputs $U = \{u(t) : |u(t)| \leq 1\}$. Given a (not necessarily zero) point $p \in \mathbb{R}^n$, we denote by X_p the set of initial conditions from which system (1) can be transferred to p in some $t > 0$. Our objective is to synthesize a time optimal feedback function $f : X_p \to U$, (necessarily bang-bang with not more than $n-1$ switches), which transfers any $x_0 \in X_p$ to p in the minimum time possible. We would also like to compute X_p for any given p. However, it was shown in [4], [9] that the above objectives are reasonable only for "constrained controllable" target points, i.e. those states which are reachable from any point in its own neighborhood. We restrict ourselves to this framework.

It is well known [1] that the time optimal control switches between the extreme admissible values (± 1) according to the so called "switching surfaces" in the state space. Hence, if the switching surfaces are known, a time optimal feedback strategy can be constructed just by checking the location of the current states with respect to these switching surfaces. By extending the results of [2] to the case of non-origin target points, it is shown that the switching surfaces can be described by semi-algebraic sets. A Gröbner basis based implicitization algorithm [10] is used to derive implicit expressions of the switching surfaces in terms of polynomials and rational functions of the state variables. Using these expressions, a feedback logic for optimal switching is synthesized. For most non-origin p's, the controller enforces limit cycles, whose period depends on p. We characterize

D. U. Patil, A. K. Mulla, D. Chakraborty and H. Pillai are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. {deepakp, ameerkm, dc, hp}@ee.iitb.ac.in

the time period of this closed trajectory for a given target point p, and give semi-algebraic expressions for the set of all target points which gives rise to limit cycles of the same time period.

Analytical properties of the switching surfaces were studied in [1], [11], [12] whereas [1], [13], [14], [15], [16] investigated the synthesis of these surfaces for limited cases. The relevant analytical framework for reachability of non-origin target points was introduced in [3], [4], [9], where analytical properties of set X_p and related topological structures were studied.

The remaining article is arranged as follows: In section II, a selection of results from [3], [4], [9] are recounted and extended, while polynomial representations of the switching surfaces are derived in section III. In section IV, implicit representations of switching surfaces are derived and a state feedback based switching algorithm is proposed. Several numerical examples are included. The properties and conditions for existence of limit cycle are studied is section V. In section VI algebraic methods are proposed to compute the set of all points reachable with constrained inputs.

II. Preliminaries and Analysis

Consider a system described by (1). It is assumed that eigenvalues of A, $\lambda_i(A) \in \mathbb{Q} \setminus \{0\}$ for $i = 1, ..., n$ and $\lambda_i(A) \neq \lambda_j(A)$ for $i \neq j$. The set of initial conditions of system (1) that can be driven to origin with input $u(t) \in U$ in time t is given by,

$$R_0(t) = \left\{ x : x = -\int_0^t e^{-A\tau} Bu(\tau) d\tau, \ \ \forall u(t) \in U \right\} \quad (2)$$

The set of all the initial conditions that can be driven to origin using $u(t) \in U$ is called the *null-controllable set* [11] and is given by $X_0 = \bigcup_{t\in[0,\infty)} R_0(t)$.

When the target point is $p \in \mathbb{R}^n$, the states of (1) must be driven from x_0 to p in $t \in [0, \infty)$ using the admissible control $u(t) \in U$. From solution to (1) we write,

$$x_0 = e^{-At} p - \int_0^t e^{-A\tau} Bu(\tau) d\tau \quad (3)$$

Definition 1. [9] The *reachable set* (denoted by $R_p(t)$) to point p at time $t > 0$ is the set of all the points $x \in \mathbb{R}^n$ that can be driven to p in time t using the admissible control $u(t) \in U$.

$$R_p(t) = \left\{ e^{-At} p - \int_0^t e^{-A\tau} Bu(\tau) d\tau : u(t) \in U \right\} \quad (4)$$

Clearly, (2) is a special case of (4) with $p = 0$. For any set $S \subset \mathbb{R}^n$ and $q \in \mathbb{R}^n$, define $S + q = \{x + q : x \in S\}$. Using this definition, $R_p(t)$ can also be defined as,

$$R_p(t) = e^{-At} p + R_0(t) \quad (5)$$

The set of all the states that can be driven to p using $u(t) \in U$ is called the *reachable set* to the point p and is $X_p = \bigcup_{t\in[0,\infty)} R_p(t)$.

Definition 2. [9] The *attainable set* (denoted by $\mathcal{A}_p(t)$) from point p at time t is the set of all the states that can be reached from the point p using admissible control $u(t) \in U$ in time $t > 0$.

$$\mathcal{A}_p(t) = \left\{ e^{At} p + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau : u(t) \in U \right\} \quad (6)$$

The set of all the states that can be reached from p using $u(t) \in U$ is called the *attainable set* from the point p and is $\mathcal{A}_p = \bigcup_{t\in[0,\infty)} \mathcal{A}_p(t)$ [9].

For a given system (1), a point $p \in \mathbb{R}^n$ is called *constrained controllable* if it is in the interior of X_p i.e. $x \in int(X_p)$. This implies that for a point p to be constrained controllable, all points in the neighborhood of p must be transferable to p. We will restrict ourselves to only such constrained controllable points as target states for time optimal feedback synthesis. The points p which are not constrained controllable might still be reachable from the corresponding X_p but it is impossible to keep the state trajectory inside some neighborhood of such a p. Hence such a target point is not too useful for practical purposes.

Let p be a constrained controllable point. We define the following functions that characterize the states that can be driven to p using bang-bang input ($u(t) = \pm 1$) with $k - 1$ switches.

$$\left.\begin{aligned} F^+_{p,k}(t_1, \dots, t_k) &= e^{-At_k} p + \Big(-\textstyle\int_0^{t_1} + \int_{t_1}^{t_2} - \\ &\quad \dots + (-1)^k \textstyle\int_{t_{k-1}}^{t_k} \Big) e^{-A\tau} B d\tau \\ F^-_{p,k}(t_1, \dots, t_k) &= e^{-At_k} p - \Big(-\textstyle\int_0^{t_1} + \int_{t_1}^{t_2} - \\ &\quad \dots + (-1)^k \textstyle\int_{t_{k-1}}^{t_k} \Big) e^{-A\tau} B d\tau \end{aligned}\right\} \quad (7)$$

Observe that, $0 \le t_1 \le t_2 \le ... \le t_k$. We define a set $V_k := \{(t_1, t_2, ..., t_k) : 0 \le t_1 \le t_2 \le ... \le t_k < \infty\}$ for $k = 1, ..., n$. The set of the states which can be driven to p in $k-1$ switches using initial input $u = 1$ can be defined as $M^+_{p,k} = \left\{ x : x = F^+_{p,k}(v), \ \ \forall v \in V_k \right\}$ and similarly, $M^-_{p,k} = \left\{ x : x = F^-_{p,k}(v), \ \ \forall v \in V_k \right\}$.

Thus the set of all states that can be driven to p in $k - 1$ switches is defined as,

$$M_{p,k} = \left\{ x : x = F^{\pm}_{p,k}(v), \forall v \in V_k \right\} = M^+_{p,k} \cup M^-_{p,k} \quad (8)$$

Lemma 3. $M_{p,0} \subset M_{p,1} \subset \ldots \subset M_{p,n}$

Proof: Consider $x \in M^+_{p,k}$ for any $0 < k \le n - 1$. Hence, from (7) $x = e^{-At_k} p + \left(-\int_0^{t_1} + \ldots + (-1)^k \int_{t_{k-1}}^{t_k} \right) e^{-A\tau} B d\tau$ and $0 \le t_1 \le t_2 \le \ldots \le t_k$. Now, let $t_{k+1} = t_k$. Then $x = e^{-At_{k+1}} p + \left(-\int_0^{t_1} + \ldots + (-1)^{k+1} \int_{t_k}^{t_{k+1}} \right) e^{-A\tau} B d\tau$ (since, $\int_{t_k}^{t_{k+1}} e^{-A\tau} B d\tau = \int_{t_{k+1}}^{t_{k+1}} e^{-A\tau} B d\tau = 0$). Therefore, $x \in M^+_{p,k+1}$. Hence, $M^+_{p,k} \subset M^+_{p,k+1}$. Similarly, it can be shown that, $M^-_{p,k} \subset M^-_{p,k+1}$ and hence $M_{p,k} \subset M_{p,k+1}$ for all $0 < k \le n - 1$. □

Lemma 4. $M_{p,n} = X_p$.

Proof: Clearly $M_{p,n} \subset X_p$. It is known that, for a system described by (1), if there exists at least one admissible control which transfers the state x_0 to p, there also exists a

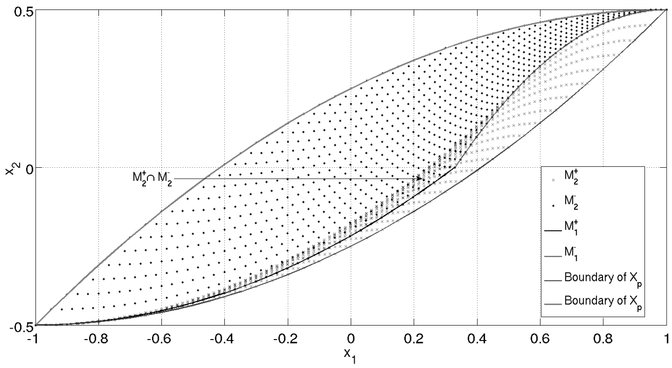

Fig. 1. $X_0 = M_2^+ \cup M_2^-$ for case of example 5

time-optimal control which transfers the state x_0 to p [1]. But, time-optimal control is guaranteed to be bang-bang with at most $n-1$ switches. Hence, $X_p \subset M_{p,n}$ implying $M_{p,n} = X_p$. □

Lemma 4 assures that the set of states that can be driven to p with only bang-bang input (with at most $n-1$ switches) is in fact the entire reachable set to p (i.e. X_p). Hence from (8), $X_p = M_{p,n}^+ \cup M_{p,n}^-$. In case of the origin as the target point ($p = 0$), $M_{0,n}^+ \setminus M_{0,n-1}^+$ and $M_{0,n}^- \setminus M_{0,n-1}^-$ are disjoint and $M_{0,n}^+ \cap M_{0,n}^- = M_{0,n-1}$ is a set of measure zero in M_n [2], [11]. However for $p \neq 0$, this need not be true. The set $M_{p,n}^+ \cap M_{p,n}^-$ is of non-zero measure and points in the set $M_{p,n}^+ \cap M_{p,n}^-$ can be driven to p by more than one different bang-bang inputs with at most $n-1$ discontinuities. However only one of them is time-optimal (by uniqueness of time-optimal control). All the other points $x \in X_p \setminus M_{p,n}^+ \cap M_{p,n}^-$, can be partitioned as $x \in M_{p,n}^+$ or $x \in M_{p,n}^-$. For such points there exists unique bang bang control with at most $n-1$ switches and it is time-optimal.

The structure of the reachable set X_p of a constrained controllable point p is demonstrated in the following example for a simple 2 state system.

Example 5. For a second order LTI system (1) with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|u| \leq 1$, the corresponding structure of X_p with $p = \begin{bmatrix} 0 & 0.33 \end{bmatrix}^T$ is shown in the figure 1. This figure illustrates that $M_2^+ \cap M_2^-$ is non-empty.

Based on the structure of $M_{p,n}$, an iterative switching logic can be defined that drives the system from any state in X_p to p in finite time using bang-bang control. For any initial condition $x_0 \in X_p$, the switching should ideally create the following sequence of events. Consider $x_0 \in M_{p,n}^+$,

- Input $u = 1$ pushes x from $M_{p,n}^+$ to the manifold $M_{p,n-1}^-$.
- As soon as $x \in M_{p,n-1}^-$, switch 2 pushes x to $M_{p,n-2}^+$
- $\vdots$
- Input at $(n-1)^{th}$ switch pushes x from $M_{p,1}^+$(if n is odd) or $M_{p,1}^-$(if n is even) to the target p.

Similar sequence is valid for $x_0 \in M_{p,n}^-$ with opposite signs of input.

For the initial conditions $x_0 \in X_p \setminus M_{p,n}^+ \cap M_{p,n}^-$ this switching logic drives the system to p in minimum time. However, for $x_0 \in M_{p,n}^+ \cap M_{p,n}^-$ we have two choices of initial control namely $u = 1$ and $u = -1$, out of which one drives the system to p in minimum time. Hence the choice of initial control is crucial in this case and can be computed using PMP. However after the initial input has been applied, all the future control inputs are computed as per the algorithm proposed. Even if choice of initial input goes wrong, the target point will be achieved in finite time. However time-optimality will be lost.

To implement the feedback switching logic given above, we need to compute $M_{p,k}^+$ and $M_{p,k}^-$ for $k = 1, ..., n$. Before we give an algorithm to compute $M_{p,k}$, note the following,

Remark 6. For system (1), A is assumed to have non-zero rational eigenvalues. For any real similarity transformation $\widehat{x} = Tx$ on a system, the corresponding set $\widehat{M}_{p,k}^+ = \{\widehat{x} = Tx \ : \ x \in M_{p,k}^+\}$ and similarly $\widehat{M}_{p,k}^- = \{\widehat{x} = Tx \ : \ x \in M_{p,k}^-\}$ [11]. Thus, computing $M_{p,k}^+$ and $M_{p,k}^-$ for the diagonalized system is enough to compute the corresponding $M_{p,k}^+$ and $M_{p,k}^-$ for all similar systems. So, we can assume A to be diagonal without loss of generality.

III. Parametric Representation

All $x \in M_{p,k}$ are characterized by the functions $F_{p,k}^+$ or $F_{p,k}^-$ (given by (7)) defined over V_k. As A is assumed to be diagonal, we can write the state variables $x_i, \ i = 1, ..., n$ as functions $f_{p,ki}^{\pm}$ defined over arguments $e^{-\lambda_i t_1}, ..., e^{-\lambda_i t_k}$ where $\lambda_i \in \lambda(A)$ for all $i = 1, ..., n$.

$$x_i = f_{p,ki}^{\pm}(e^{-\lambda_i t_1}, ..., e^{-\lambda_i t_k}); \ \forall i = 1, \ldots, n \qquad (9)$$

By simple substitution of variables these functions can be represented as polynomials or rational functions. So the switching surfaces $M_{p,k}, \ k = 1, ..., n-1$ and the reachable set $X_p = M_{p,n}$ can be represented as polynomials or rational functions in parametric form. These polynomials or rational functions can be implicitized using Gröbner basis techniques to get polynomial equalities and inequalities involving only the state variables. Recall that, $\lambda(A) \in \mathbb{Q} - \{0\}$. Hence we can write $\lambda_i = \frac{n_i}{d_i}$. Let $l = lcm(d_1, ..., d_n)$. Substitute $z_i = e^{-\frac{t_i}{l}}, \ \forall i = 1, ..., k$ in (9), we get either either polynomial or rational function representations depending on the signs of the eigenvalues of A.

We consider a general case wherein some eigenvalues of A are positive and others negative. Without loss of generality we assume, $\lambda_1, ..., \lambda_q$ are positive and the remaining $\lambda_{q+1}, ..., \lambda_n$ are negative. Substituting $z_i = e^{-\frac{t_i}{l}} \ \forall i = 1, ..., k$ we get, $x_i = f_{p,ki}^+(z_1^{p_i}, \ldots, z_k^{p_i})$, for $i = 1, ..., q$ and $x_i = f_{p,ki}^+(z_1^{-p_i}, \ldots, z_k^{-p_i})$, for $i = q+1, ..., n$

Observe that x_i's for $i = 1, ..., q$ are polynomials in $z_1, ..., z_k$ but for $i = q+1, ..., n$, x_i's are rational functions in $z_1, ..., z_k$. Hence, in general,

$$x_i = \frac{N_{p,ki}^+(z_1, ..., z_k)}{D_{p,ki}^+(z_1, ..., z_k)}, \qquad i = 1, ..., n$$

where $\frac{N^+_{p,ki}}{D^+_{p,ki}} = f^+_{p,ki}$ and $D^+_{p,ki} = 1$ for $i = 1, ..., q$. This gives rational representations of $M^+_{p,k}$ and $M^-_{p,k}$:

$$M^+_{p,k} = \{(x_1, \ldots, x_n) : x_i = \frac{N^+_{p,ki}(z_1, ..., z_k)}{D^+_{p,ki}(z_1, ..., z_k)} \quad \forall i = 1, \ldots, n, \;\; 0 < z_k \le z_{k-1} \le \ldots \le z_1 \le 1\} \tag{10}$$

The expression (10) involve $z_1, ..., z_k$ and the state variables. Thus these expressions cannot be used directly for state based switching. The variables $z_1, ..., z_k$ must be eliminated from these expressions and represent $M^+_{p,k}$ and $M^-_{p,k}$ only in terms of the state variables $x_1, ..., x_n$. Such a representation can then be used directly for a state-feedback based switching of the input values between ± 1. We use an implicitization method based on Gröbner basis to eliminate $z_1, \ldots, z_k$.

IV. Switching Surface and Switching algorithm

In this section, we describe the implicitization procedure for the set $M^{\pm}_{p,n-1}$. If required, a similar procedure can be followed for computing the lower dimensional surfaces also.

A. Implicitization

For different cases of the eigenvalues of A, parametric representations of $M^{\pm}_{p,n-1}$ is more generally given by (10). Notice that the process of implicitization of rational functions can also be used for polynomial functions. So, we only describe the implicitization procedure for rational function case. In this case,

$$x_i = \frac{N^+_{p,n-1,i}(z_1, ..., z_k)}{D^+_{p,n-1,i}(z_1, ..., z_k)}, \qquad i = 1, ..., n \tag{11}$$

$$0 < z_{n-1} \le z_{n-2} \le \ldots \le z_1 \le 1 \tag{12}$$

parametrically describe the switching surface $M^+_{p,n-1}$. Using standard implicitization steps (e.g. see [10]) given below, we eliminate $z_1, ..., z_{n-1}$ from (11). For simplicity of notation, we drop the subscript p from $N^+_{p,n-1}$ and $D^+_{p,n-1}$.

1) Form an ideal $J^+_{p,n-1} = \langle D^+_{n-1,1}x_1 - N^+_{n-1,1}, \ldots, D_{n-1,n}x_n - N^+_{n-1,n}, 1 - D^+_{n-1,1}D^+_{n-1,2}...D^+_{n-1,n}y\rangle$. The additional variable y is introduced in order to prevent the denominators $D^+_{n-1,k}, k = 1, .., n$ from becoming zero.
2) Compute Gröbner basis $G^+_{p,n-1}$ of $J^+_{p,n-1}$ w.r.t. lexicographic ordering as $y \succ z_1 \succ z_2 \succ \ldots \succ z_{n-1} \succ x_1 \succ \ldots \succ x_n$.
3) The element $g^+_{p,n-1} \in G^+_{p,n-1} \cap \mathbb{Q}[x_1, \ldots, x_n]$ defines the smallest variety containing the parametric representation (11).

Remark 7. In general, the variety defined by g^+_{n-1} may be larger than the parametrized surface defined by $M^+_{p,n-1}$. However, by uniqueness of time-optimal control, it follows that there are unique switching instants $t_1, \ldots, t_{n-1}$ and hence $z_1, \ldots, z_{n-1}$. Thus all variables $x_1, \ldots, x_n \in X_p$ satisfying $g^+_{p,n-1}(x_1, \ldots, x_n) = 0$ are always extendable to entire variety.

For completely describing $M^+_{p,n-1}$ and guaranteeing that $(x_1, \ldots, x_n) \in X_0$, condition (12) needs to be imposed. For that we need inequality (12) in terms of $x_1, x_2, \ldots, x_n$. This can be achieved by solving each z_k in terms of $x_1, x_2, \ldots, x_n$ and then imposing inequality (12). The technique for this is described next.

Let G^+_{p,z_k} be the Gröbner basis obtained by using ordering $.. \succ .. \succ \ldots \succ z_k \succ x_1 \succ \ldots \succ x_n$. Consider the set $G^+_{p,z_k} \cap \mathbb{Q}[z_k, x_1 \ldots, x_n]$. Let there be m elements in this set. Denote these elements by $g^+_{p,1}, g^+_{p,2}, ..., g^+_{p,m}$. Clearly, each of these elements is a polynomial in the variables $(z_k, x_1, \ldots, x_n)$ and the equations $g^+_{p,i} = 0 \;\; \forall i = 1, \ldots, m$ can be solved for z_k in terms of $(x_1, \ldots, x_n)$. For $M^+_{p,n-1}$ we denote the respective z_k by z^+_k for $k = 1, \ldots, n-1$. Thus, $M^+_{p,n-1}$ can now be defined as follows:

$$M^+_{p,n-1} = \{(x_1, \ldots, x_n) : g^+_{p,n-1}(x_1, x_2, \ldots, x_n) = 0 \quad 0 < z^+_{n-1} \le z^+_{n-2} \le \ldots \le z^+_1 \le 1\}$$

By following the same method, we can also find the expression for $M^-_{p,n-1}$.

B. Switching algorithm

The expressions obtained above, can be used to check if a point x in the state-space belongs to the switching surface $M_{p,n-1}$. The switching logic described in section II, requires that a bang-bang input trajectory starting from M_n should change its sign when it intersects $M_{p,n-1}$. As we now have the expression for $M_{p,n-1}$, this logic can be implemented by algorithm 1

Algorithm 1 Switching logic

Require: current $x(t)$ and past $(u(t^-))$
  if $x(t) \in M_{p,n-1}$ **then**
    $u = -u(t^-)$
  else
    $u = u(t^-)$
  end if

In this implementation, ideally, the sign of the input should change as soon as the algorithm 1 computes that state trajectory has reached $M_{p,n-1}$. Now, the state trajectory is in $M_{p,n-1}$ and will travel along $M_{p,n-1}$ until it hits $M_{p,n-2}$ and so on for all the lower dimensional surfaces. Hence, it appears that for implementing the switching logic ideally, we need to compute all $M_{p,k}$, $k = 0, ..., n-1$. However, in simulation or practice, unavoidable delays in computation causes the state-trajectory to overshoot $M_{p,n-1}$. Thus, the state-trajectory, instead of staying in $M_{p,n-1}$ overshoots in $M_{p,n}$ slightly before the first switch occurs. Now, as state trajectory is in $M_{p,n}$, we can use $M_{p,n-1}$ for the second switch again. This situation repeats for all the $n-1$ switches and algorithm 1 gives entire switching law required for implementation using only $M_{p,n-1}$.

Even though not required in practice, the lower dimensional switching surfaces i.e. $M_{p,n-k}$, $k = 2, ..., n-1$ can

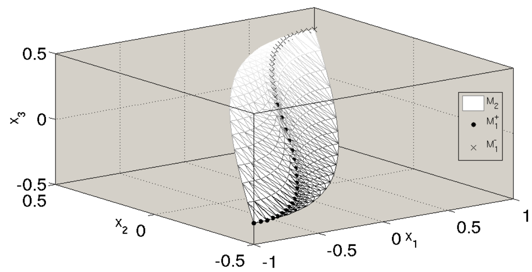

Fig. 2. Points on the Switching Surface (Example 8)

be computed from $M_{p,n-1}$ by substituting $z_k = 1$ in the inequality (12). This forces $z_i = 1 \ \forall i \leq k-1$. Then,

$$M^+_{p,n-k} = \{(x_1, \dots, x_n) : g^+_{n-1}(x_1, x_2, \dots, x_n) = 0 \\ 0 < z^+_{n-1} \leq z^+_{n-2} \leq \dots \leq z^+_k = \dots = z^+_1 = 1\}$$

Now lower dimensional switching surfaces can be added to algorithm 1 and used for estimating control switching as described in section II. However, this would require instant switching with no computational delay.

C. Examples

Example 8. Consider a system of type (1) with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Let the target state be $p = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T$. The switching surface is $M_2 = M_2^+ \bigcup M_2^-$. Using the notation introduced in section II, the parametric expression and the inequalities $0 \leq z_2 \leq z_1 \leq 1$ that describe the surface M_2^+ are computed. Using the implicitization method, we eliminate z_1 and z_2. First form an ideal $J^+ = \langle x_1 - 2z_1 + \frac{9}{10}z_2 + 1, x_2 - z_1^2 + \frac{2}{5}z_2^2 + \frac{1}{2}, x_3 - \frac{2}{3}z_1^3 + \frac{7}{30}z_2^3 + \frac{1}{3} \rangle$ from the polynomials of F_2^+. Then eliminate z_1 and z_2 and form implicit equation for M_2^+. The Gröbner basis G^+ with ordering $z_1 \succ z_2 \succ x_1 \succ x_2 \succ x_3$ for the ideal J^+ is calculated [1]. Due to lack of space we omit Gröbner basis. We select an element from Gröbner basis which is only in terms of x_1, x_2 and x_3 denoted as $g_2^+(x_1, x_2, x_3)$. Now, $g_2^+(x_1, x_2, x_3) = 0$ and condition $0 \leq z_2 \leq z_1 \leq 1$ together describe M_2^+. Further, we also select an element from Gröbner basis which is linear in z_2. Solving this equation gives $z_2 = -\frac{-\frac{2675}{8014}x_1^3 - \frac{8025}{8014}x_1^2 + \frac{22}{4007}x_1x_2 - \frac{8003}{8014}x_1 + \frac{22}{4007}x_2 + \frac{168507}{160280}x_3 + \frac{3109}{160280}}{x_1^2 + 2x_1 - \frac{163609}{80140}x_2 - \frac{3329}{160280}}$. Similarly, the expression for z_1 can be found from the Gröbner basis of ideal J with ordering $z_2 \succ z_1 \succ x_1 \succ x_2 \succ x_3$. Hence,

$$M_2^+ = \{(x_1, x_2, x_3) : g_2^+(x_1, x_2, x_3) = 0, \ \ 0 < z_2 \leq z_1 \leq 1\}$$

M_2^- can be calculated by following same procedure. Figure 2 shows points on M_2^+ and M_2^-. The feedback logic can be implemented as per algorithm 1.

[1]Computed using a computer algebra package Singular[17]

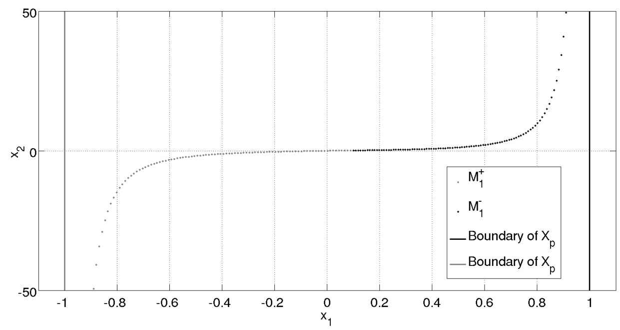

Fig. 3. Switching surface and reachable region X_p(Example 9)

We compare the switching instants generated using algorithm 1 with those generated by optimal open loop input computed using PMP for sample initial condition, $x_0 = \begin{bmatrix} -0.2 & -0.14 & -0.1 \end{bmatrix}^T$. The initial value of the optimal input is $u(0) = +1$ and the optimal open-loop switching instants are $t_1 = 0.0704$ and $t_2 = 0.1953$. At $t_3 = 0.6419$ the system reaches p. We infer from the closed loop response, that the respective open loop switching instants are matching with the closed loop switching instants which are $t_1 = 0.0704$ and $t_2 = 0.1953$. At $t_3 = 0.6419$ $x(t_3) = p$.

Example 9. Next, consider a two state system of the form (1) with mixed eigenvalues. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the target state be $p = [\ 0.1 \quad 0.1\ ]^T$. The switching surface in this case is $M_1 = M_1^+ \cup M_1^-$. We form parametric representation for M_1^+ and M_1^- using methods described in section 2. Eliminating z from these parametric expressions we get $g_1^+(x_1, x_2) = x_1^2 x_2 - \frac{x_1^2}{2} + 2x_1x_2 - x_1 + x_2 - \frac{2}{125}$. Similarly, we get $g_1^-(x_1, x_2) = x_1^2 x_2 + \frac{x_1^2}{2} - 2x_1x_2 - x_1 + x_2 + \frac{7}{500}$. The reachable set X_p in this case is unbounded in the direction of x_2-axis and is a proper subset of $\mathbb{R}^2$ as shown in the figure 3. For initial condition $x_0 = [0.2\ 6]^T$, closed loop switching instant is $t_1 = 0.32$ and at $t_2 = 1.26$ $x(t_2) = p$.

V. Limit Cycles

If the origin is the target, then it was shown in [2] that the trajectory can be transferred to, and kept within any arbitrarily small neighborhood of the origin by accurate switching. However, for arbitrary target points the trajectory, after passing through the target might go in to a limit cycle about the target. The occurrence of such a limit cycle depends on the relative directions of the vector fields associated with the two inputs (± 1) at the target point. In the case of non-ideal switching (with delays), the trajectory can be made to pass arbitrarily close to the target by reducing the delays. However, the period of the limit cycle might increase with larger switching delays. The conditions for occurrence of such a limit cycle are analyzed and an algorithm is proposed for characterizing the period of the limit cycle in this section.

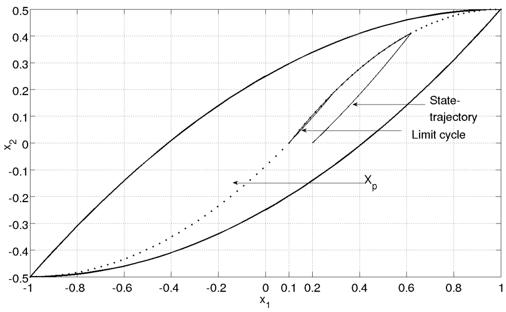

Fig. 4. The limit cycle near $p = [0.1\ 0]^T$

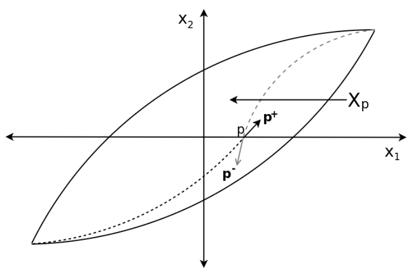

Fig. 5. The vector fields corresponding to the two inputs ± 1

A. Existence of a Limit Cycle

The switching algorithm proposed in section IV repeatedly forces the state trajectory back to p everytime it overshoots p. This gives rise to a limit cycle about the target point p. The situation is explained in the following example.

Example 10. Consider example 5 from section II with final state $p = [\ 0.1 \quad 0\ ]^T$. State-trajectories are generated using the switching algorithm 1. Figure 4 shows the phase diagram corresponding to the state-trajectory starting with the initial condition $x_0 = [\ 0.2 \quad 0\ ]^T$.

As seen in the figure, state-trajectory after reaching point p is forced into a limit cycle (with a period of 0.410 seconds). This is because the switching curve is not smooth at the target p and, as shown in figure 5, the velocity vectors $\mathbf{p}^+$ and $\mathbf{p}^-$ corresponding to the two inputs ± 1 are not exactly opposed to one another at p. Hence after reaching p the state trajectory is forced to go in the direction of $\mathbf{p}^+$ until it reaches the switching surface again, after which the next input switch brings it back to p along the switching curve.

Let the input for the system be $u = 1$ for time Δt_1 followed by $u = -1$ for Δt_2. The change in the state value Δx in time $\Delta t = \Delta t_1 + \Delta t_2$ is given by $\Delta x = \dot{x}_+ \Delta t_1 + \dot{x}_- \Delta t_2$. Let $\Delta t_1 = \alpha \Delta t$ for some $\alpha \in (0, 1)$. We get, $\frac{\Delta x}{\Delta t} = \alpha \dot{x}_+ + (1 - \alpha)\dot{x}_-$. Taking limit as $\Delta t \to 0$, we define the average velocity as $\dot{x}_{avg} = \alpha \dot{x}_+ + (1-\alpha)\dot{x}_-$.

The state trajectory can be kept in an arbitrarily small neighbourhood of the target point, if and only if there exists $\alpha \in (0, 1)$ such that $\dot{x}_{avg} = 0$ at target p (This fact follows from the analysis given in chapter 3 section 1.1 of [18]).

Now, $\dot{x}_{avg} = \alpha \dot{x}_+ + (1 - \alpha)\dot{x}_- = 0$ i.e. $\dot{x}_+$ and $\dot{x}_-$ are linearly dependent and are pointing in opposite direction. Hence $Ax + B = -\frac{1-\alpha}{\alpha}(Ax - B)$ i.e. $x = (1-2\alpha)A^{-1}B$ (since A is invertible). Therefore the set of points for which it is possible to avoid limit cycle are characterized by the set $L_0 := \{x | x = (1-2\alpha)A^{-1}B,\ 0 < \alpha < 1\}$. Clearly, $0 \in L_0$ for $\alpha = \frac{1}{2}$. Moreover, L_0 is a line passing through the two shifted equilibrium points $x = -A^{-1}B$ and $x = A^{-1}B$ (corresponding to the two inputs $u = 1$ and $u = -1$ respectively) and the origin. For the points lying outside L_0 since there exist no $\alpha \in (0, 1)$ to create $\dot{x}_{avg} = 0$, the limit cycle is unavoidable.

B. Time period of the Limit Cycle

After a target point $p \notin L_0$ is reached, the proposed switching law forces the state-trajectory into a limit cycle. To compute the time period of limit cycle, we use equation (7) where initial condition is same as target p. This results into following equation :

$$p = e^{-At_n}p \pm (-\int_0^{t_1} + ... + (-1)^n \int_{t_{n-1}}^{t_n} e^{-A\tau}Bd\tau) \quad (13)$$
$$p = (I - e^{-At_n})^{-1}F^{\pm}_{0,n}(t_1, ..., t_n)$$

Definition 11. The set of all the target points for which the corresponding limit cycle is of time period $t_n \leq T$ is given by, $L_T = \{p = (I - e^{-At_n})^{-1}F^{\pm}_{0,n}(t_1, ..., t_n) : 0 < t_1 \leq ... \leq t_n \leq T\}$ Also the boundary of set L_T is $\partial L_T = \{p = (I - e^{-At_n})^{-1}F^{\pm}_{0,n}(t_1, ..., t_n) : 0 < t_1 \leq ... \leq t_n = T\}$.

Theorem 12. *For $T_1 > T_2$, $L_{T_2} \subset L_{T_1}$.*

Proof: Consider $p \in L_{T_2}$. Then $p = \pm(I - e^{-At_n})^{-1}(-\int_0^{t_1} + \int_{t_1}^{t_2} - ... + (-1)^n \int_{t_{n-1}}^{t_n} e^{-A\tau}Bd\tau)$ for some $0 < t_1 \leq ... \leq t_n \leq T_2$. But since $T_2 < T_1$, $p \in L_{T_1}$. Therefore $L_{T_2} \subset L_{T_1}$. □

Theorem 13. *L_T is a convex set.*

Proof: $R_0(T)$ is a convex set [11]. We observe from the definition of ∂L_T that $(I - e^{-AT})^{-1}\partial R_0(T) = \partial L_T$ and $(I - e^{-AT})^{-1}R_0(T) = L_T$. Also, for $T_1 < T_2 < T$, $R_0(T_1) \subset R_0(T_2) \subset R_0(T)$ [11]. Now, by convexity of $R_0(T)$, for $\lambda \in [0, 1]$, we have $\lambda \partial R_0(T_1) + (1 - \lambda)\partial R_0(T_2) = R_0(T_2)$. Thus, $(I - e^{-AT_2})^{-1}(\lambda \partial R_0(T_1) + (1-\lambda)\partial R_0(T_2)) = L_{T_2}$ i.e. $\lambda(I - e^{-AT_2})^{-1}\partial R_0(T_1) + (1 - \lambda)\partial L_{T_2} = L_{T_2} \subset L_T$. But, since $\partial R_0(T_1) \subset R_0(T_2)$, we have $(I - e^{-AT_2})^{-1}\partial R_0(T_1) \subset L_{T_2} \subset L_T$. Now choose $p_1 \in [I - e^{-AT_2}]^{-1}\partial R_0(T_1) \subset L_T$ and $p_2 \in \partial L_{T_2} \subset L_T$. Therefore, $\lambda p_1 + (1 - \lambda)p_2 \in L_T$. □

Let $L_T^+ = \{(I - e^{-At_n})^{-1}F^{+}_{0,n} : 0 < t_1 \leq ... \leq t_n \leq T\}$ (L_T^- is then defined accordingly). This parametric representation is converted to rational or polynomial parametric representation (depending upon eigenvalues of A) by procedure described in section III. Compute Gröbner basis G_n^+ and G_n^- for L_T^+ and L_T^- respectively using standard implicitization steps described in section IV. Using G_n^+ and G_n^-, we compute the time period of limit cycle for a given target p. Moreover, we also provide a sufficient condition to check whether a target point $p \in L_T$ for a given time period T.

1) Computation of time period: Given target point p, we use following procedure to compute time period of the limit cycle.

1: Select $g_n^+ \in G_n^+ \cap \mathbb{Q}[z_n, p_1, ..., p_n]$.Solve $g_n^+ = 0$ for z_n.

2: Select $g_{n-1}^+ \in G_n^+ \cap \mathbb{Q}[z_n, z_{n-1}, p_1, ..., p_n]$.Using values of z_n obtained in previous step, solve $g_{n-1}^+ = 0$ for z_{n-1}.

$\vdots$

n: Select $g_{n-1}^+ \in G_n^+ \cap \mathbb{Q}[z_n, z_{n-1}, ..., z_1, p_1, ..., p_n]$. Using values of $z_n, z_{n-1}, ..., z_2$ obtained in previous step, solve $g_1^+ = 0$ for z_1.

Repeat the above procedure for G_n^- and identify combinations of $z_1,\ z_2,\ \cdots,\ z_n$ such that $0 < z_n \leq z_{n-1} \leq \cdots \leq z_1 \leq 1$ is satisfied. The value of $t_i = -l\log(z_i)$ and the corresponding $t_1,\ t_2,\ \cdots,\ t_n$ give the switching instants for a limit cycle about p and the *time period of limit cycle is* $T = t_n = -l\log(z_n)$ (refer to section III).

2) Characterizing L_T: We now characterize the set of points p that give a limit cycle of period T i.e. ∂L_T . Substitute the value of $z_n = e^{-\frac{T}{l}}$. Then $g_n^+(z_n, p_1, ..., p_n) = 0$ defines the smallest variety containing the parametric representation of ∂L_T^+ for given period T. Similarly, $g_n^-(z_n, p_1, ..., p_n) = 0$ defines the smallest variety containing the parametric representation of ∂L_T^- . Moreover, by convexity of L_T following sufficient condition follows:

Theorem 14. *If for all $\mu \in [0,1]$, $g_n^+(\mu p) \neq 0$ and $g_n^-(\mu p) \neq 0$ then $p \in L_T$.*

Example 15. Consider the system from Example 5 in Section II. For the target point $p = [p_1\ p_2]^T$, the solution of the system for limit cycle (i.e. with initial condition also being p) with $u = 1$ for $0 \leq t \leq t_1$, we get, for $0 < t_1 \leq t_2 < \infty$

$$
\begin{aligned}
(1 - e^{-t_2})p_1 + 1 - 2e^{-t_1} + e^{-t_2} &= 0 \\
(1 - e^{-2t_2})p_2 + \frac{1}{2} - e^{-2t_1} + \frac{1}{2}e^{-t_2} &= 0
\end{aligned}
$$

Substituting $e^{-t_i} = z_i$ for $i = 1, 2$ we get polynomial expressions and the inequality becomes $0 < z_2 \leq z_1 < 1$. Using the procedure described above we eliminate z_1 and obtain G_2^+ as:

$$
\begin{aligned}
s_1 &= z_2^2p_1^2 - 2z_2^2p_1 + 4z_2^2p_2 - z_2^2 - 2z_2p_1^2 \\
&\quad +2z_2 + p_1^2 + 2p_1 - 4p_2 - 1 \\
s_2 &= 2z_1 + z_2p_1 - z_2 - p_1 - 1
\end{aligned}
$$

Here $g_2^+ = s_1$. Similarly, we can find g_2^- using $u = -1$ for $0 \leq t \leq t_1$ and then switching accordingly. For $p = [0.3\ \ 0]^T$, $g_2^+ = (1.51z_2 - 0.31)(z_2 - 1) = 0$. The solution $z_2 = 1$ is not a meaningful solution since it gives a limit cycle of period $t_2 = 0$. The solution $z_2 = \frac{31}{151}$, gives $z_1 = \frac{218}{302}$ and the inequality $0 < z_2 \leq z_1 < 1$ is satisfied for this solution. Hence at $p = [0.3\ \ 0]^T$, we get a limit cycle with period $t_2 = 1.5833$. This can be verified from Figure 6 . Also for a given period, say $t_2 = T = 1$, the set of all the points p around which the state trajectory goes into a limit cycle of period $T = 1$ with starting input $u = 1$ can be computed from g_2^+. For $t_2 = 1$, we have $z_2 = e^{(-1)} = 0.3679$. With this value of z_2, $s_1 = 0$ gives $p_1^2 + 4.3276p_1 - 8.6554p_2 - 1 = 0$. Similarly, for starting input $u = -1$, the set of such points can be calculated from g_2^-. This set is shown in figure 7.

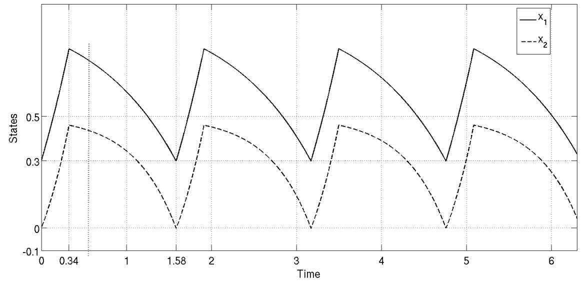

Fig. 6. Limit cycle with $p = [0.3\ \ 0]^T$and period $t = 1.5833$

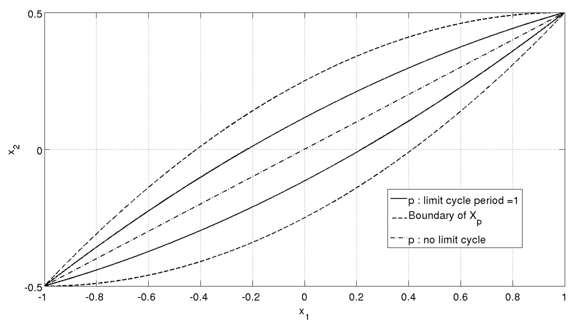

Fig. 7. Set of points p around which limit cycle has period 1

VI. The Set of Constrained Controllable Points (C) and The Reachable Set (X_p)

Let $C = \{p \in \mathbb{R}^n : p \in \text{int}\ X_p\}$ be the set of all constrained controllable points for (1). We describe an algebraic method to compute C. Recall the definition of the null controllable set X_0 and the attainable set $\mathcal{A}_0$ from section II. Under the assumption that system (1) is controllable, the set C can be specified for completely stable and anti-stable systems as follows [9]:

Lemma 16. *1) If for system (1), origin lies in C and $\lambda(A) < 0$ then, $X_0 = \mathbb{R}^n$ and $C = X_0 \bigcap \mathcal{A}_0 = \mathcal{A}_0$.*

2) If for system (1), origin lies in C and $\lambda(A) > 0$ then, $\mathcal{A}_0 = \mathbb{R}^n$ and $C = X_0 \bigcap \mathcal{A}_0 = X_0$.

It seems, that to compute C, one needs to have separate methods for computing X_0 (for anti-stable systems) and $\mathcal{A}_0$ (for stable systems). However, using (4) and (6), and we have,

$$
X_0 = \bigcup_{t \geq 0} \{x = -\int_0^t e^{-A\tau}Bu(\tau)d\tau : u(\tau) \in U\} \quad (14)
$$

$$
\mathcal{A}_0 = \bigcup_{t \geq 0} \{x = \int_0^t e^{A\tau}Bu(\tau)d\tau : u(\tau) \in U\} \quad (15)
$$

From (14) and (15), it can be clearly seen that the $\mathcal{A}_0$ for stable system can be calculated directly from the X_0 of a related system:

Lemma 17. *Consider two systems given by (1), one with $A = L$ and another with $A = -L$, where $L \in \mathbb{R}^{n\times n}$ and $\lambda(L) \in \mathbb{Q} - \{0\}$. Then the reachable set X_0 for one system (with $A = L$) is same as the attainable set $\mathcal{A}_0$ for the other system (with $A = -L$) and vice a versa.*

An algebraic technique for computing X_0 was already developed in [2]. Along with Lemma 16 and 17, this already developed method can be used to compute C for completely stable and anti-stable system.

For the case in which, some eigenvalues of A are positive and remaining are negative, the following results hold:

Lemma 18. *[19], [20] If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, with $A_1 \in \mathbb{R}^{n-m\times n-m}$ anti-stable $(\lambda(A_1) > 0)$ and $A_2 \in \mathbb{R}^{m\times m}$ stable $(\lambda(A_2) < 0)$, and B is partitioned accordingly as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ then, $X_0 = X_0^1 \times \mathbb{R}^m$ where X_0^1 is the null-controllable region of the anti-stable system $\dot{x}_1 = A_1x_1 + B_1u$.*

Theorem 19. *If for system (1), $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, with $A_1 \in \mathbb{R}^{n-m\times n-m}$ anti-stable $(\lambda(A_1) > 0)$ and $A_2 \in \mathbb{R}^{m\times m}$ stable $(\lambda(A_2) < 0)$, B is partitioned accordingly as $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and origin lies in C, then $C = X_0^1 \times \mathcal{A}_0^2$ where X_0^1 is the null controllable set for anti-stable system $\dot{x}_1 = A_1x_1 + B_1u$ and $\mathcal{A}_0^2$ is attainable set for the stable system $\dot{x}_2 = A_2x_2 + B_2u$.*

Proof: Using lemma 17 and 18, $X_0 = X_0^1 \times \mathbb{R}^m$. Similarly $\mathcal{A}_0 = \mathbb{R}^{n-m} \times \mathcal{A}_0^2$. Therefore, $C = X_0 \cap \mathcal{A}_0 = (X_0^1 \times \mathbb{R}^m) \cap (\mathbb{R}^{n-m} \times \mathcal{A}_0^2) = (X_0^1 \cap \mathbb{R}^{n-m}) \times (\mathbb{R}^m \cap \mathcal{A}_0^2) = X_0^1 \times \mathcal{A}_0^2$. □

Thus, to check whether given $x = [\ x_1 \quad x_2\ ]^T \in C$, we independently check whether $x_1 \in X_0^1$ and $x_2 \in \mathcal{A}_0^2$. Hence, by lemma 18, computation of null-controllability region X_0 for anti-stable systems suffices to characterize C for any given system.

Next we characterize the reachable set X_p for a point $p \in C$. The following lemma from [9] can be used directly.

Lemma 20. $X_p = X_0$ *if and only if* $p \in \mathcal{A}_0 \bigcap X_0 = C$.

Therefore, it is clear that the semi-algebraic characterization of X_p for any point $p \in C$ is same as that of null-controllability region X_0 of the given system. As mentioned above, X_0 can be computed using the method proposed in [2].

VII. Conclusions

In this article, we use a Gröbner basis based implicitization technique to get a semi-algebraic description of switching surfaces for constructing time optimal state feedback. The proposed method extends previous synthesis results to the case of non-zero target states. In practice, the switching feedback law drives the system to a some neighborhood of p, in near optimal time. The presence of limit cycles, even for time optimal transfer, is an interesting feature of this development. A characterization of target points for which limit cycle exists is provided. Properties of the limit cycle for different target points are studied. The proposed method is currently limited to controllable diagonalizable systems with non-zero rational eigenvalues. Relaxing some of these assumptions is the subject of current and future research.

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