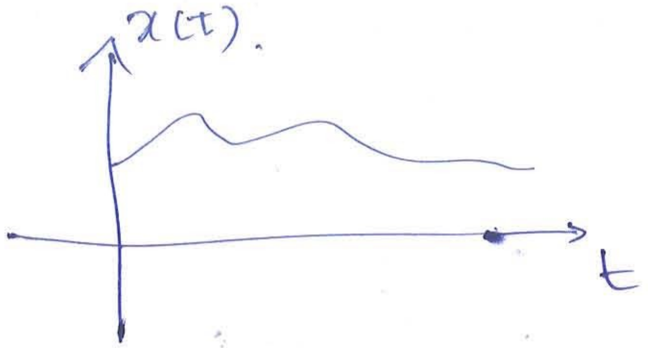
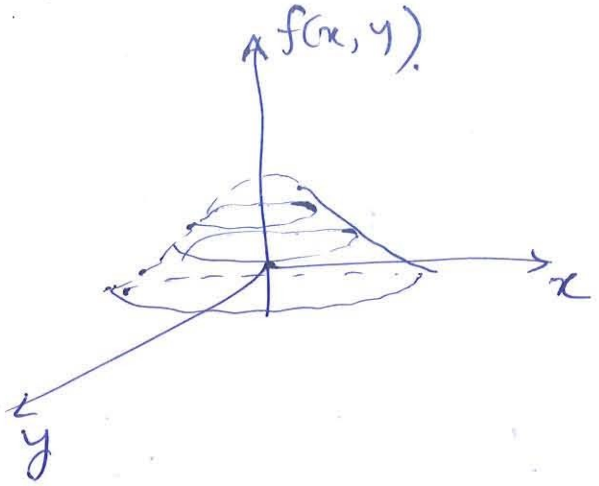
Signals & systems. (Week 1) ①

- What are Signals? Traffic signal [green, yellow, red]. Sound, Temperature, Pressure, velocity, acceleration, etc.

- A measurement of physical quantities represented as their variation with time, space, or any other independent direction.





- Most applications requires us to deal with or treat them as a system that operates on signals and gives out the desired signal.

- Broadly can be classified as analysing signals, Extracting, shaping signals.

eg:- - Predict weather Conditions based on past observations of pressure, temperature, wind speed etc.

- Seismic activity monitoring
- Voltage, currents, load angles on power grid. predicting worst case scenarios.

- get estimate of some indirectly available signals. Medical applications (ultrasound, ECG, MRI, etc)

- shape signals to get desired behaviour. noise cancelling headphones, hearing aids, etc.

This Course is building block which will teach to ~~analyse, design~~ how to effectively deal with signals & system

Signal & its representation

Signal is a function of "possibly" Several independent variables.

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \longmapsto \begin{bmatrix} f_1(x_1, x_2, \dots, x_m) \\ f_2(x_1, x_2, \dots, x_m) \\ \vdots \\ f_n(x_1, x_2, \dots, x_m) \end{bmatrix}$$

eg: Video signal is a signal in (x, y) and t where (x,y) is pixel location and t is frame number

$\begin{bmatrix} V_C(x, y, t) \\ V_I(x, y, t) \end{bmatrix}$ gives the Color & intensity of pixel (x, y) at frame t.

$$V : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

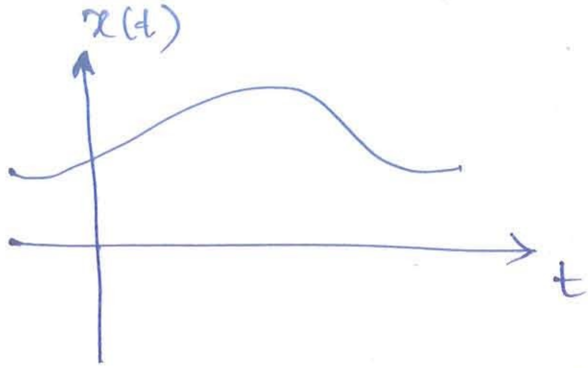
In this course we will deal with signals which are functions of "time" variable.

notation: $f(t)$, $u(t)$, ~~x~~, etc.

Signals can be classified in two categories

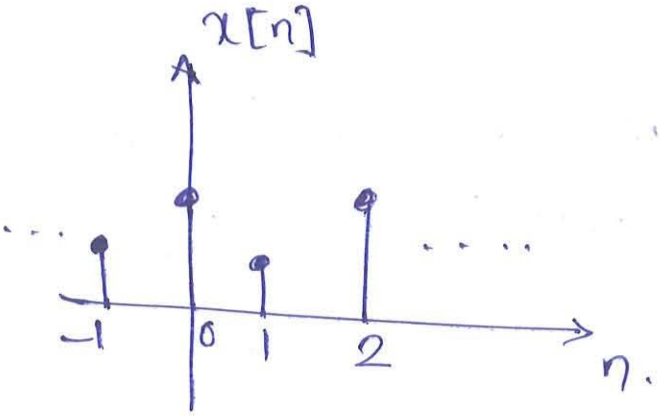
(i) Continuous time signals. [notation $v(t)$, $x(t)$, $y(t)$]

(ii) Discrete time signals. [notation. $x[n]$, $v[n]$, $i[n]$]



Continuous time signals are the ones where the value of signal is specified at every $t \in \mathbb{R}$

eg: Analog Signals.



Discrete time signals are the ones where value of signal is specified only at some discrete "samples" of time.

eg. Digital signals can be represented in this manner.

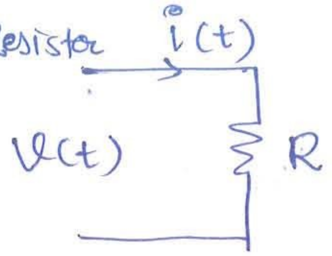
Also those quantities that cannot be monitored continuously gives rise to discrete time signals.

for eg: – Stock market indices, Blood glucose levels, etc.

- Signal Energy and Power.

[May not be always relatable to physical energy/power notions. Must be treated in abstract manner. But these notions are indeed "inspired" (or generalized) from the physical notions].

example : Instantaneous Power supplied to Resistor



is . $p(t) = \frac{v^2(t)}{R}$

Then $\int_{t_1}^{t_2} \frac{v^2(t)}{R} dt$ ~~would~~ is be the total energy expended in $t \in [t_1, t_2]$

and $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt$ is average power in this interval

Let $R = 1$. Then,
We can say that a signal $v(t)$ has total energy content in $[t_1, t_2]$ to be

$\int_{t_1}^{t_2} v^2(t) dt$. and

~~total~~ average power $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v^2(t) dt$

Generalizing for any Signal $x(t)$.

Total energy on interval $[t_1, t_2]$:

$$\int_{t_1}^{t_2} |x(t)|^2 \, dt$$

$|x(t)|$ is the absolute value of Complex no. at time t. (In case of Complex signal)

Time Averaged ~~tran~~ power

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

Analogously for Discrete time Signals.

Energy on samples between $[n_1, n_2]$

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

average power

$$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

We may be sometimes interested in Energy over infinite interval the so-called full line ~~integrals~~. intervals $(-\infty, \infty)$.

Continuous time :

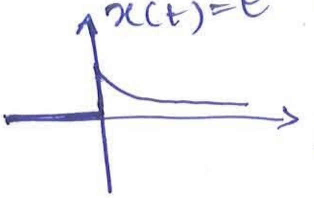
$$E_\infty := \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

[Integration is finite then finite energy signal.

Discrete time :

$$E_\infty := \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2$$ [Summation finite finite energy signal]

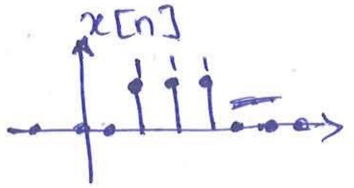
Examples :-



$$x(t) = \begin{cases} 0, & t<0 \\ e^{-\alpha t}, & t\geq 0 \end{cases} \quad \alpha > 0$$

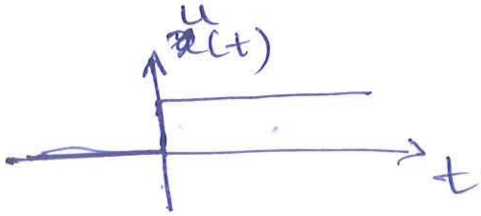
is a finite energy signal.

Or



is also a finite energy signal.

but



$$x(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

is infinite energy signal.

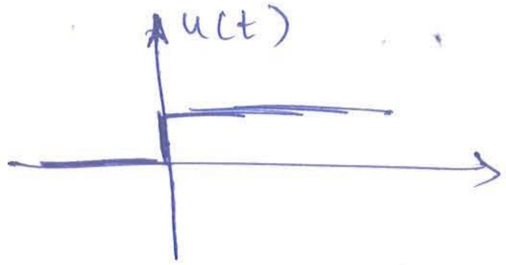
Time averaged power over infinite interval.

Continuous time:

$$P_\infty := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Discrete time:

$$P_\infty := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$



is a finite average power signal.

[But infinite energy]

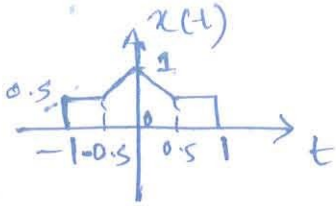
$\boxed{\text{Verify}}$ for $(-\infty, \infty)$ interval.

- Finite energy signal gives zero average power.
- Finite average power signal leads to infinite energy.

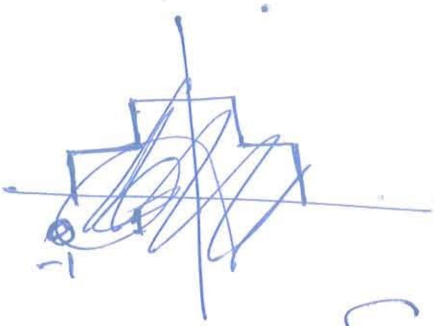
What about $x(t) = t$?

Signal Transformations.

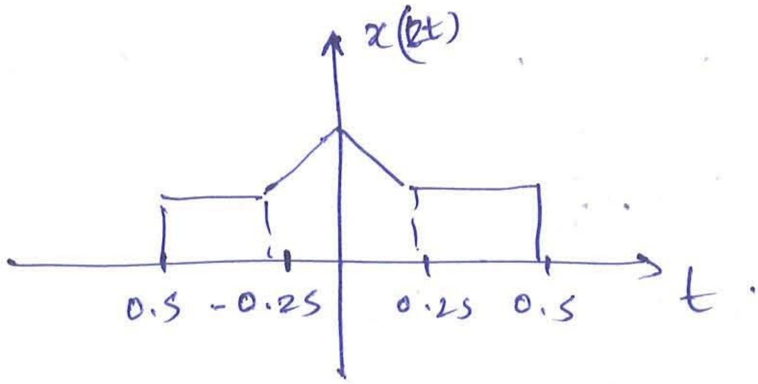
$x(t)$



$$x(t) = \begin{cases} 0, & t < -1 \\ 0.5, & -1 \le t \le -0.5 \\ t+1, & -0.5 \le t \le 0. \\ t-1, & 0 \le t < 0.5 \\ 0.5, & 0.5 \le t \le 1 \\ 0 & t > 1 \end{cases}$$



$$x(2t) = \begin{cases} 0, & 2t < -1 \equiv t < -0.5 \\ 0.5 & -1 \le 2t \le -0.5 \equiv -0.5 \le t \le -0.25 \\ 2t+1, & -0.5 \le 2t \le 0 \equiv -0.25 \le t \le 0. \\ 2t-1, & 0 \le 2t < 0.5 \equiv 0 \le t < 0.25 \\ 0.5 & 0.5 \le 2t \le 1 \equiv 0.25 \le t \le 0.5. \\ 0 & 2t > 1 \equiv t > 0.5 \end{cases}$$



Similarly Calculate $x\left(\frac{t}{2}\right)$, $x(-t)$, $x(t-1)$, $x(t+1)$

and so on.

Periodic Signals.

Most often encountered
- Harmonic Oscillators, ~~Electrical~~
- Seasonal cycles, predator prey.
- Clocks — Extensively used in Fourier analysis (later).

Designing a stable clock was a big problem in past

A signal $x(t)$ is a periodic signal if there is a positive no. $T > 0$ s.t.

$$x(t+T) = x(t)$$

if $x(t+T) = x(t)$

then $x(t+mT) = x(t)$ for any integer $m \in \mathbb{Z}$

$\mathbb{Z}$ is a set of integers

- which means any mT is time period of $x(t)$ if T is a time period.

- The least positive number $T_0 > 0$ for which $x(t+T_0) = x(t)$ is known as the fundamental period of $x(t)$.

eg: $\sin(2\pi t) = \sin(2\pi(t+m1))$ for any $m \in \mathbb{Z}$

But $T_0 = 1$ is least no. s.t

$$\sin(2\pi(t+T_0)) = \sin 2\pi t$$

Thus 1 is the fundamental time period of $\sin 2\pi t$

In Discrete time signals one needs to be careful.

A discrete time signal $x[n]$ is periodic if there exists a positive "<u>integer</u>" $N > 0$ s.t.

$$x[n+N] = x[n].$$

Just like in continuous case the least positive "<u>integer</u>" N_0 s.t

$$x[n+N_0] = x[n]$$ is fundamental period.

eg: $\cos\left(\frac{2\pi n}{3}\right)$ is periodic with period. $N_0 = 3$

$$\cos\left(\frac{2\pi}{3}(n+3)\right) = \cos\left(\frac{2\pi n}{3} + 2\pi\right) = \cos\left(\frac{2\pi n}{3}\right)$$

But $\cos\left(\frac{n}{2}\right)$ is not periodic because there is no "integer" s.t $\cos\left(\frac{n+N}{2}\right) = \cos\left(\frac{n}{2}\right)$

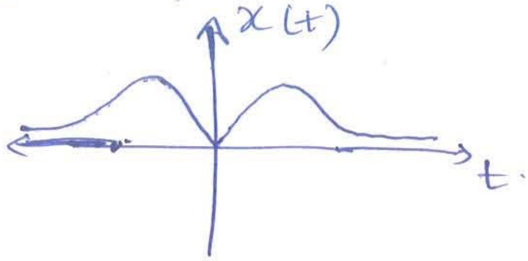
- Even and Odd Signals.

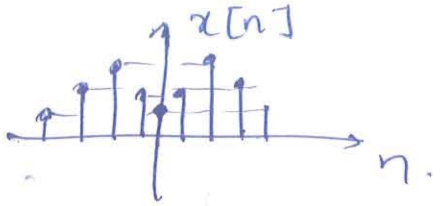
Even signal. $x(t)$ Satisfies

$x(t) = x(-t)$ (continuous time)

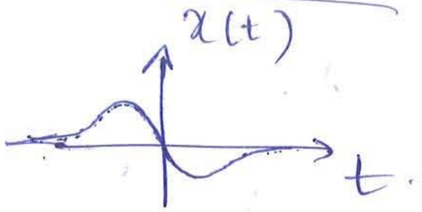
$x[n] = x[-n]$ (Discrete time)

Symmetric about $t=0$ (or $n=0$)



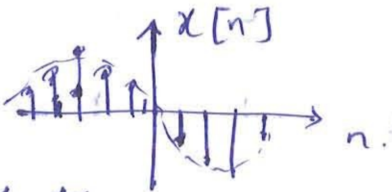


Odd signals.



$x(t) = -x(-t)$

$x[n] = -x[-n]$.



for odd signals notice that

$x(0) = -x(-0) \Rightarrow \quad x(0) = 0$

and $\quad x[0] = 0.$

Necessarily zero at the origin

$$x(t) = \underbrace{\frac{x(t) + x(-t)}{2}}_{\substack{Ev(x(t)) \\ \text{Even part}}} + \underbrace{\frac{x(t) - x(-t)}{2}}_{\substack{Od(x(t)) \\ \text{odd part}}}$$

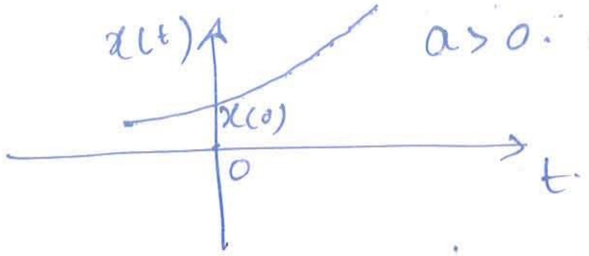
Since $\frac{x(-t) + x(t)}{2}$, $\frac{x(-t) - x(+t)}{2} = -\frac{(x(t) - x(-t))}{2}$

Commonly Encountered signals.

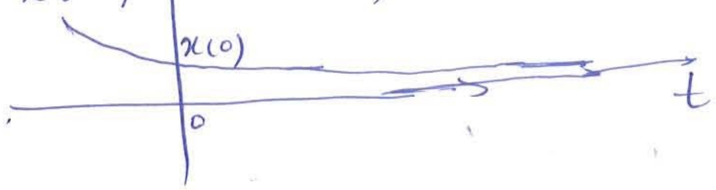
(i) Exponential. Signals.

$x(t) = x(0)\, e^{at}$ ["a" can be complex.]

$a > 0$: real.



$x(t)$ $a < 0$, real.



a complex.

then $x(t) = e^{(\sigma + j\omega)t}\, x(0)$

$= e^{\sigma t}\left(\cos \omega t + j \sin \omega t\right) x(0)$

[Euler's identity
$e^{j\omega t} = \cos \omega t + j \sin \omega t$]

now if $\sigma > 0$

$e^{\sigma t} \sin \omega t \, x(0)$

Similary plot $e^{\sigma t} \cos \omega t$

$$e^{j\omega_0 t} = e^{j\omega_0\left(t + \frac{2\pi}{\omega_0}\right)}.$$

Thus, $T_0 = \frac{2\pi}{\omega_0}$ is fundamental period.

In general, $T>0$ s.t. $\omega_0 T = 2\pi m$, $m \in \mathbb{Z}$ is a time period for $e^{j\omega_0 t}$. but Least T is $T_0 = \frac{2\pi}{|\omega_0|}$ fundamental period.

Sine & Cosine with Phase.

$$x(t) = A\cos(\omega_0 t + \phi).$$

$T_0 = \frac{2\pi}{\omega_0}$ is fundamental period.

DISCRETE TIME

$$x[n] = e^{j\omega_0 n}$$

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

$$\Rightarrow e^{j\omega_0 N} = 1$$

will be satisfied. if $\omega_0 N = 2\pi m$, $m \in \mathbb{Z}$

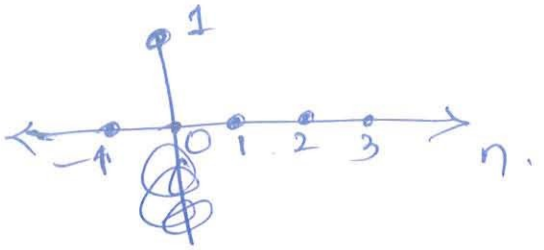
Thus. if $N = \frac{2\pi m}{\omega_0}$ is an integer

Only then N is the period of signal $x[n]$

If integer, $N_0 = \frac{2\pi}{\omega_0}$ is fundamental period.

- Unit impulse. [Discrete time].

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



plot $\delta[n-1]$
$\delta[n+2]$?

For any sequence or signal $x[n]$. Pointwise multiplication.

$$x[n]\delta[n] = x[0]\delta[n].$$

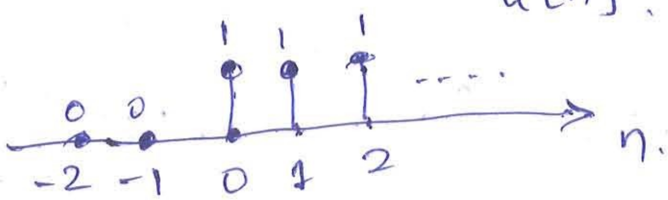
$$\underbrace{x[n]\,\delta[n-2]}_{\text{Pointwize multiplic..}} = \underbrace{x[2]}_{\text{value of } x[n] \text{ at } n=2}\ \underbrace{\delta[n-2]}_{\text{signal}}$$

$\delta[n]$ samples $x[n]$ at $n=0$.

$\delta[n-2]$ samples $x[n]$ at $n=2$.

- unit step.

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0. \end{cases}$$



note:

- $\delta[n] = u[n] - u[n-1].$

- $u[n] = \sum_{m=-\infty}^{n} \delta[m]$ } Cumulative. Sum of values of $\delta[m]$ at n.

- $u[n] = \sum_{k=\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$

Continuous time unit step

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

undefined at $t=0$ (by convention?)

Dirac - delta (Ideal. Impulse)

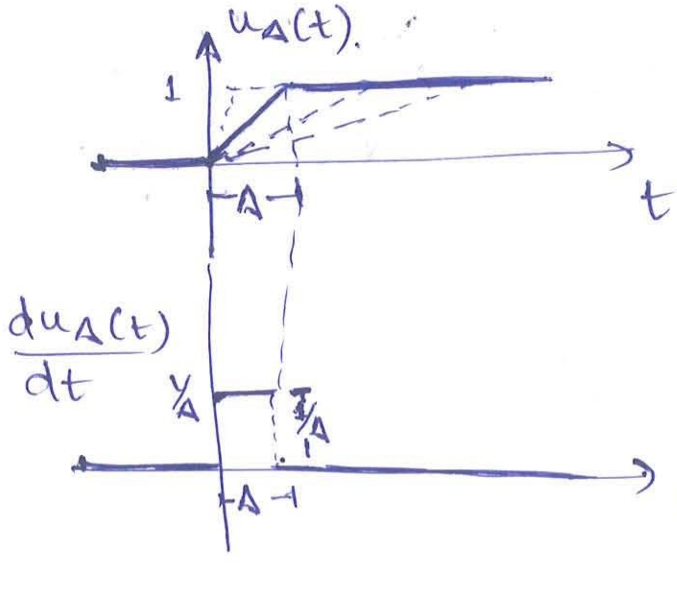
$$\boxed{\frac{du(t)}{dt} = \delta(t)} \longrightarrow \text{Strange!}$$

notice in discrete time case:
$\delta[n] = u[n]-u[n-1]$

[derivative is to be understood in limiting sense]

[More precise defn. can be done by using "theory of distributions" (Schwarz, 1951)].

Consider an approximate unit step. $u_\Delta(t)$.



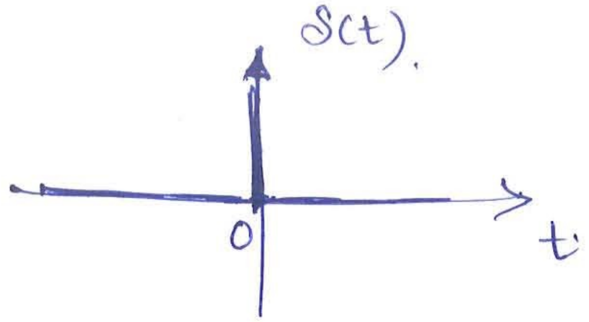
$$u_\Delta(t) = \begin{cases} 0 & t<0 \\ \left(\frac{1}{\Delta}\right)t & 0 \le t \le \Delta \\ 1 & t \ge \Delta \end{cases}$$

$$\frac{du_\Delta(t)}{dt} = \begin{cases} 0, & t<0 \\ \frac{1}{\Delta}, & 0<t \le \Delta \\ 0, & t \ge \Delta \end{cases}$$

Then $\frac{du(t)}{dt} = \lim_{\Delta \to 0} \frac{du_\Delta(t)}{dt} = \delta(t).$

It is in this limiting sense that the derivating $\frac{du(t)}{dt}$ is to be understood.

We represent $\delta(t) = \begin{cases} \text{infinity}, & t = 0 \\ 0, & t \neq 0 \end{cases}$

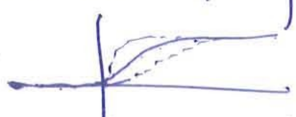


The ~~magnitude~~ strength of $\delta(t)$ is measured by ~~the~~ its area

Note that we can approximate $\delta(t)$ by $\delta_\Delta(t)$ which is $\frac{du_\Delta(t)}{dt}$. (see previous page)

Then $\int_{-\infty}^{t} \delta(t)\,dt = u(t)$ for $t > 0$.

We could have very well use function



to approximate unit step. and arrived at same limiting case.

because $\int_{-\infty}^{t} \delta_\Delta(t)\,dt = 1$ any. for $t \mp \Delta > 0$ and as $\Delta \to 0$. $\int_{-\infty}^{t} \delta_\Delta(t)\,dt \to \int_{-\infty}^{t} \delta(t)\,dt$ and $\int_{-\infty}^{t} \delta_\Delta(t)\,dt = 1$ so is $\uparrow$

Thus $\int_{-\infty}^{t} \delta(t) = u(t)$

and $\int_{0}^{\infty} \delta(t-\sigma)\, d\sigma = u(t)$.

Properties of $\delta(t)$

$$\delta(-t) = \delta(t).$$

Infact $\delta(t)$ is sometimes defined as an object which acts on signal $x(t)$ and gives its value at particular instance. for example.

$$x(0) = \int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt$$

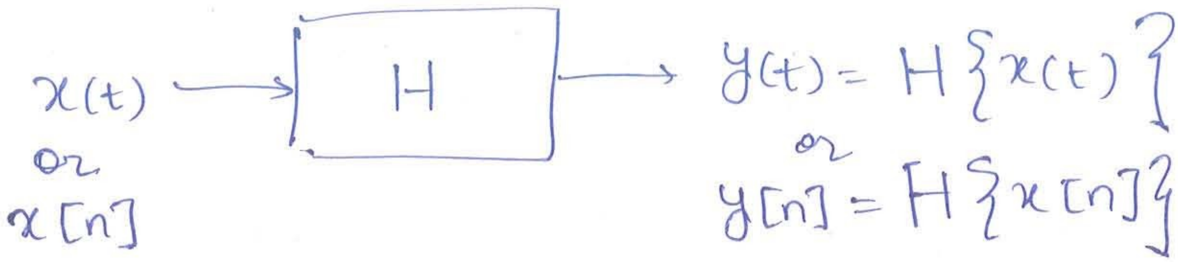
or $x(\tau) = \int_{-\infty}^{\infty} x(t)\, \delta(t-\tau)\, dt$

This property we will derive and use later to represent any signal $x(t)$ in terms of continuum of Dirac-delta impulses.

Systems.

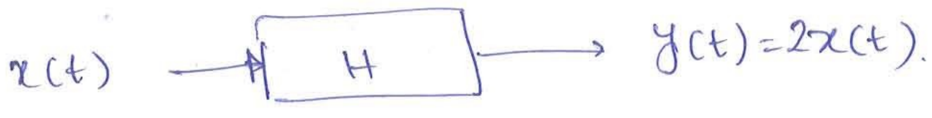
- Systems operate on signals and gives out another signals.

One can treat a system as a blackbox or a function H

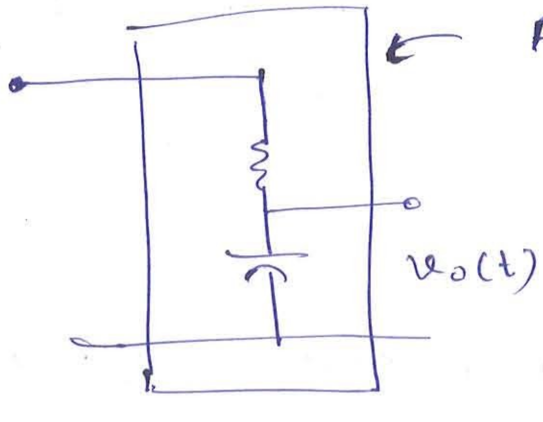


Ex:

(i)



(ii)



A slightly more complicated System.

- A ~~system~~ Very useful class of systems that we frequently encounter are the systems that posses ~~proper~~ following properties

(i) Linearity

(ii) Time invariance.

- Linear System. [A similar defn holds for discrete time systems]

A system H is said to be linear

if it satisfies.

(i) for any signals $x_1(t)$ and $x_2(t)$

Suppose $y_1(t) = H\{x_1(t)\}$ and

$y_2(t) = H\{x_2(t)\}$.

then. $y(t) = H\{x_1(t) + x_2(t)\}$

$= H\{x_1(t)\} + H\{x_2(t)\}$

$= y_1(t) + y_2(t)$.

holds. [additivity/ Superposition]

(ii). for any scalar $\alpha \in \mathbb{C}$, and

any signal $x(t)$

Suppose $y(t) = H\{x(t)\}$.

then $\bar{y}(t) = H\{\alpha x(t)\}$

$= \alpha H\{x(t)\} = \alpha y(t)$.

holds.

[Homogeneity property]

Note in particular ~~for~~ for $x(t) \equiv 0$, $y(t) \equiv 0$

(choose $\alpha = 0$ and apply homogeneity) eg: $y(t) = H\{x(t)\} = t\,x(t)$

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = t\,x(t)$$

Consider any $x_1(t)$ and $x_2(t)$

s.t $y_1(t) = t\,x_1(t)$

$y_2(t) = t\,x_2(t)$

then indeed. for $x(t) = x_1(t) + x_2(t)$

$$\begin{aligned} y(t) &= H\{x_1(t) + x_2(t)\} \\ &= t\left(x_1(t) + x_2(t)\right) \\ &= y_1(t) + y_2(t). \end{aligned}$$

Similarly for any α and $x(t)$

if $y(t) = H\{x(t)\} = t\,x(t)$

then $$\begin{aligned} \bar{y}(t) &= H\{\alpha\, x(t)\} \\ &= t\left(\alpha\, x(t)\right) \\ &= \alpha\, t\, x(t) = \alpha\, y(t). \end{aligned}$$

hence. $y(t) = t\,x(t)$ is a linear system

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example 2.

Is $x(t) \longrightarrow \boxed{H} \longrightarrow y(t) = 2x(t) + 1$

linear? Answer no! (why?)

Time invariant system.

A system H is said to be time invariant if for any input $x(t)$ we get $y(t) = H\{x(t)\}$ then for $x(t-t_0)$ (for any $t_0 \in \mathbb{R}$) we must get $y(t-t_0)$ as the output.

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$x(t-t_0) \longrightarrow \boxed{H.} \longrightarrow \bar{y}(t) = y(t-t_0)$$

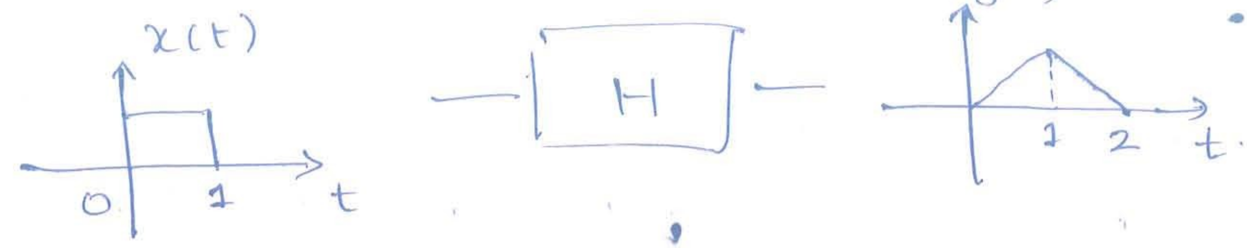
In discrete time case (also called shift invariance)

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n].$$

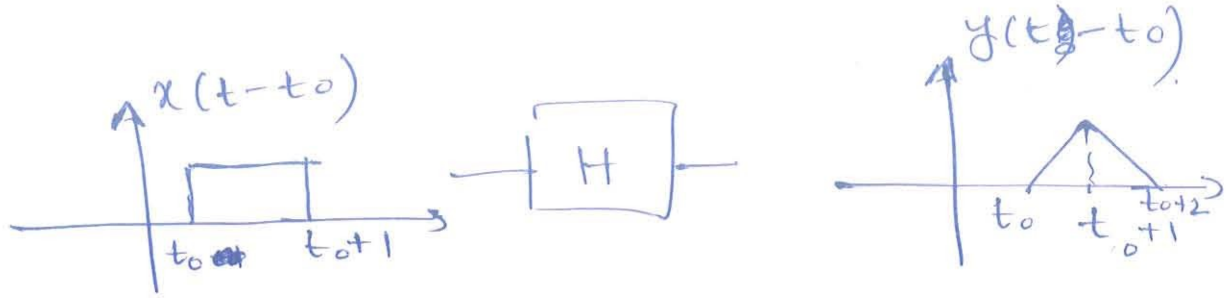
$$x[n-n_0] \longrightarrow \boxed{H} \longrightarrow \bar{y}[n] = \bar{y}[n-n_0].$$

eg:

if



then



- Systems with and without memory

(i) Memoryless systems:

Output depends upon only the input value at current instance of time.

eg: $y(t) = 2x(t)$

(ii) Memory.

Output has dependence on past / future / both input value.

ex. 1) $y(t) = x(t-1) + x(t+1)$

2) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

3) $y(t) = \frac{d}{dt} x(t)$

Invertibility of system.

A system is invertible if it is possible to uniquely infer input given the output signal.

In other words

Distinct inputs leads to distinct output.

Ideally we would like ~~to~~ Systems to be invertible (Think of encoding - decoding).

[But not for adversaries]

- Causal Systems.

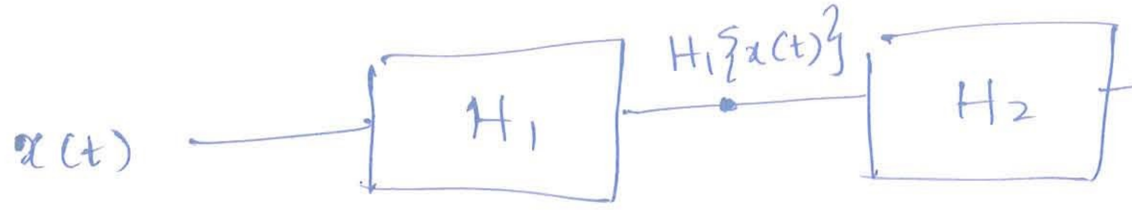
  - Output at any time depend only on input values at present and/or those seen in the past.

  - Non- anticipating nature.

  - $y(t) = x(t+1)$ (is not Causal system)
  - $y[n] = x[n] + 2x[n-1]$ is causal.
  - $y(t) = x(t-1)$? $y(t) = \frac{d}{dt}x(t)$?

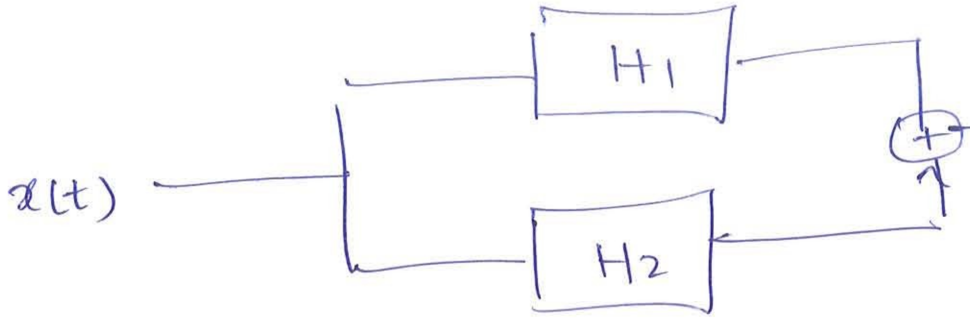
Various interconnections of systems.

a) Cascade inter connection.



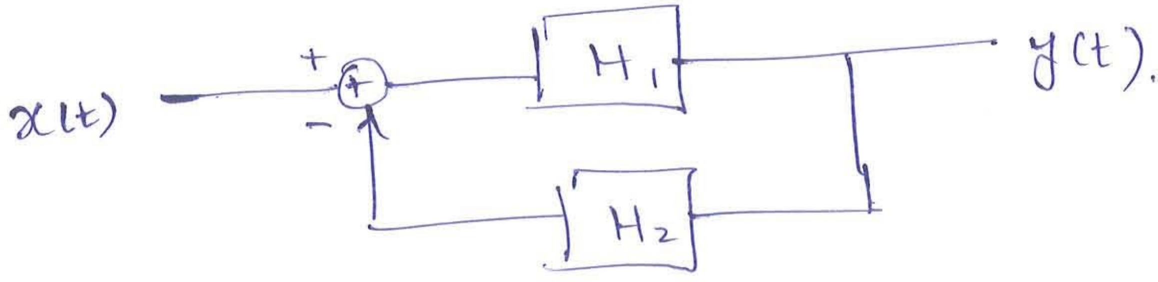
$y(t) = H_2\{H_1\{x(t)\}\}$

b) Parallel interconnection.



$y(t) = H_1\{x(t)\} + H_2\{x(t)\}$.

c) Feedback interconnection.



$$y(t) = H_1\left\{x(t) - H_2\{y(t)\}\right\}.$$

d) Combination of a, b, c.