Z- transform.

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} z^{n-k}\, h[k]$$

$$= z^n H(z)$$

$$H(z) := \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}.$$

For. :

$z = e^{j\omega}$ with $\omega \in \mathbb{R}$

~~We got~~ $|z| = 1$

and $$H(e^{j\omega}) = \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}}_{\text{DTFT of } h[n].}$$

Z- Transform. of $x[n]$

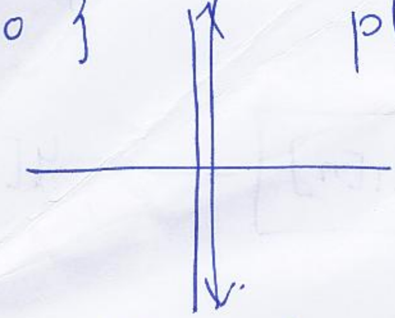
$$x[n] \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}.$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n}\, e^{-j\omega n}.$$

It is DTFT of $x[n]\, r^{-n}$.

Z- transform. reduces to Fourier Tr. when $r = 1$

(Laplace. " " $\sigma = 0$)

Just like line $\{\sigma = 0\}$  played a

$s = j\omega;\ \omega \in \mathbb{R}$

roled in Laplace Transform.

Analogous role is played by $r = 1$

or $z = e^{j\omega}$.

Ex: $x[n] = a^n u[n]$.

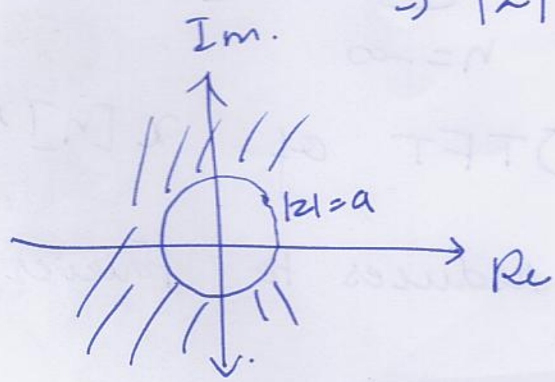
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=-\infty}^{\infty} (a z^{-1})^n u[n]$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}.$$

But only possible if $|a z^{-1}| < 1$

$$\Rightarrow |z^{-1}| < \frac{1}{|a|}$$

$$\Rightarrow |z| > |a|$$



$x[n] = -a^n u[-n-1]$.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n u[-n-1] z^{-n}.$$

$$\cancel{= \sum_{n=1}^{\infty} a^{-n} z^n. = \sum_{n=1}^{\infty} (a^{-1} z)^{+n}}$$

$$\cancel{= \frac{1}{1-a^{-1}z} =}$$

$$= -\sum_{n=1}^{\infty} (a^{-1} z)^n = -1 - \sum_{n=1}^{\infty} (a^{-1} z)^n + 1$$

$$= -\sum_{n=0}^{\infty} (a^{-1} z)^n + 1$$

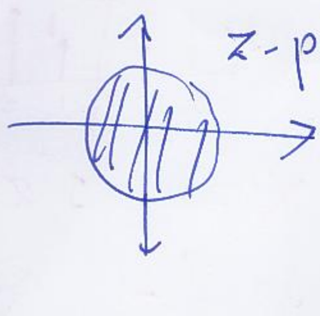
$$= \frac{-1}{1-a^{-1}z} + 1 = \frac{a^{-1}z}{1-a^{-1}z}$$

$$= \frac{-z}{a-z} = \frac{z}{z-a}.$$

$|a^{-1}z| < 1$

$|z| < a$

z-plane.



$$x[n] = +a^{-n}u[-n-1] + a^{n}u[n] = a^{|n|},\ a>0.$$

$$X(z) = \frac{-z}{z-a^{-1}} + \frac{z}{z-a}.$$

$$= \frac{-z(z-a) + z(z-a^{-1})}{(z-a^{-1})(z-a)}$$

$$= \frac{az - a^{-1}z}{(z-a^{-1})(z-a)} = \left(a - \frac{1}{a}\right)\frac{z}{(z-a^{-1})(z-a)}$$

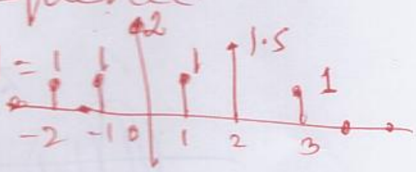
ROC. $a < |z| < a^{-1}$

for $a < 1$

for $a > 1$ not defined for any z.

- Finite sequence.

$x[n] =$



$$X(z) = 1z^{2} + 1z + 2 + z^{-1} + 1.5z^{-2} + z^{-3}$$

- $x[n] = \begin{cases} 1 & n \in [-3, 3] \end{cases}$ $\quad X(z) = z^{-3} + z^{-2} + z^{-1} + 1 + z$

For exponentials / complex exponentials.

$$X(z) = \frac{N(z)}{D(z)}$$

$N(z)$ roots are zeros of $X(z)$

$D(z)$ roots are poles of $X(z)$

Property of ROC.

(i) ROC of $X(z)$ consists of a ring in z-plane centered around the origin

$x[n] = a^n u[n]$

(ii) ROC does not contain any poles.

(iii) For finite duration signal $x[n]$ $X(z)$'s ROC is entire z-plane except $z=0$ and / or $z=\infty$.

(iv) $x[n]$ right sided then ROC (unbounded) opens outside a circle.

(v) $x[n]$ left sided then ROC is inside circle (Bounded)

$x[n] = a^n u[-n-1]$

(xvi). for two sided sequence.

ROC is annular region between two circles.

Inverse Z-transform.

Z- transform Fourier Transform of

$$X(z)= F\left(x[n]\, r^{-n}\right) = \sum_{n=-\infty}^{\infty}\left[x[n]\, r^{-n}\right] e^{-j\omega n}.$$

For. $z = r e^{j\omega}$ in ROC we get.

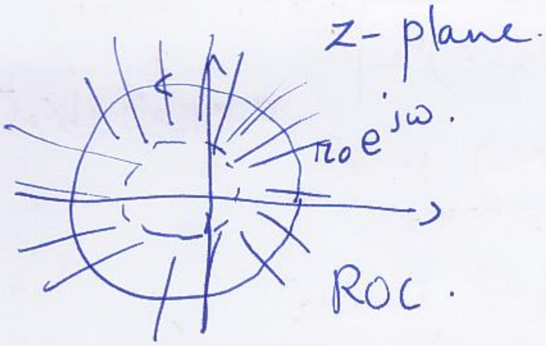
$$F^{-1}\left(X(re^{j\omega})\right) = x[n]\, r^{-n}.$$

$$x[n]\, r^{-n} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^{n} d\omega.$$

let $z = re^{j\omega}$ then $dz = re^{j\omega} j\, d\omega$
$= z\, j\, d\omega.$

$$x[n] = \frac{1}{2\pi j}\oint_{2\pi} X(z)\, z^{n-1}\, dz$$



Integral over a circle of radius r_0 defined by $z = r_0 e^{j\omega} \subseteq$ ROC.

Once again just as in case of Laplace Transforms we only use known tables of ~~Lap~~ Z-transform $\leftrightarrow$ Inverse Z-transform pairs.

eg: $X(z) = \sum_{i=1}^{m} \frac{A_i}{1 - a_i z^{-1}}$, $|z| > \max_i \{|a_i|\}$ $\Rightarrow$ $x[n] = \sum_{i=1}^{m} A_i (a_i)^n u[n]$

$X(z) = 4z^2 + 2 + 3z^{-1}$, $0 < |z| < \infty$

$$x[n] = \begin{cases} 4, & n = 2 \\ 0, & n = 1 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$X(z) = \frac{1}{1 - az^{-1}}$, $|z| > |a|$

$$\begin{array}{r} 1 + az^{-1} \\ 1 - az^{-1} \,\big)\, 1 \quad\quad\quad \\ \underline{1 - az^{-1}} \\ az^{-1} \\ \underline{az^{-1} - a^2 z^{-2}} \\ a^2 z^{-2} \end{array}$$

$= 1 + az^{-1} + a^2 z^{-2} + \dots$

$x[n] = a^n u[n]$.

$X(z) = \ln(1 + az^{-1})$, $|z| > |a|$

$\ln(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n z^{-n}$

$x[n] = -\frac{(-a)^n}{n} u[n-1]$

$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$

$= \frac{A}{1 - \frac{1}{4} z^{-1}} + \frac{B}{1 - \frac{1}{3} z^{-1}}$

$= \frac{A - \frac{A}{3} z^{-1} + B - \frac{B}{4} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$

$\frac{(A + B) - \left(\frac{A}{3} + \frac{B}{4}\right) z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$

$A + B = 3$

$\frac{A}{3} + \frac{B}{4} = \frac{5}{6}$.

~~$X(z)$~~ $=$

$= \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$; $x[n] = (\frac{1}{4})^n u[n] + (\frac{1}{3})^n u[n]$

Properties.

(i) Linearity $x_1[n] \leftrightarrow X_1(z)$ ROC $= R_1$

$x_2[n] \leftrightarrow X_2(z), R_2$

$a x_1[n] + b x_2[n] \overset{?}{\leftrightarrow} a X_1(z) + b X_2(z) \quad R_1 \cap R_2$

(ii) Time Shifting $x[n] \leftrightarrow X(z)$ ROC $= R$.

$x[n-n_0] \leftrightarrow z^{-n_0} X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}, = \sum_{l=-\infty}^{\infty} \underbrace{x[l] z^{-n_0} z^{-l}}$$

$n - n_0 = l$

$= z^{-n_0} X(z)$

(iii) Scaling in Z-domain

$x[n] \overset{Z}{\leftrightarrow} X(z)$. ROC $= R$

$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$ ROC $= |z_0| R$

(iv) $x[n] \leftrightarrow X(z)$ ROC $= R$.

$x[-n] \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ ROC $= \frac{1}{R}$.

(v) Time expansion.

$x[n] \leftrightarrow X(z)$

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right] & \text{if } k \text{ divides } n. \\ 0 & \text{Otherwise.} \end{cases}$$

$x_{(k)}[n] \overset{Z}{\leftrightarrow} X(z^k)$ ROC $= R^{1/k}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

$$X(z^k) = \sum_{n=-\infty}^{\infty} x[n] z^{-nk}$$

$$= \cancel{\sum_{n=-\infty}^{\infty} x\left[\frac{n}{k}\right] z^{-n}}$$

$$= \sum_{m=-\infty}^{\infty} x\left[\frac{m}{k}\right] z^{-m}.$$

for $\frac{m}{k}$ integer.

(vi) Conjugation.

$x[n] \overset{Z}{\leftrightarrow} X(z)$ ROC $= R$

$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*)$ ROC $= R$

for real $x[n]$

$X(z) = X^*(z^*)$

(vii) Convolution.

$x_1[n] \leftrightarrow X_1(z), R_1$

$x_2[n] \leftrightarrow X_2(z), R_2$

$x_1[n] * x_2[n] \overset{Z}{\leftrightarrow} X_1(z) X_2(z)$

ROC $= R_1 \cap R_2$

(viii) $x[n] \overset{Z}{\leftrightarrow} X(z)$ ROC $= R$

$n x[n] \overset{Z}{\leftrightarrow} -z \frac{dX(z)}{dz}$ ROC $= R$.

(xi)

$$x[0] = \lim_{z \to \infty} X(z)$$ for $x[n]$ s.t $x[n] = 0, n < 0$

A Discrete time LTI system can be represented (~~replaced~~) by its impulse response z-transform.

$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$

$x[n]$ $\qquad h[n] \qquad y[n] = h[n] * x[n]$.

$H(z)$ is gain offered to signal $x[n] = z^n$
(sort of eigen value defining function)

$H(z)$ - Transfer function. $H(z) = \dfrac{N(z)}{D(z)}$, Roots of $N(z)$ - Zeros of system
Roots of $D(z)$ - poles of system.

- Causality of system by its Transfer function

A Discrete time LTI system is Causal if $h[n] = 0$ for $n < 0$.

This means that in z-transform. ~~zo~~.
expansion in terms of summation.
$z, z^2, \ldots, z^n$ terms won't appear.

Which means $\lim\limits_{z \to \infty} H(z)$ must be finite

- A discrete time LTI system is Causal

$\Updownarrow$

ROC of $H(z)$ is exterior of a disc with center at origin & also includes infinity

If $H(z)$ is Rational i.e of the for $\frac{N(z)}{D(z)}$

then degree $(N(z)) \leq$ degree $(D(z))$ for Causality and vice . a . versa.

- STABILITY.

DT LTI system is stable if & only if:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$ (absolutely summability of impulse response)

$\Updownarrow$

DT Fourier transform of $h(n)$ exists

$\Updownarrow$

$H(z)$ ROC must contain unit circle.

- For Rational $H(z)$

poles of $H(z)$ must lie within unit circle. i.e. magnitude of poles must be less than 1.

- Difference Equation based systems. Can be replaced by its transfer fun.

$$x[n] \longrightarrow \boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]} \longrightarrow y[n]$$ (Assuming zero initial conditions)

$X(z)$ $\qquad Y(z)$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Since $y[n] \leftrightarrow Y(z)$

$y[n-k] \leftrightarrow z^{-k} Y(z)$

$$Y(z) = \underbrace{\boxed{\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}}_{H(z)} X(z).$$

— With Non-Zero initial conditions

look for unilateral Z-transform.

$$Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n}$$ Start everything from 0.

Let $w[n] = y[n-1]$.

then. $$W(z) = \sum_{n=0}^{\infty} y[n-1] z^{-n}.$$

$$= y[-1] + \sum_{n=1}^{\infty} y[n-1] z^{-n}.$$ put $m = n-1$

$$= y[-1] + \sum_{m=0}^{\infty} y[m] z^{-(m+1)}$$

$$= y[-1] + z^{-1} Y(z).$$

$$y[n-2] \longleftrightarrow y[-2] + z^{-1} y[-1] + z^{-2} Y(z)$$

$$\cancel{y[n-k] \longleftrightarrow \sum_{l=0}^{k-1} z^{-(k-1-l)} y[-k+1+l] + z^{-K}} Y(z)$$

$$y[n-3] \longleftrightarrow y[-3] + z^{-1} y[-2] + z^{-2} y[-1] + z^{-3} Y(z)$$

$$y[n-k] \longleftrightarrow \sum_{l=0}^{k-1} z^{-l} y[-k+l] + z^{-K} Y(z).$$

0. and so. initial conditions can be tackled in solving difference eq.

Consider.

$$y[n] + 2y[n-1] = x[n]$$

with $y[-1] = \beta$

$$Y(z) + 2\left[y[-1] + z^{-1} Y(z) \right] = X(z)$$

$$Y(z)(1 + 2z^{-1}) + 2y[-1] = X(z)$$

$$Y(z) = \underbrace{\frac{X(z)}{1+2z^{-1}}}_{\text{Zero initial condition. Response.}} + \underbrace{\frac{2y[-1]}{1+2z^{-1}}}_{\text{Zero input response.}}$$

$$\boxed{y[n] = Z^{-1}\left\{ \frac{X(z)}{1+2z^{-1}} \right\} + 2Z^{-1}\left\{ \frac{y[-1]}{1+2z^{-1}} \right\}}$$

Procedure to solve difference eqns.

a) Compute zero initial condition response (By assuming zero initial conditions).

b) Compute zero input response by assuming zero input. (so called soln to homogeneous eq).

c) $y[n] = y_{(a)}[n] + y_{(b)}[n]$.