– Linear Time Invariant Systems. (LTI systems)

(Time Domain Representation)

<u>Discrete time case</u>

One of the crucial aspects of unit impulse i.e. the sifting property can be used to represent any signal $x[n]$ in terms of a train of ~~white~~ impulses of varying magnitudes. Then a response of an LTI system to this unit impulse can be used to ~~dep~~ represent a LTI system. This representation goes under the name "Impulse response".

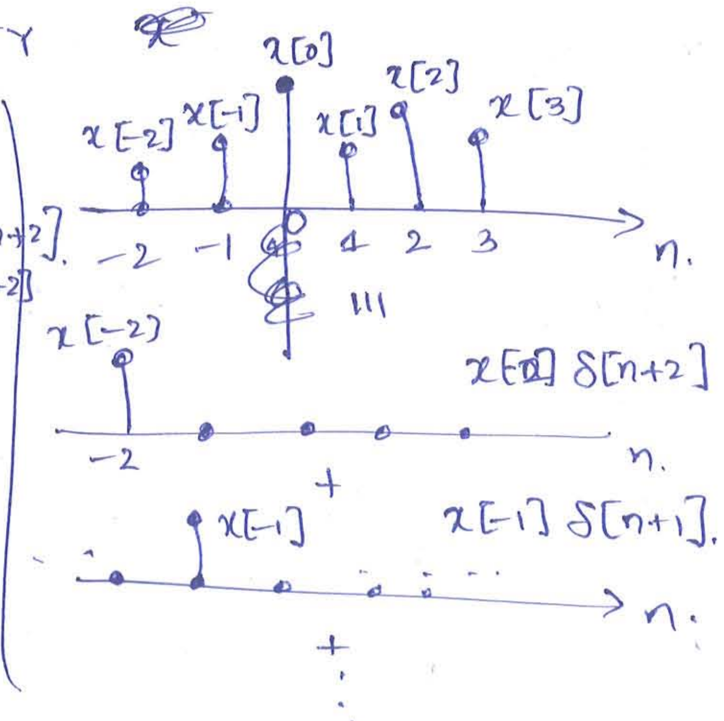
– SIFTING PROPERTY

Consider $x[n]$

then $x[n]\delta[n+2] = x[-2]\delta[n+2]$.
$x[n]\delta[n-2] = x[2]\delta[n-2]$
and so on.

Thus

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].$$



$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]} \quad (*)$$

$$= \ldots + x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + \ldots$$

(*) is the representation of $x[n]$ as a train of impulses with varying magnitudes.

An impulse response of System H

Now let us consider a system

$y[n] = H\{x[n]\}$.

Then $y[n] = H\left\{ \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \right\}$.

If H is linear then by additivity and homogeneity we get

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, H\{\delta[n-k]\}.$$

Let $h_k[n] = H\{\delta[n-k]\}$.

then $y[n] = \sum_{k=-\infty}^{\infty} h_k[n]\, x[k]$

~~Notes~~ 2

The signal $h_K[n]$ is the ~~impulse~~ response of H to a unit impulse $\delta[n-k]$.

In case ~~we were considering~~ of time varying systems ~~then~~, $h_K[n]$ for all $K \in \mathbb{Z}$ are unrelated (or no direct relation ~~is~~ seen).

But, in case of time invariant systems.

$$h_K[n] = h_0[n-k]$$ (By definition of time invariance)

where $h_0[n]$ is the response of H to unit impulse $\delta[n]$.

And Thus for any input signal $x[n]$ we obtain the response of H as follows.

$$\boxed{y[n] = H\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]\, h_0[n-k]}$$

Knowledge of $h_0[n]$ is sufficient to determine response of H to any signal $x[n]$.

The operation between two signals $x[n]$ and $h_o[n]$ given by

$$\sum_{k=-\infty}^{\infty} x[k]\, h_o[n-k] := x[n] * h_o[n]$$

is ~~called~~ defined to be the Convolution of $x[n]$ with $h_o[n]$. " $*$ " is used to denote this binary operation.

Thus, for any signal $x[n]$ the response of a system H is the convolution of signal $x[n]$ with the impulse response of H

$$H\{x[n]\} = x[n] * h_o[n].$$

Remark: The impulse response $h_o[n]$ completely characterizes H (LTI system)

One more property of unit impulse.

$$H\{\delta[n]\} = \delta[n] * h_o[n] = h_o[n].$$

$$\Rightarrow \delta[n] * h_o[n] = h_o[n].$$

Convolution of unit impulse $\delta[n]$ & any signal $x[n]$ results in $x[n]$ itself.

~~Stability~~ -

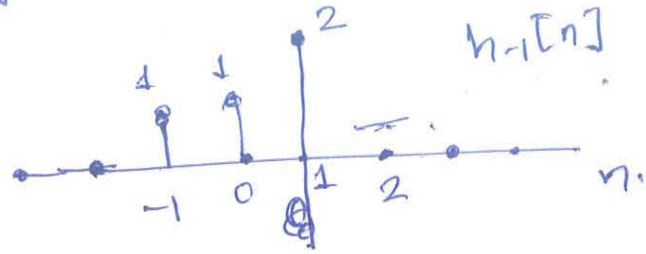
Example : Time Varying System

Can be specified by responses to

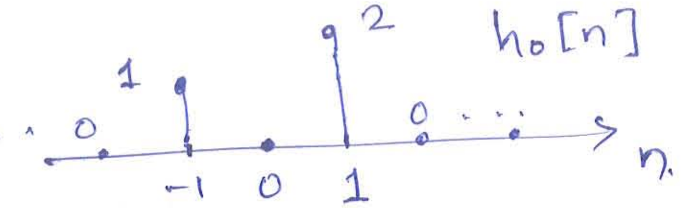
$\delta[n-k]$ for $k \in \mathbb{Z}$

i.e. $H\{\delta[n-k]\} = h_k[n]$ for $k \in \mathbb{Z}$

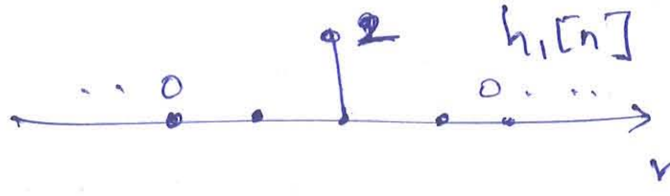
Let. $h_{-1}[n]$



$h_0[n]$

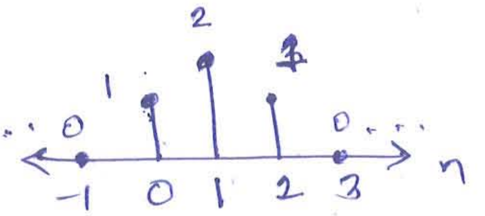


$h_1[n]$



and $h_k[n] \equiv 0$ for $k \in \mathbb{Z} - \{-1, 0, 1\}$.
(all k except $-1, 0, 1$)

then. $x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = H\{x[n]\}$



$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$= x[0]\,\delta[n-0] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2]$$

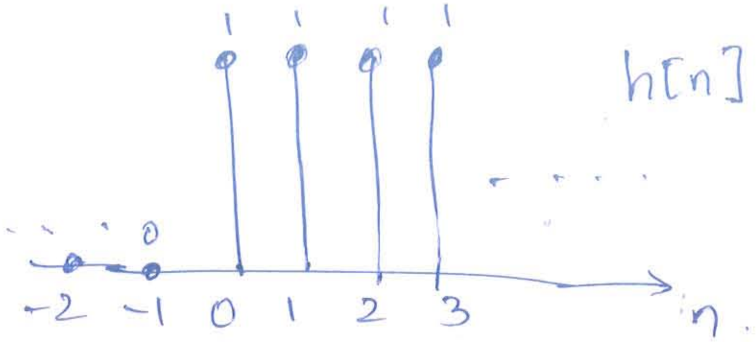
$$H\{x[n]\} = x[0]\,H\{\delta[n-0]\} + x[1]\,H\{\delta[n-1]\} + x[2]\,H\{\delta[n-2]\}$$

$$= x[0]\,h_0[n] + x[1]\,h_1[n] + x[2]\,h_2[n].$$

$$= h_0[n] + 2h_1[n] + h_2[n].$$ (pointwise addition).

Example Linear Time Invariant System

Consider $h[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



Consider any $x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$.

and so

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$$

$$= \dots + x[-1]\, h[n+1] + x[0]\, h[n] + x[1]\, h[n-1] + \dots$$

Let $x[n] \doteq 0$ for $n < 0$. but be non-zero for $n \geq 0$.

then

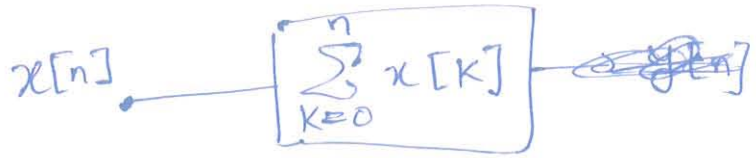
$$y[n] = x[0]\, h[n] + x[1]\, h[n-1] + \dots + x[r]\, h[n-r] + \dots$$

This gives us.

$y[0] = x[0]$, $y[1] = x[0] + x[1]$

$y[2] = x[0] + x[1] + x[2]$, $\dots$, $y[r] = \sum_{k=0}^{r} x[k]$, and so on.

The impulse response $u[n]$ represent a system that is an accumulator.



But let us spend some discussion on convolution. operation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[n] * h[n].$$

Now we can expand this sum and say that output is just the weighted and summed impulse responses shifted by amount k.

This is only one ~~way~~ interpretation.

There is another useful way to look at this sum.

At a particular instance 'n' consider $x[k]\, h[n-k]$ as a sequence in k. i.e., fix the value of 'n' and treat $x[k]$ as a sequence in 'k' and $h[n-k]$ too.

Notice in earlier cases we needed $x[k]$ to be value of $x[n]$ at fixed k.

Let $g_n[k] = x[k]\, h[n-k]$ be a sequence.

Then $y[n]$ at our fixed value n is $\sum_{k=-\infty}^{\infty} g[k]$.

For demonstration. Let $n=0$ be fixed.

Then. $g_0[k] = x[k]\, h[-k]$. is a sequence in k at $n=0$

and $y[0] = \sum_{k=-\infty}^{\infty} g_0[k]$.

Similarly for $n=1$

$$g_1[k] = x[k]\, h[1-k]$$

and $y[1] = \sum_{k=-\infty}^{\infty} g_1[k]$.

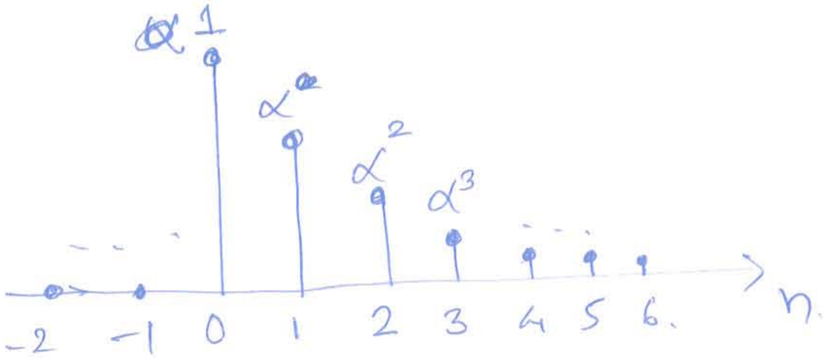
and so on. for $n=r$

$$g_r[k] = x[k]\, h[r-k]$$

and $y[r] = \sum_{k=-\infty}^{\infty} g_r[k]$.

Example $x[n] = \alpha^n u[n]$

$h[n] = u[n]$.



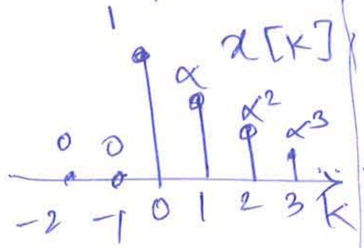
Now we expect to get cumulative sums. i.e. $y[r] = 1 + \alpha + \alpha^2 + \dots + \alpha^r$

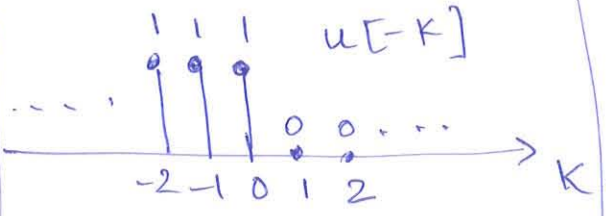
$$= \frac{1 - \alpha^{r+1}}{1 - \alpha}$$

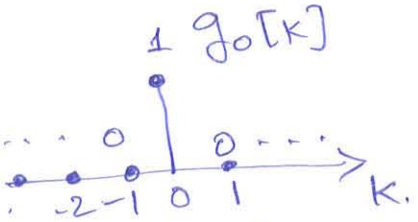
for $r \geq 0$ since $x[n]$ is non-zero from $n=0$ onwards and is zero before.

Let $n = 0$.

then $g_0[k] = x[k]\, h[-k] = x[k]\, u[-k]$



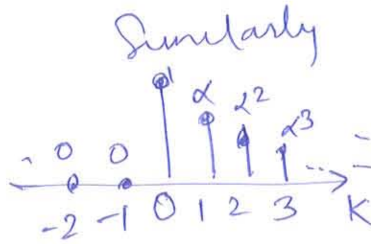


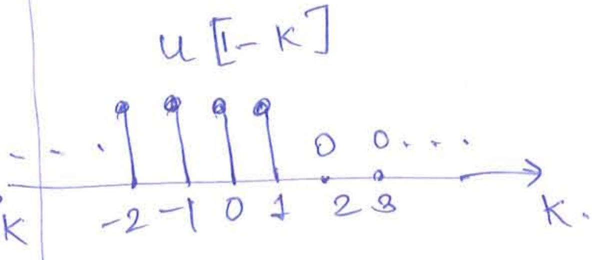


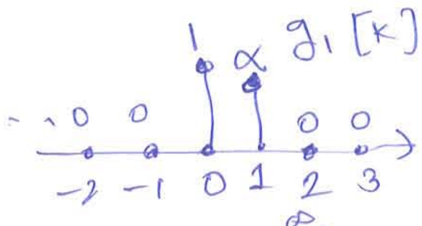
and $y_0[k] = \sum_{k=-\infty}^{\infty} g_0[k]$

$= 1$

Similarly



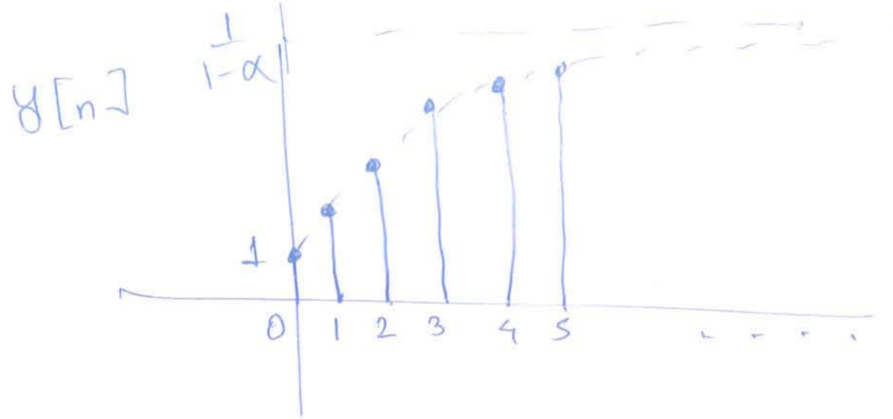




and $y_1[k] = \sum_{k=-\infty}^{\infty} g_1[k]$

$= 1 + \alpha$.

and so on.



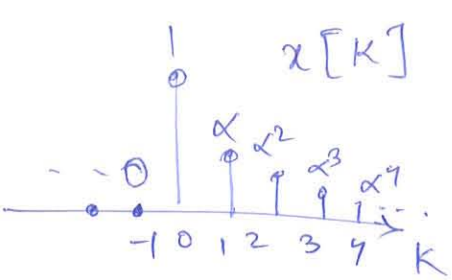
$$y[n] = \begin{cases} \dfrac{1-\alpha^{n+1}}{1-\alpha} & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

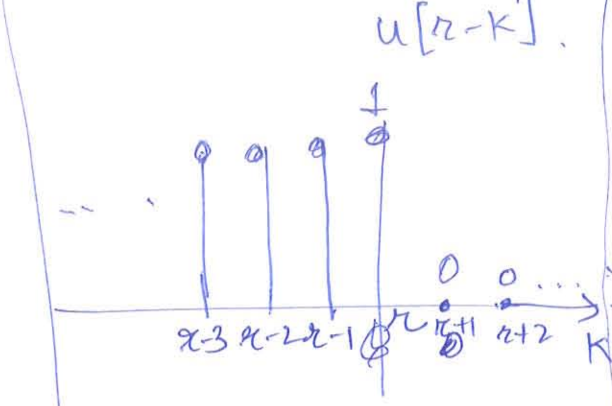
$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

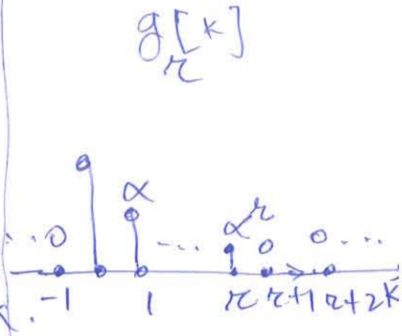
and. as $n \to \infty$, $y[n] \to \frac{1}{1-\alpha}$

~~as~~ ~~y[r]~~

for $y[r]$.







$$y[r] = \sum g_r[k]$$
$$= 1 + \alpha + \alpha^2 + \cdots + \alpha^r$$
$$= \frac{1-\alpha^{r+1}}{1-\alpha}.$$

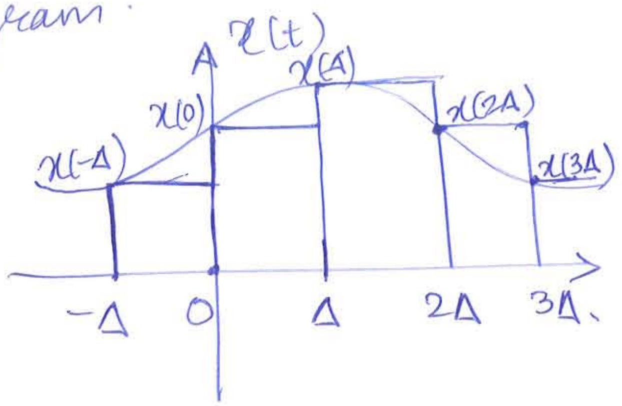
- Continuous time ~~[illegible]~~ LTI system.

(Time Domain Representation)

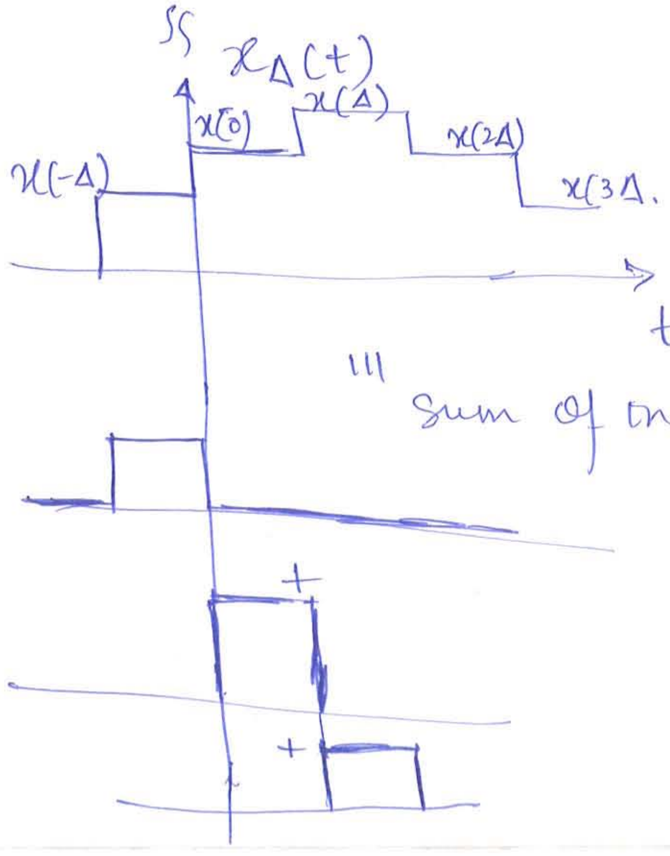
- Signal Representation

Any Signal $x(t)$, can be decomposed into a continuum of Dirac-delta $\delta(t)$ (which we saw briefly in earlier class).

Signal $x(t)$ can be approximated by ~~[illegible]~~ a sample and hold procedure as depicted in following diagram.



- Sample $x(t)$ at $t = -\Delta, 0, \Delta, 2\Delta, \ldots$ and hold the value in interval $[-\Delta, 0)$, $[0, \Delta)$, $[\Delta, 2\Delta)$ and so on.



$\equiv$ sum of individual pulses of width Δ and height $x(t)$.

Now let us consider the approximation to Dirac-delta $\delta(t)$ considered earlier which is

$$\delta_\Delta(t) := \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_\Delta(t-\Delta) := \begin{cases} \frac{1}{\Delta}, & -\Delta \le t < 0 \\ 0, & \text{Otherwise} \end{cases}$$

and so on

$$\delta_\Delta(t-k\Delta) := \begin{cases} \frac{1}{\Delta}, & k\Delta \le t < (k+1)\Delta \\ 0, & \text{otherwise.} \end{cases}$$

Using $\delta_\Delta(t)$ we can represent $x_\Delta(t)$ as follows.

$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t-k\Delta)\, \Delta$$

Now as $\Delta \to 0$, $x_\Delta(t) \to x(t)$ and $\delta_\Delta(t) \to \delta(t)$ and this summation can be replaced by integration

~~Consider~~ i.e.,

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t-k\Delta)\, \Delta$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

This gives ~~these the to~~ us a way to represent any signal $x(t)$ in terms of Dirac-delta $\delta(t)$ as follows.

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

Analogous to what we did for Discrete time systems now we can characterize LTI system by its impulse response.

For that first consider the response of system H to a pulse $\delta_\Delta(t)$.

Let $H\{\delta_\Delta(t-k\Delta)\} = h_{k\Delta}(t)$.

Then response of this system to the approximate signal $x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta$

would be

$$y_\Delta(t) = H\{x_\Delta(t)\} = H\left\{\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta\right\}$$

$$= \sum_{k=-\infty}^{\infty} x(k\Delta)\,H\{\delta_\Delta(t-k\Delta)\}\,\Delta = \sum_{k=-\infty}^{\infty} x(k\Delta)\,h_{k\Delta}(t)\,\Delta$$

Now as $\Delta \to 0$, $\delta_\Delta(t) \to \delta(t)$, ~~$h_{k\Delta}(t) \to h_\tau(t)$ at $\tau = k\Delta$~~

$x_\Delta(t) \to x(t)$, $y_\Delta(t) \to y(t)$.

$$y(t) = \lim_{\Delta \to 0} y_\Delta(t)$$

$$= \lim_{\Delta \to 0} H\{x_\Delta(t)\}$$

$$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_{k\Delta}(t)\, \Delta$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h_\tau(t)\, d\tau$$

~~Thus~~ For time varying system $h_\tau(t)$ would be the impulse response at various τ

In case of Time invariant systems

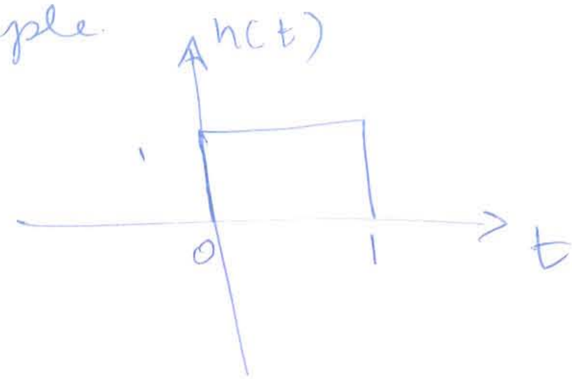
$$h_\tau(t) = h_0(t-\tau) \quad \left[\begin{array}{l}\text{Since } h_0(t) = H\{\delta(t)\} \\ \text{and } h_\tau(t) = H\{\delta(t-\tau)\}\end{array}\right]$$

and Thus for LTI system the impulse response $h_0(t-\tau)$ gives a way to Calculate output $y(t)$ for any input $x(t)$ as follows.

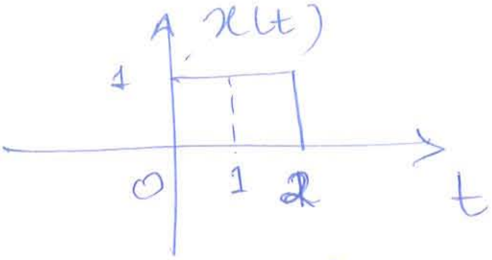
$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h_0(t-\tau)\, d\tau}$$

This operation of $x(t)$ and $h_0(t)$ is ~~defined~~ ~~called~~ ~~as~~ Continuous time Convolution

Example



$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \in [0, 1] \\ 0, & t > 0 \end{cases}$$



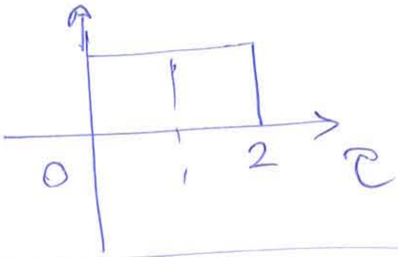
$$x(t) = \begin{cases} 0, & t < 0 \\ 1, & t \in [0, 2] \\ 0, & t > 0 \end{cases}$$

for $t \le 0$, $x(\tau) h(t-\tau) = 0$ because $x(\tau) = 0$ for $\tau < 0$

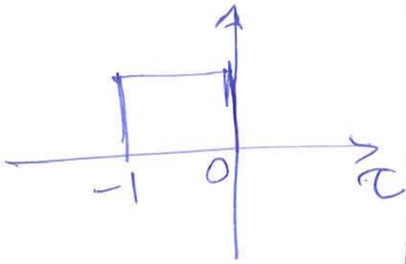
Consider

~~x(2τ)~~

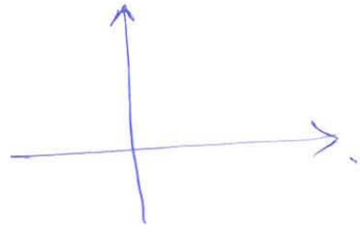
$x(\tau)$



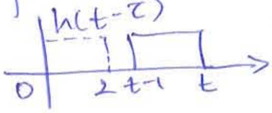
$h(-\tau)$



$x(\tau) h(-\tau)$

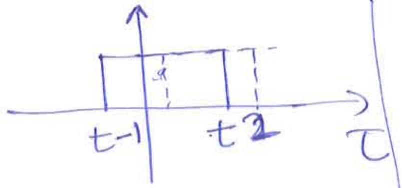


for $t > 3$



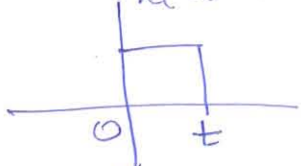
and so $y(t) = 0$

$h(t-\tau)$,



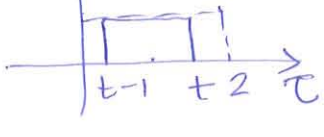
$t \in [0, 1]$

$x(\tau) h(t-\tau)$

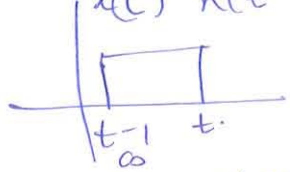


$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) = \int_0^t 1 \, d\tau = t$$

for $t \in [1, 2]$



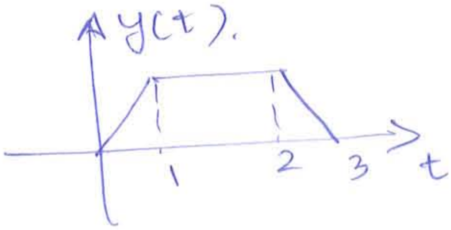
$x(\tau) h(t-\tau)$



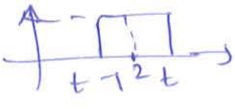
$$y(t) = \int_{\infty}^{\infty} x(\tau) h(t-\tau) = \int_{t-1}^{t} 1 \, d\tau = t - (t-1) = 1$$

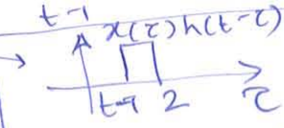
Thus

$$y(t) = \begin{cases} 0, & t < 0 \\ t & t \in [0, 1] \\ 1 & t \in [1, 2] \\ 3-t & t \in [2, 3] \\ 0, & t > 3 \end{cases}$$



for $t \in [2, 3]$





$$y(t) = \int_{t-1}^{2} 1 \, d\tau = 2 - t + 1 = 3 - t$$

Example.

$$y(t) = x(t - t_0).$$

$$h(t) = \delta(t - t_0) \quad \text{delay by } t_0$$

inverse of $h(t)$. $\rightarrow$ $h_1(t) = \delta(t + t_0)$ advance by t_0

$$h(t) * h_1(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, \delta(t + t_0 - \tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau - t_0)\, \delta(t - (\tau - t_0))\, d\tau$$

$$\cancel{= \delta(t)}. = \int_{-\infty}^{\infty} \delta(w)\, \delta(t - w)\, dw$$

$$= \delta(t). \quad [\text{Sifting property}]$$

Properties of LTI systems

- Memorylessness (LTI system).

Since output depends on current instance input $y[n] = k\,x[n]$

impulse response is $h[n] = k\,\delta[n]$

$$y(t) = k\,x(t)$$

impulse response is $h(t) = k\,\delta(t)$.

- If $h[n]$ (respectively $h(t)$) is non-zero at $n \neq 0$ (respectively $t \neq 0$) then system has memory.

Properties of LTI systems / Convolution.

(i) Commutative.

$$x[n] * h[n] = h[n] * x[n].$$

$$x(t) * h(t) = h(t) * x(t).$$

Proof is left as excercise.

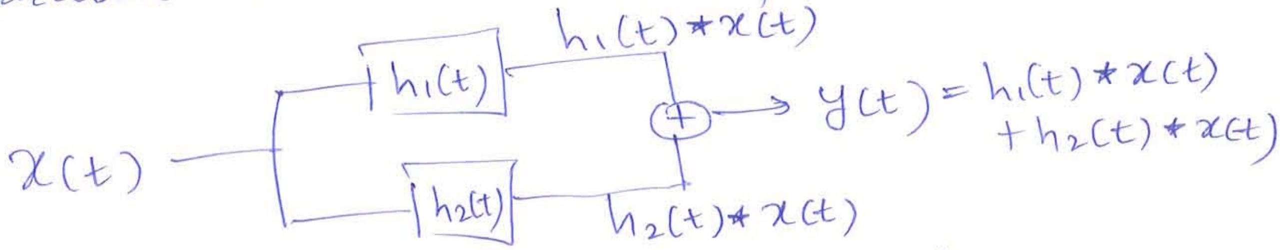
(ii) Distributive Property.

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

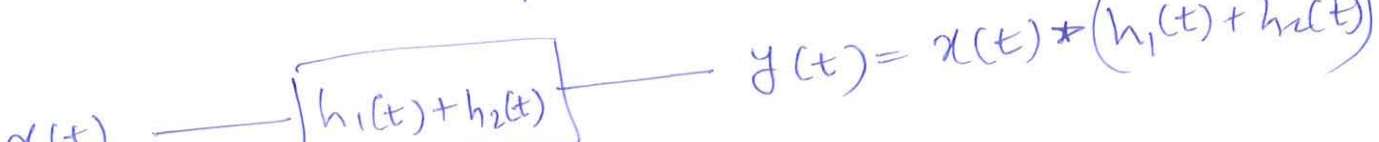
$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

[Proof is left as excercise].

This property helps in simplifying parallel interconnection of LTI systems.



can be replaced by single block.

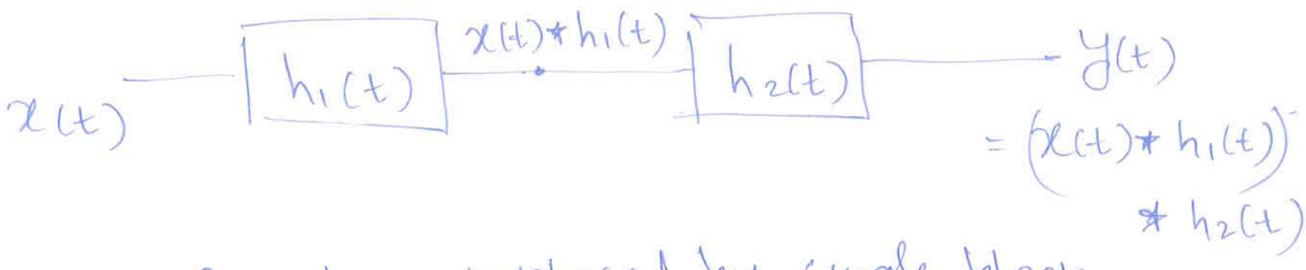


(iii) Associative Property.

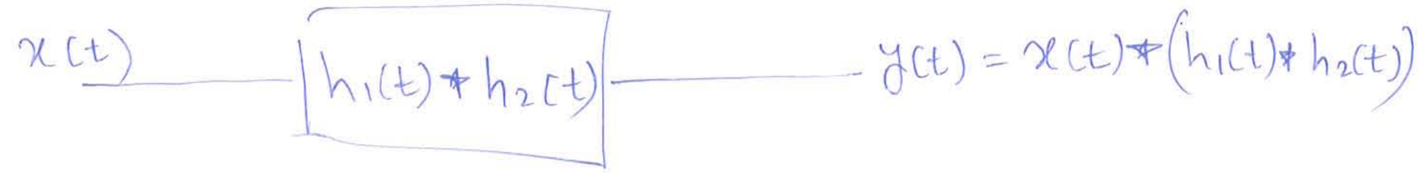
$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$$

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t).$$

This property helps in simplifying cascade interconnection. of LTI systems.



Can be replaced by single block



(iv) Invertibility -

A system $h_1(t)$ is called as the inverse system of $h(t)$ if for any signal $x(t)$, $[x(t) * h(t)] * h_1(t) = x(t)$.

In particular for $x(t) = \delta(t)$

$$h(t) * h_1(t) = \delta(t).$$

Similarly for discrete time systems

$$h[n] * h_1[n] = \delta[n]$$

$\delta(t)$, $\delta[n]$ are identity elements in the space of continuous time & discrete time signals respectively.

Example

$y(t) = x(t - t_0)$

$h(t) = \delta(t - t_0)$ impulse response
(delay by t_0)

$h_1(t) = \delta(t + t_0)$ is the inverse system of $h(t)$.

Note. $h(t) * h_1(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, \delta(t + t_0 - \tau)\, d\tau$

Let $w = \tau - t_0$, then, $dw = d\tau$

$$= \int_{-\infty}^{\infty} \delta(w)\, \delta(t - w)\, dw$$

$$= \delta(t) \quad \text{[by sifting property]}$$

Causality of LTI system.

Causal if $y[n]$ does not depend on $x[k]$ for $k > n$ (future input values).

Consider $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

Independence of $y[n]$ from $x[k]$ for $k > n$ if and only if Impulse response $h[n-k]$ ~~should~~ does not pick non-zero $x[k]$ with $k > n$

That means. $h[n-k] = 0$ for $k > n$

$$\Rightarrow h[n] = 0 \text{ for } n < 0.$$

and $y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$.

Analogously.

in Continuous time

Causal if $h(t) = 0$ for $t < 0$.

then $y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$

$= \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$

- Stability of systems.