Stability of LTI systems from impulse response.

Defn: [Bounded input Bounded ~~output~~ Stability notion] [BIBO Stability]

A system is BIBO stable if "every" Bounded input produces a response (Output) which is bounded. ~~output~~.

- Discrete System: BIBO Stable iff $\sum_{K=-\infty}^{\infty} |h[K]| < \infty$. $(\Leftrightarrow)$

Consider input $x[n]$ which is bounded for all $n \in \mathbb{Z}$

i.e. $|x[n]| < B$ for all $n \in \mathbb{Z}$

Then we want that the output $y[n]$ be also bounded.

Assume that impulse response is absolutely summable i.e.,

$$\sum_{K=-\infty}^{\infty} |h[K]| = D < \infty \quad \left(\begin{array}{c} D \text{ is of course} \\ +ve \end{array}\right).$$

Then

$$|y[n]| = \left| \sum_{K=-\infty}^{\infty} h[K]\, x[n-K] \right|$$

$$\leq \sum_{K=-\infty}^{\infty} |h[K]|\,|x[n-K]| \leq \sum_{K=-\infty}^{\infty} B\,h[K]$$

$$= BD < \infty$$

Thus $|y[n]|$ is bounded for bounded input $x[n]$

Thus, $\sum_{K=-\infty}^{\infty} |h[K]| < \infty \Rightarrow$ System is BIBO Stable

But does system being BIBO Stable ~~mean~~ imply that $\sum_{K=-\infty}^{\infty} |h[K]| < \infty$?

Yes!

Suppose system is BIBO stable but its impulse response $h[n]$ is not absolutely summable

Now, $y[n] = \sum_{K=-\infty}^{\infty} h[K]\, x[n-K]$

Consider $y[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k]$

Now choose $x[k] = \begin{cases} -1 & h[-k] < 0 \\ 1 & h[-k] > 0. \end{cases}$

$|x[k]| \leq 1$, hence input is bounded.

Then $y[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k]$

$$= \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$ ~~which~~ is

~~note~~

Thus $y[n]$ is unbounded for a

Bounded input.

Contradiction to our supposition that system was BIBO Stable.

Thus BIBO Stable $\Rightarrow \sum_{k=-\infty}^{\infty} |h[n]| < \infty$.

and

~~Or~~ Impulse response absolutely summable if and only if System is BIBO Stable.

Continuous time Case :

Assume input $x(t)$ s.t

$$|x(t)| < B \quad \text{for all } t \in \mathbb{R}$$

Then $$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

$$\leq B \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

Then if impulse response is absolutely integrable we ~~get~~ know that

We have BIBO stable system.

Does BIBO Stable $\Rightarrow \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

yes!

Excercise : Choose $x(t) = \begin{cases} -1 & \text{if } \cancel{h(t)}\ h(-t) < 0 \\ 1 & \text{if } h(-t) > 0. \end{cases}$

LTI Systems governed by Differential & difference equations.

LCCDE — Linear constant coefficient differential equation.

- Almost all systems involve some rate of change or recursive pattern. Such System's Behaviour can be modelled using differential equations.

Our goal is to represent this differential equation into a black box that takes in some signal as input, churns it ~~[illegible]~~ according to LCCDE and produces output signal. Also while we replace LCCDE by this new i/p - o/p representation we should also maintain the useful properties such as linearity, time-invariance which are useful in simplifying big block diagrams to one block.

First let us consider a basic system i.e. the differentiator.

$$x(t) \longrightarrow \boxed{\frac{dx}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}.$$

- Linear Time invariant.
- Memory. – Not Stable $\frac{du(t)}{dt} \Rightarrow \delta(t)$
- Causal? This is tricky. Depends on how you define it!

a) $$\frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$ not Causal.

b) $$\frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}$$ Causal.

c) $$\frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t-h)}{2h}.$$

~~no chance of being~~ non-Causal.

If $x(t)$ is Continuously differentiable everywhere then all three definitions are equivalent and one can effectively stick to using (b) and $\frac{d}{dt}$ in that case can be considered as Causal.

5. Invertible

No!

$x(t) = \sin t \longrightarrow y(t) = \cos t$

$x(t) = \sin t + 5 \longrightarrow y(t) = \cos t.$

For output $y(t) = \cos t$ we have many possible unequal inputs that could have been applied.

Not invertible.

Another basic system that is represented by an LCCDE is the integrator

$$x(t) \longrightarrow \boxed{\frac{dy(t)}{dt} = x(t)} \longrightarrow y(t)$$

here it is possible to see why non-zero initial conditions lead to this blackbox being not a linear system.

Consider an input starting at t_0.

Then. $\displaystyle\int_{-\infty}^{t} dy(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \Rightarrow y(t) - y(t_0) = \int_{t_0}^{t} x(\tau)\,d\tau$

$$\Rightarrow y(t) = y(t_0) + \int_{t_0}^{t} x(\tau)\,d\tau$$

This is not a linear system.

~~Since~~.

Let us consider $x_1(t)$ and $x_2(t)$ (two arbitrary signals)

Then response of this blackbox with non-zero initial condition to these inputs is

$$y_1(t) = y(t_0) + \int_{t_0}^{t} x_1(\tau)\, d\tau$$

$$y_2(t) = y(t_0) + \int_{t_0}^{t} x_2(\tau)\, d\tau$$

[note that initial condition $y(t_0)$ of blackbox is same in both cases (as we are not considering different blackboxes)]

Response for $x_3(t) = x_1(t) + x_2(t)$ is

$$y_3(t) = y(t_0) + \int_{t_0}^{t} \left(x_1(\tau) + x_2(\tau)\right) d\tau$$

$$= y(t_0) + \int_{t_0}^{t} x_1(\tau)\, d\tau + \int_{t_0}^{t} x_2(\tau)\, d\tau$$

$$\neq y_1(t) + y_2(t)$$

Exercise: Verify that it is also not time invariant!

But, if we choose initial condition from whenever the input first became non-zero (time instant)

we get $y(t) = \int_{t_0}^{t} x(\tau)\,d\tau$ which is

linear and time invariant.

Thus this assumption ~~[illegible]~~ of initial conditions being zero or initial rest assumption is crucial in simplifying the differential equation into an LTI input output system.

Impulse Response:

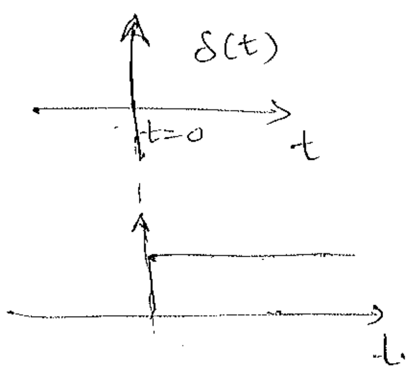
Just as we saw previously it is possible to replace LCCDE by its impulse response. We need to solve differential equation for ~~its~~ ~~inepu~~ dirac impulse as input.

For example consider the integrator again

$$\delta(t) \longrightarrow \boxed{\frac{dy}{dt} = x(t)} \longrightarrow y(t)$$

Then the impulse response of the integrator ~~to~~ is

$$y(t) = \int_{-\infty}^{t} \delta(t) = u(t) \quad \text{Unit Step.}$$



One can say that at $t = 0^-$

$y(t) = 0$ and at $t = 0^+$

$y(t)$ jumped to 1

i.e. $y(0^+) = 1$

Then one can solve $\frac{dy}{dt} = 0$ for $t > 0$

with initial condition $y(0^+) = 1$.

This example, although, a very simple example outlines a procedure to compute impulse response for LCCDE.

(i) Assume zero initial conditions. (for $t<0$).

(ii) Compute what happens at the time instance right after application of impulse at $t=0$. i.e., compute the changed initial state right at $t=0^+$.

(iii) Solve homogenous differential equation with this changed initial state as new set of initial conditions.

Let us use these steps on a more complicated example now.

Consider $\frac{dy(t)}{dt} + 2y(t) = x(t)$ with $y(0^-)=0$.
also, $y(t)=0$ for $t<0$.

Consider $x(t) = \delta(t)$

$$\frac{dy(t)}{dt} + 2y(t) = \delta(t)$$

$$\int_{-\infty}^{\infty} \frac{dy(\tau)}{d\tau} d\tau + 2\int_{-\infty}^{\infty} y(\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

we are interested in what happens at

$t = 0^+$, we know $y(0^-) = 0$ and
$y(t) = 0, t < 0$

then:

$$\int_{0^-}^{0^+} dy(\tau) + 2 \times 0 = 1$$

$$y(0^+) - y(0^-) = 1$$

but $y(0^-) = 0 \Rightarrow y(0^+) = 1$

Now solve. $\frac{dy}{dt} + 2y = 0$ for $y(0^+) = 1$
for $t > 0$.

we get, $h(t) = y(t) = e^{-2t} y(0^+) = e^{-2t} u(t)$
for $t > 0$
and for $t < 0, h(t) = 0$.

Thus, $h(t) = e^{-2t} u(t)$

Now we can now replace ① by a black box

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau$$

which computes output for any input.

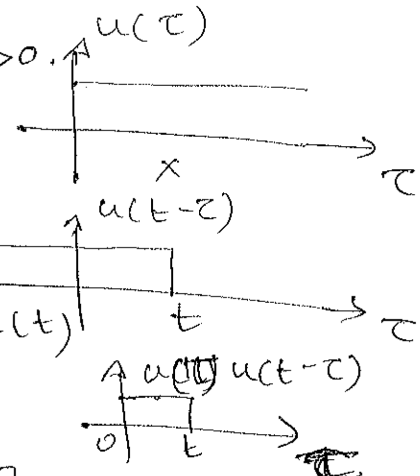
for instance if $x(t) = e^{3t}u(t)$

then $$y(t) = \int_{-\infty}^{t} e^{3\tau} u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau$$

$$= e^{-2t} \int_{-\infty}^{t} e^{(3\tau + 2\tau)} u(\tau)\, u(t-\tau)\, d\tau$$

$$= e^{-2t} \int_{0}^{t} e^{5\tau} d\tau \text{ for } t>0.$$

$$= e^{-2t} \left\{ \left[\frac{e^{5t}}{5} \right]_0^t - \frac{1}{5} \right\} u(t)$$



$$= \frac{1}{5} \left[\frac{e^{3t}}{5} - \frac{e^{-2t}}{5} \right] u(t)$$

Taking insights from this example let us consider a general LCCDE given by.

$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{l=0}^{M} b_l \frac{d^l x(t)}{dt^l}} \longrightarrow y(t)$$

Now what can be done is that we can break this system into a cascade of two systems. (This breaking down ~~seen~~ is possible only if we assume zero initial conditions.

Consider first an operator which consists of linear combination of

$$\frac{d}{dt}, \quad \frac{d^2}{dt^2}, \quad \ldots, \quad \frac{d^k}{dt^k}$$

That is,
Consider

$$A\left(\frac{d}{dt}\right) = a_0 + a_1 \frac{d}{dt} + \cdots + a_N \frac{d^N}{dt^N}$$

$$= \sum_{k=0}^{N} a_k \frac{d^k}{dt^k}$$

Then

$$A\left(\frac{d}{dt}\right) w(t) = a_0 w(t) + a_1 \frac{dw}{dt} + \cdots + a_N \frac{d^N}{dt^N} w(t)$$

Similarly we can have.

$$B\left(\frac{d}{dt}\right) = \sum_{\ell=0}^{M} b_k \frac{d^\ell}{dt^\ell}$$

Now consider ($*$) again. In our new notation.

$$x(t) \longrightarrow \boxed{A\left(\frac{d}{dt}\right) y(t) = B\left(\frac{d}{dt}\right) x(t)} \longrightarrow y(t).$$

Now, we can break this into two systems as follows.

$$x(t) \longrightarrow \boxed{A\left(\frac{d}{dt}\right) w(t) = x(t)} \xrightarrow{\;w(t)\;} \boxed{y(t) = B\left(\frac{d}{dt}\right) w(t)} \longrightarrow y(t).$$

Informally In Laplace domain

$$X(s) = A(s)\, W(s)$$
$$Y(s) = B(s)\, W(s).$$

Thus
$$Y(s) = \frac{B(s)}{A(s)} X(s).$$
$$A(s)\, Y(s) = B(s)\, X(s).$$

Now just consider first block

$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N} a_k \frac{d^k w(t)}{dt^k} = x(t)} \longrightarrow w(t).$$

Consider its impulse response.

i.e $$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} w(t) = \delta(t)$$

Assume $w(t) = 0$ for $t < 0$.

Thus. $$\left.\frac{d^{N-1} w(t)}{dt^{N-1}}\right|_{t=0^-} = \cdots = \left.\frac{dw(t)}{dt}\right|_{t=0^-} = w(0^-) = 0$$

We are interested in what happens to these initial conditions after application of impulse $\delta(t) := \begin{cases} 0 , & t<0 \\ \infty ; & t=0 \\ 0 , & t>0. \end{cases}$

and $\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$

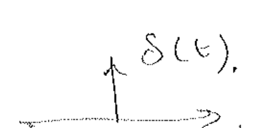
Integrating

$$\int_{-\infty}^{t} \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} w(\tau)\, d\tau = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

We split into intervals. $(-\infty, 0^-), (0^-, 0^+), (0^+, \infty)$

for $t<0$ we have. $\int_{-\infty}^{t} \delta(\tau)\, d\tau = 0$

Thus $w(t) \equiv 0$ for $t<0$ is soln.



Now in $(0^-, 0^+)$

$$\int_{0^-}^{0^+} \sum_{k=0}^{N} a_k \frac{d^k}{d\tau^k} w(\tau)\, d\tau = \int_{0^-}^{0^+} \delta(\tau)\, d\tau = 1$$

$$\underbrace{\sum_{k=1}^{N} a_k \int_{0^-}^{0^+} \frac{d^k}{d\tau^k} w(\tau)\, d\tau}_{} + \underbrace{\int_{0^-}^{0^+} a_0 w(\tau)\, d\tau}_{0} = 1.$$

encloses zero measure area.

$$\sum_{k=1}^{N} a_k \frac{d^{k-1}}{dt^{k-1}} w(0^+) - \underbrace{\sum_{k=1}^{N} a_k \frac{d^{k-1}}{dt^{k-1}} w(0^-)}_{= 0.} = 1$$

ϕ

$$\Rightarrow \sum_{k=1}^{N} a_k \frac{d^{k-1}}{dt^{k-1}} w(0^+) = 1$$

Now in particular, we can choose

$$w(0^+) = 1, \text{ and } \frac{dw}{dt}(0^+) = \frac{d^2 w}{dt^2}(0^+) = \cdots = \frac{d^{N-1}}{dt^{N-1}} w(0^+) = 0.$$

So that $\displaystyle\sum_{k=1}^{N} a_k \frac{d^{k-1}}{dt^{k-1}} w(0^+) = 1$

and Solve

$$\sum_{k=0}^{N} a_k \frac{d^k w(t)}{dt^k} = 0 \quad \text{with initial Conditions.}$$

$$w(0^+) = 1, \quad \dot{w}(0^+) = \ddot{w}(0^+) = \cdots = w^{(N-1)}(0^+) = 0$$

to obtain impulse response

$h_1(t)$ of . $\sum_{k=0}^{N} a_k \frac{d^k w(t)}{dt^k} = x(t)$

Then impulse response

$h(t)$ of (*) is

$$h(t) = B\left(\frac{d}{dt}\right) h_1(t) = \sum_{l=0}^{M} b_l \frac{d^l}{dt^l} h_1(t)$$

(*) —

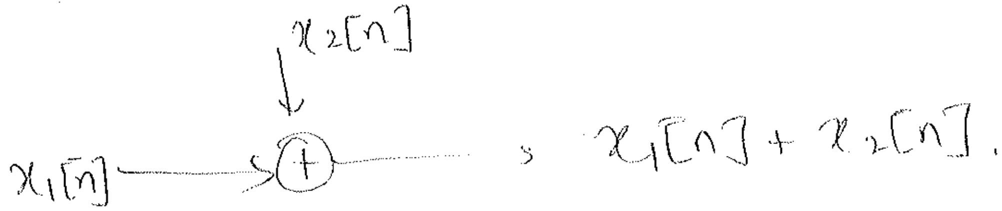
$$x(t) \longrightarrow \boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{l=0}^{M} b_k \frac{d^l x(t)}{dt^l}} \longrightarrow y(t)$$

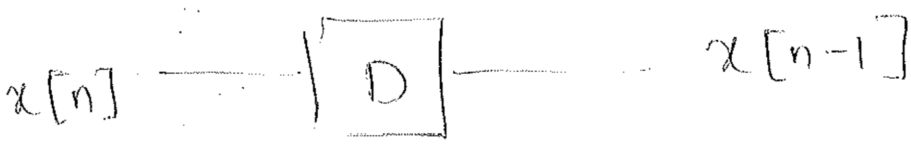
$\delta(t)$ gives. $h(t)$

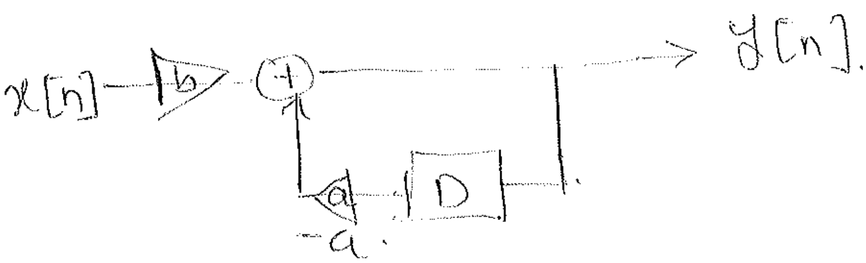
Thus for any $x(t)$ we get.

$$y(t) = \int_{-\infty}^{t} h(t-\tau)\, x(\tau)\, d\tau$$

Block diagrams:

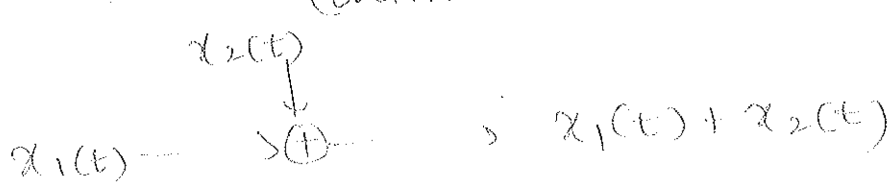


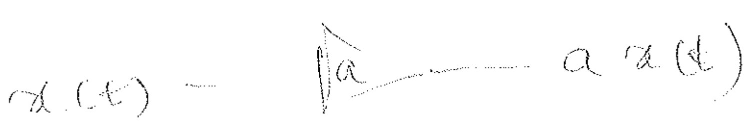


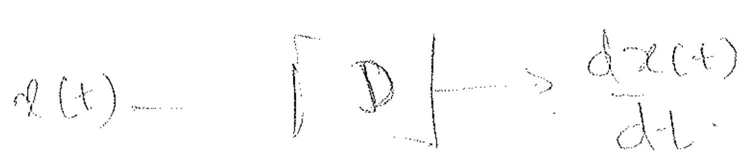


$$y[n] = b x[n] - a\,y[n-1].$$

Continuous time case.







$y(t) = b\left(x(t) - a\frac{dy}{dt}\right)$

$\frac{dy}{dt} = \frac{b}{\ } x(t) - \frac{1}{\ } y(t)$

Impulse response of non-recursive equation.

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0 & n < 0. \end{cases}$$

FIR system (Finite Impulse response)

Example: $y[n] - \frac{1}{2} y[n-1] = x[n].$

$\Rightarrow y[n] = x[n] + \frac{1}{2} y[n-1].$

Impulse response with ~~zero~~ initial rest

i.e. $y[-1] = 0$

and $x[n] = \delta[n].$

$h[0] = y[0] = x[0] + 0 = 1$

$h[1] = y[1] = x[1] + \frac{1}{2} y[0] = \frac{1}{2}$

$h[2] = y[2] = x[2] + \frac{1}{2} y[1] = \left(\frac{1}{2}\right)^2$

$\vdots$

$h[n] = y[n] = \left(\frac{1}{2}\right)^n.$

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

(IIR) - Infinite Impulse Response.

Again we need to specify initial conditions to get one input - output relationship which is LTI & Causal.

That happens when one chooses initial rest conditions

In discrete time case things are slightly easier to analyse because rearranging — (*) leads to.

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

(Recursive equation).

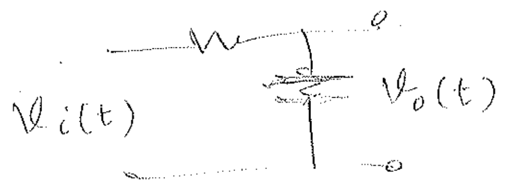
Thus ~~at y[n]~~ we immediately see need to specify $y[-N], y[-N+1], \ldots, y[-1]$ to know $y[0]$.

[Special Case when $N = 0$)

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right) x[n-k].$$

Non-recursive equation. (No need of past outputs)

Example



Let $v_o(0^-) = 2V$

If $v_i(t) = x_0 \Rightarrow v_o(t) = 2V$

$\cancel{v_i(t) = x_0} \Rightarrow \cancel{v_o(t)} = 2V$

$v_i(t) = 2x_0 \Rightarrow v_o(t) = 2V \neq 4V.$

not - linear.

In discrete time case we have linear constant coefficient difference equations.

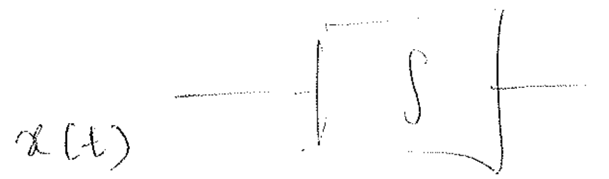
$$\sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{N} a_k y[n-k]$$

Again we have a particular solution and solution to homogeneous eq.

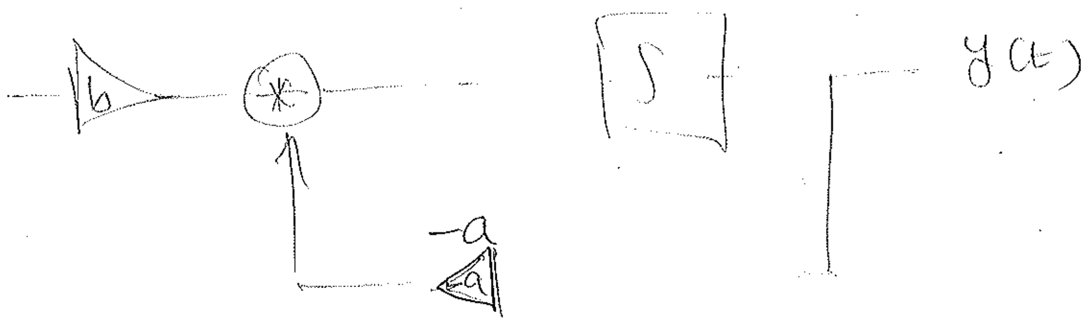
$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

and sum of the solutions gives us a solution $y[n] = y_p[n] + y_h[n]$.

One can also use integrator



$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$



$$y(t) = \int_{-\infty}^{t} \left(-a\, y(\tau) + b\, x(\tau)\right) d\tau$$

- Unit doublets.

$$y(t) = \frac{d}{dt} x(t).$$

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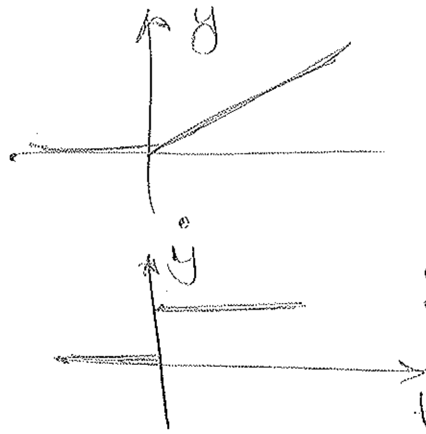
$$\delta \longrightarrow \boxed{\ddot{y} = x} \longrightarrow y = t\,u(t)$$



suddenly imparting some velocity

Recap

Signals — Continuous time / Discrete time — Systems.

eg:- (i) Dirac-delta, unit impulse.

(ii) Unit step

(iii) Sinusoidal / exponential signals.

- Any signal can be represented in terms of unit impulses / Dirac impulses

- For LTI systems, impulse response characterises it entirely.

- Convolution operation.

- System properties
  - Causality
  - Stability
  - Invertibility
  - Linearity
  - Time-invariance

- Simplification of Block diagrams possible for LTI-systems
  - additivity
  - Associativity
  - Commutativity

  Convolution Properties

- LCCDE systems are LTI if assumed to be at rest initially (zero-initial conditions)
- Can compute impulse response to LCCDE by specifying initial conditions at $t=0^+$ & solving homogeneous eqn.

~~Similarly in Discrete domain~~

Some more special Signals / generalized functions.

$$\boxed{\begin{array}{l}\text{unit doublet:}\\ u_1(t) = \frac{d}{dt}\delta(t).\end{array}}$$

$$u_2(t) = \frac{d^2}{dt^2}\delta(t) = \frac{d}{dt}u_1(t)$$

$\vdots$

$$u_k(t) = \frac{d^k}{dt^k}\delta(t) ..$$

$$u_{-1}(t) = \int_{-\infty}^{t}\delta(\tau)\,d\tau = u(t)$$

$$u_{-2}(t) = u(t) * u(t) = t\,u(t)$$

$\vdots$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t).$$