Fourier Series, Transforms

Eigen functions

Now that we have characterized the input-output system interms of its impulse response, lets try to find its eigenfunctions. One guess is the Complex exponential signals of the form e^{st} where $s \in \mathbb{C}$

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau)\, d\tau$$

$$y(t) = e^{st} \boxed{\int_{-\infty}^{\infty} e^{-s\tau} h(\tau)\, d\tau} = e^{st} H(s)$$

We recover back a scaled version of e^{st} and the scaling factor is $H(s)$.

Later we will see that $H(s)$ forms what is known as the transfer function.

Similarly in discrete time case

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} z^{n-k} h[k]$$

$$y[n] = z^n \underbrace{\sum_{k=-\infty}^{\infty} z^{-k} h[k]}_{\rightarrow H(z)}$$

$$y[n] = z^n \underline{H(z)}$$

Thus z^n is the eigenfunction of $h[n]$ with $H(z)$ as the scaling (also later we will see it as transfer function)

Now we can input some linear combinations of these signals.

$$a e^{s_1 t} + b e^{s_2 t} + c e^{s_3 t} \longrightarrow \boxed{h(t)} \longrightarrow a e^{s_1 t} H(s_1) + b e^{s_2 t} H(s_2) + c e^{s_3 t} H(s_3)$$

So $x(t) = \sum_k a_k e^{s_k t}$

$$y(t) = \sum_k a_k e^{s_k t} H(s_k)$$

and $x[n] = \sum_k a_k z_k^n$ then $y[n] = \sum_k a_k H(z_k) z_k^n$.

It is easier to study response of system if we are able breakdown a signal in terms of these basic signals i.e. the complex exponentials.

- <u>Fourier Series Representation of Continuous time periodic Signals</u>

We call a signal $x(t)$ to be periodic if $\exists\ T>0$ s.t $x(t+T) = x(t)$ for all t

The least positive ~~real~~ ~~no~~ non-zero real no. T s.t $x(t+T) = x(t)$ for all t is "<u>fundamental period</u>". The quantity $\omega_0 = \frac{2\pi}{T}$ will be referred to as "<u>fundamental frequency</u>".

In complex exponentials $x(t) = e^{j\omega_0 t}$ is one such periodic signal. From this signal. We can generate harmonically related signals ~~which~~ that are ~~are~~ having frequencies which are integer multiples of fundamental frequency

$\phi_K(t) = e^{jw_0Kt}$ for $K = 0, \pm 1, \pm 2, \ldots$

This set of signals $\phi_0, \phi_1, \phi_2 \ldots$

all have fundamental period that is a fraction of T, fundamental period of $\phi_1(t) = e^{jw_0t}$.

In Fourier Series representation we try to represent a periodic signal as a linear combination of these harmonically related signals. $\{\phi_0, \phi_1, \ldots, \phi_K, \ldots\}$.

$$\boxed{x(t) = \sum_{K=-\infty}^{\infty} a_K \phi_K = \sum_{K=-\infty}^{\infty} a_K e^{jKw_0t}}$$

FOURIER SERIES OF $x(t)$

$$= \cdots + a_{-N} e^{-jNw_0t} + \cdots + a_{-1} e^{-jw_0t} + a_0 + a_1 e^{jw_0t} + a_2 e^{2jw_0t} + \cdots + a_N e^{jNw_0t} + \cdots$$

$a_0 \longrightarrow$ Constant term.

$K = \pm 1 \longrightarrow$ fundamental Component or first harmonic.

$K = \pm N \longrightarrow$ N^{th} harmonic Component

Example. [Oppenheim Ex. 3.2]

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

$$= a_{-3} e^{-j6\pi t} + a_{-2} e^{-j4\pi t} + a_{-1} e^{-j2\pi t} + a_0$$
$$+ a_3 e^{j6\pi t} + a_2 e^{j4\pi t} + a_1 e^{j2\pi t}$$

if $a_{-3} = a_3 = \frac{1}{4}$, $a_{-2} = a_2 = \frac{1}{3}$, $a_{-1} = a_1 = \frac{1}{2}$, $a_0 = 1$

then.

$$x(t) = 2a_3 \cos 6\pi t + 2a_2 \cos 4\pi t + 2a_1 \cos 2\pi t + a_0.$$

$$= \frac{1}{2} \cos 6\pi t + \frac{2}{3} \cos 4\pi t + \cos 2\pi t + 1$$

Now since $x(t)$ was real. Let us see what happens if we take the complex conjugate of the series.

$$[x(t)]^* = x(t) = \left[\sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}\right]^*$$

$$= \left[\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right] = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

(relabeling).

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

we get $a_k = a_{-k}^*$

~~Now~~, we can also write the series as.

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right)$$

$$= a_0 + \sum_{k=1}^{\infty}\left(\underbrace{a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}}_{\text{Conjugates of each other}}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left[a_k e^{jk\omega_0 t}\right]$$

Let $a_k = A_k e^{j\theta_k}$ in polar form

then.

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}$$

(1)

$$\boxed{x(t) = a_0 + \sum_{k=1}^{\infty} 2A_k \cos(k\omega_0 t + \theta_k)}$$

Yet another form can be found by letting $a_k = B_k + jC_k$.

Then.

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{(B_k + jC_k)\, e^{jk\omega_0 t}\right\}.$$

(2)

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos(k\omega_0 t) - C_k \sin k\omega_0 t\right]}$$

(3) In original form that we started with

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.}$$

Now let us look at these basic harmonically related signals ϕ_K ~~let us~~

~~call them~~

$$B = \left\{ \phi_K = e^{jK\omega_0 t}, \; K \in \mathbb{Z} \right\}.$$

(i) Consider a space of all periodic signals with fundamental period T (and equivalent fundamental freq. $\omega_0 = \frac{2\pi}{T}$).

Fourier said that all these signals can be decomposed into a linear combination of elements in B.

Thus B forms ~~the~~ a "basis" for ~~the~~ periodic signals of fundamental period T

(ii) $$\langle \phi_k, \phi_m \rangle = \int_0^T \phi_k \phi_m^{*} \, dt$$ where ϕ_m^{*} is complex conjugate of ϕ_m.

$$= \int_0^T \left(e^{jl\omega_0 t} e^{-jm\omega_0 t} \right) dt = \int_0^T e^{j(l-m)\omega_0 t} \, dt$$

$$= \begin{cases} 0 & \text{for } l-m \neq 0 \\ T & \text{for } l-m = 0 \end{cases}$$

~~This gives us a way to compute~~

With respect to the inner product $\langle \phi_l , \phi_m \rangle$ we have just shown that B forms a orthogonal set. ~~of f~~

Then we can use this to find coefficients a_k of the Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

$$x(t) e^{-j\omega_0 l t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 (k-l)t}$$

$$\int_0^T x(t) e^{-j\omega_0 l t} = T a_l$$

$$\Rightarrow \boxed{a_l = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 l t} \quad \text{for } l \in \mathbb{Z}}$$

I can also write $a_l = \dfrac{1}{T} \displaystyle\int_{-T/2}^{T/2} x(t) e^{-j\omega_0 l t}$

Infact result won't change on any interval of length T.
Since $x(t)$ is Periodic with period T.

TO Summarize

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

with $$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

FOURIER SERIES.

Example :

$$x(t) = \sin\omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

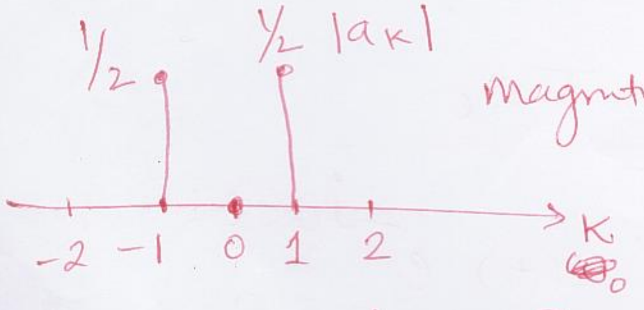
$$= \sum_{k=-1}^{1} a_k e^{jk\omega_0 t}$$

with $a_{-1} = \frac{-1}{2j} = \frac{1}{2}j$

$a_1 = \frac{1}{2j} = -\frac{1}{2}j$

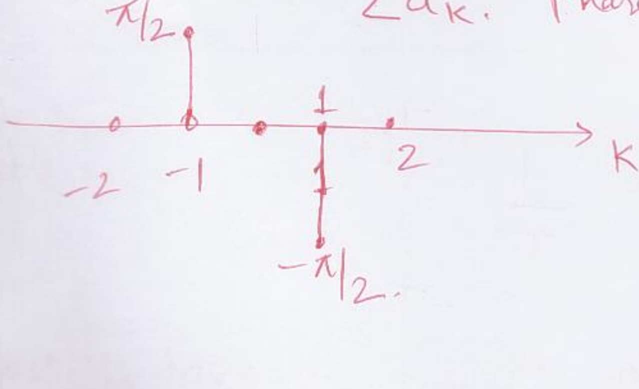
$a_0 = 0$.

and $a_k = 0$ for $k \neq -1, 1, 0$



magnitude.

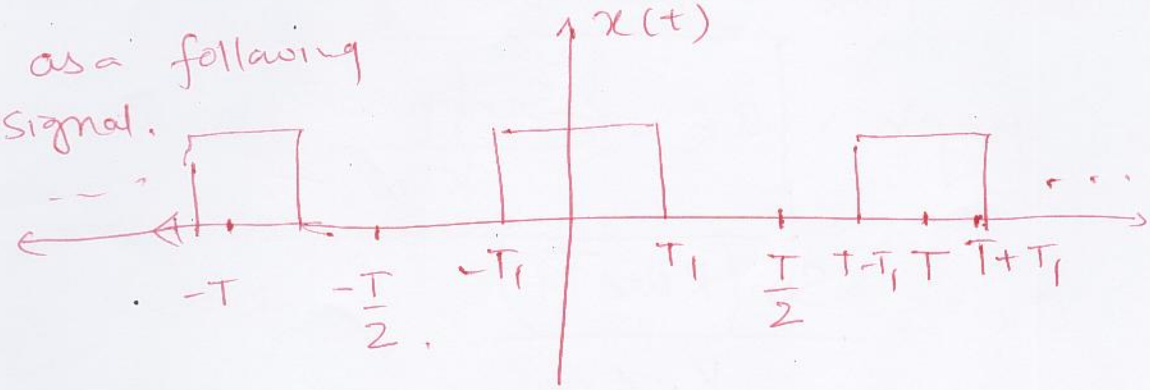
$\angle a_k$. Phase plot.



Example. 3.5 (oppenheim).

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Treat it as a following periodic signal. with period. of T



$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt$$

$$= \frac{2T_1}{T}$$

$$a_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} \cancel{x(t)}\,e^{j\omega_0 t k}\,dt$$

$$= \frac{1}{T}\, 2\,\frac{e^{j\omega_0 kT_1} - e^{-j\omega_0 kT_1}}{2j\omega_0 K} = \frac{2\sin K\omega_0 T_1}{K\omega_0 T}$$

for $K \neq 0$.

$$= \frac{\cancel{2}\sin K\omega_0 T_1}{\cancel{2}\pi\ K} = \frac{\sin K\omega_0 T_1}{\pi K}$$

In general fourier coefficients can be Complex. In this example it turns out that they are all real valued. As a consequence in this case we can plot them in One graph.

$$\boxed{a_K = \frac{2\sin(K\omega_0 T_1)}{K\omega_0 T}}$$

$$a_K = \frac{\sin(K\omega_0 T_1)}{K\pi} \qquad K \neq 0.$$

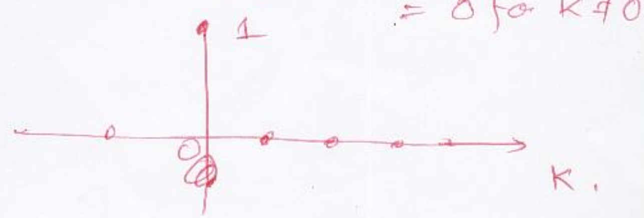
$$a_0 = \frac{2T_1}{T}.$$

Assume $T_1 = T/2$

$a_0 = 1$

$$a_K = \frac{\sin\left(K\omega_0 \frac{T}{2}\right)}{K\pi} = \frac{\sin(K\pi)}{K\pi}$$

$= 0$ for $K \neq 0$.

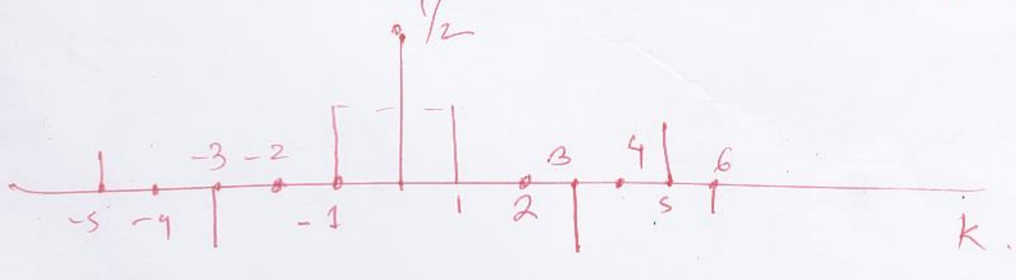


Assume $T_1 = T/4$.

~~Then. $2\pi T_1 = \frac{\pi}{2}$, $\frac{T_1}{T} = \frac{1}{4}$~~

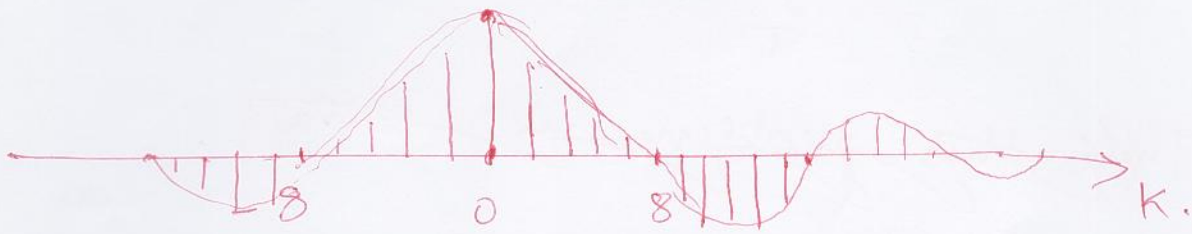
Then, $a_0 = \frac{1}{2}$

$$a_K = \frac{\sin(K\omega_0 T/4)}{K\pi} = \frac{\sin\left(K\left(\frac{2\pi}{\pi 2}\right)\right)}{K\pi} = \frac{1}{2}\frac{\sin\left(\frac{\pi K}{2}\right)}{\left(\pi K/2\right)}$$



Try for $T_1 = \frac{T}{8}$, $T_1 = \frac{T}{16}$.

we get



More & more samples of function $\frac{2 \sin \omega T_1}{\omega}$.

Convergence of Fourier Series.

Let us take partial sums.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Let $e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{j\omega_0 k t}$

Let energy of error signal over one time period.

$$E_N := \int_T |e_N(t)|^2 dt$$

if $E_N \rightarrow 0$ as $N \rightarrow \infty$ we have

that $x_N(t) \rightarrow x(t)$

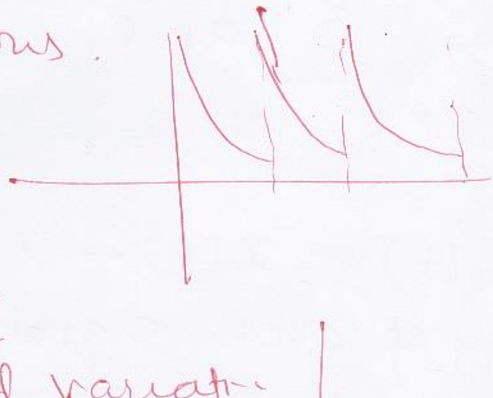
Problem 1: Coefficients a_k may become infinite.

Restrict ~~for the~~ to signals $x(t)$ s.t.
~~Assume~~ $\int_T |x(t)|^2 dt < \infty$ i.e. finite.

~~Thus this problem is solved~~.

There is another problem of pointwise convergences which we will see later.

Dirichlet's Conditions - Guarantees that $x(t) =$ Fourier Series representation except at few isolated points t at which $x(t)$ is discontinuous.



Cond. 1: $\int_T |x(t)| < \infty$.

non-example $x(t) = 1/t$

2. $x(t)$ of bounded variati..

non-example: $x(t) = \sin\left(\frac{2\pi}{t}\right)$

3. In any finite interval function $x(t)$ has finite no. of discontinuities

$$x(t) = \begin{cases} 1 & 0 \le t \le 4 \\ 1/2 & 4 \le t < 6 \\ 1/4 & 6 \le t \le 7 \end{cases}$$

3.5. Properties of Continuous time Fourier series.

$$x(t) \xleftrightarrow{FS} a_k.$$

(i) Linearity:

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k.$$

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k.$$

(ii) Time shifting.

$$x(t) \xleftrightarrow{FS} a_k. \quad \Big| \quad x(t-t_0) \xleftrightarrow{F(s)} e^{-j\omega_0 t_0} a_k$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(t-t_0) \longleftrightarrow b_k = \frac{1}{T} \int_T x(t-t_0) e^{-jk\omega_0 t} dt$$

$t - t_0 = \tau$

$$= \frac{1}{T} \int_{-t_0}^{T-t_0} x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau$$

$$= e^{-j\omega_0 t_0} \left[\frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 (\tau)} d\tau \right]$$

$$= e^{-j\omega_0 t_0} a_k.$$

(iii) Time Reversal.

$$x(t) \xleftrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega_0 k t}\, dt$$

$$x(-t) \longleftrightarrow b_k = \frac{1}{T}\int_T x(-t)\, e^{-j\omega_0 k t}\, dt$$

$$= -\frac{1}{T}\int_{0}^{-T} x(\tau)\, e^{+j\omega_0 k \tau}\, d\tau$$

$$= \frac{1}{T}\int_{-T}^{0} x(\tau)\, e^{j\omega_0 k \tau}\, d\tau.$$

$$= \frac{1}{T}\int_T x(\tau)\, e^{j\omega_0 k \tau}\, d\tau$$

$$= a_{-k}$$

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

(iv) Time Scaling

$$x(t) \longleftrightarrow a_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 k t}\, dt$$

$$\underbrace{x(\alpha t)}_{\substack{\text{Fundamental} \\ \text{period become} \\ T/\alpha,\ (\alpha\omega_0\ \text{freq.})}} \longleftrightarrow b_k = \frac{1}{T}\int_0^T x(\alpha t)\, e^{-j\omega_0 \cancel{\alpha} k \frac{t}{\cancel{\alpha}}}\, dt$$

$$\tau = \alpha t.$$

$$= \alpha \frac{1}{T}\int_0^{T/\alpha} x(\tau)\, e^{-j\frac{\omega_0 k}{\alpha}\tau}\, d\tau$$

$$= \cancel{\alpha} \frac{1}{T/\alpha}\int_0^{T/\alpha} x(\tau)\, e^{-j\frac{\omega_0 k}{\alpha}\tau}\, d\tau$$

$$= a_k.$$

(v). Multiplication

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \xleftrightarrow{FS} b_k.$$

both have same period T

$$x(t)\,y(t) \longleftrightarrow h_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{-j\omega_0 k t}$$

$$z(t) = x(t)\,y(t) = \sum_{k=-\infty}^{\infty} c_k e^{-j\omega_0 k t}$$

$$= \left(\sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t}\right)\left(\sum_{k=-\infty}^{\infty} b_k e^{-j\omega_0 k t}\right)$$

To obtain c_n multiply by $e^{-j\omega_0 n t}$ and integrate over one time period.

$$c_n = \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t}\right) \sum_{k=-\infty}^{\infty} b_k e^{+j\omega_0 (k-n) t}\, dt.$$

$$= \cancel{\frac{1}{T}\left[\frac{1}{T}\int \sum_{k=-\infty}^{\infty} \frac{a_k}{j\omega_0 k} e^{-j\omega_0 k t} \sum_{k=\infty}^{\infty} j\omega_0 (k-n)\, b_k e^{j\omega_0 k n}\right.}$$

$$\frac{1}{T}\int_T e^{j\omega_0 (K-n)t} = \begin{cases} 1 & K = n \\ 0 & K \neq n \end{cases} = \delta[n-K].$$

<u>hint</u>: treat one summation as impulse response.

(v) Conjugation & Conjugate Symmetry

if $x(t) \xleftrightarrow{FS} a_K.$

then $x^*(t) \longleftrightarrow a^*_{-K}.$

(7) <u>Parseval's Relation [exploits orthogonality]</u>

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} |a_K|^2.$$

Note $\frac{1}{T}\int_T |a_K e^{jK\omega_0 t}|^2 dt = |a_K|^2$

Thus sum of average power of each harmonic is the total average power of this signal $x(t)$

$x(t) \xleftrightarrow{FS} a_k$

$\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k.$

$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \frac{1}{jk\omega_0} a_k.$

This completes CT Fourier series