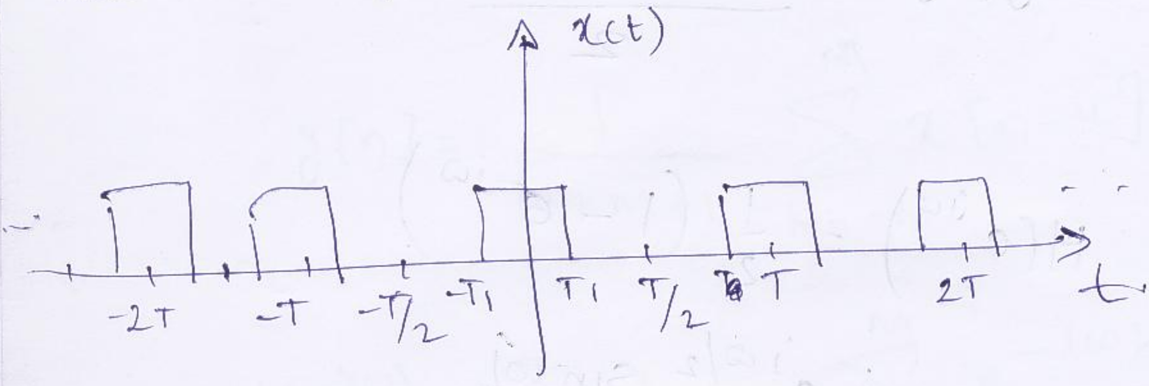
FOURIER TRANSFORM

Representation of Aperiodic signals.

Recall Example from Fourier Series (1)



over one period.

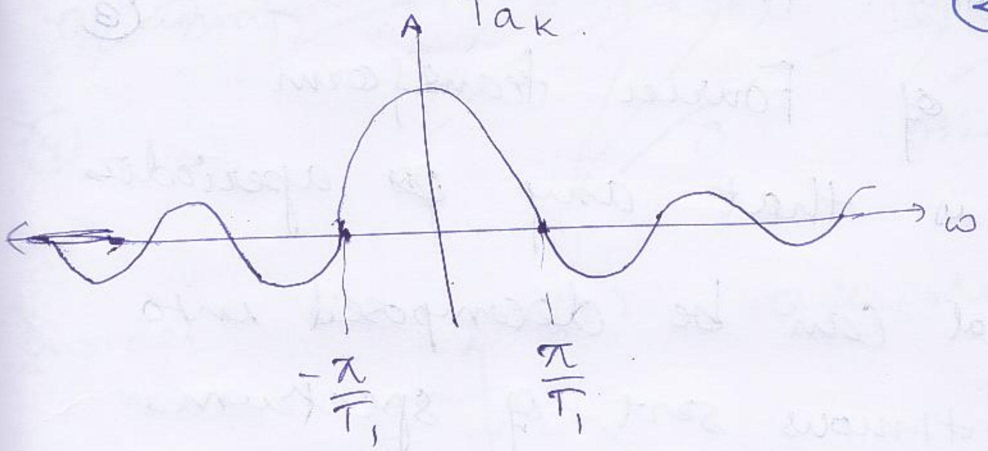
$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

For this case we obtained

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \frac{2\sin(\omega T_1)}{\omega}\Bigg|_{\omega = k\omega_0}$$



If T_1 is fixed then this function of ω remains fixed.

Getting Fourier series coeffecients is just taking samples of this function.

$$\frac{2\sin(\omega T_1)}{\omega}$$

As T increases or as ω_0 decreases. We are taking finer & finer samples of $\frac{2\sin(\omega T_1)}{\omega}$, in $k\omega_0$ as $\omega_0 \downarrow$ no. of samples increase.

As $T \to \infty$ i.e. signal $x(t)$ approaches the aperiodic pulse

We can "loosely" say that we are sampling $\frac{2\sin(\omega T_1)}{\omega}$ very finely enough to be considered continuous. This leads to the

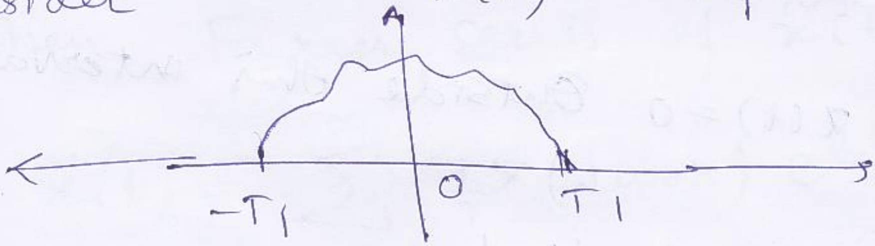
(3)

notion of Fourier transform.
Which is that any ~~p~~ aperiodic
signal can be decomposed into
a continuous sort of spectrum

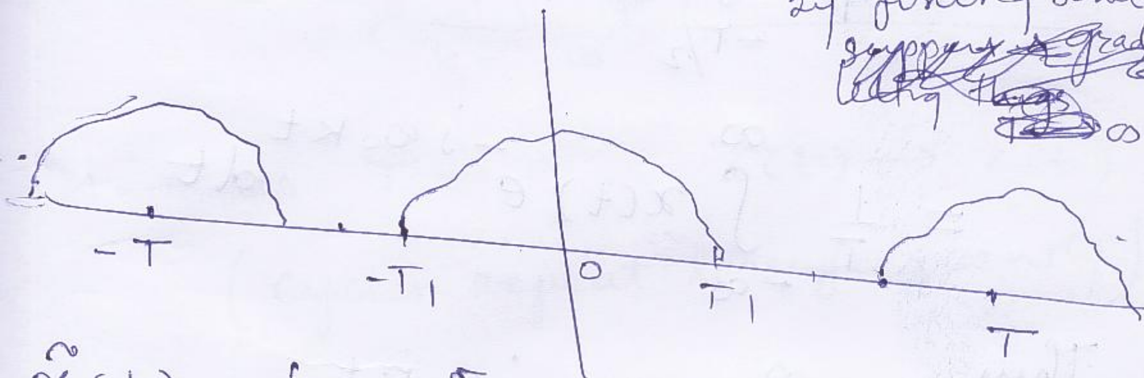
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

Notice similarity with Fourier series.

Consider $x(t)$ aperiodic signal



$\tilde{x}(t)$ — made periodic. by fixing some period. ~~[illegible] gradually letting [illegible] $\to \infty$~~



$\tilde{x}(t)$ has Fourier series given by.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

Where $\omega_0 = \frac{2\pi}{T}$.

$\tilde{x}(t) = x(t)$ on interval $[-T/2, T/2]$

and

~~Best~~ $x(t) = 0$ Outside this interval.

then it holds that

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\omega_0 k t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-j\omega_0 k t} dt$$

Thus,

$$Ta_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega_0 k t} dt$$

Just like in that example let us define now the envelope to be

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega} dt$$

Then Ta_k are samples of $X(j\omega)$

$$a_k = \frac{1}{T} X(j\omega_0 k)$$

Then Fourier series of $\tilde{x}(t)$ becomes

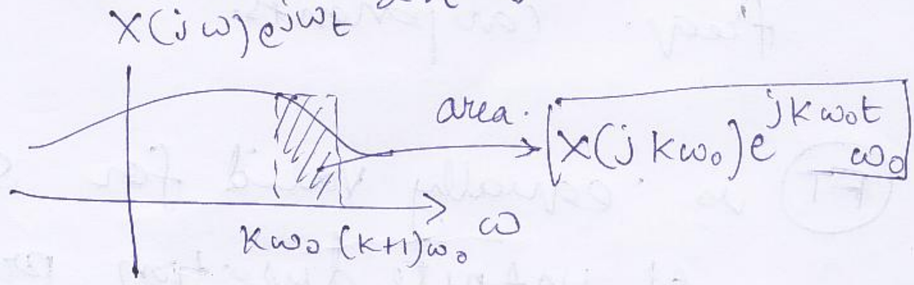
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(j\omega_0 k)\, e^{j\omega_0 k t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \underline{X(j\omega_0 k)\, e^{j\omega_0 k t}\, \omega_0}$$

Now as $T \to \infty$, $\tilde{x}(t) \to x(t)$
(cycles repeat less frequently)

and as $T \to \infty$, $\omega_0 \to 0$.

$$\tilde{x}(t) \to x(t) = \frac{1}{2\pi} \int X(j\omega)\, e^{j\omega t}\, d\omega$$



Thus we have. Inverse Fourier Transfor of $X(j\omega)$

Synthesis eqn: $$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega} \quad \text{— IFT}$$

and (also called Fourier Integral)
$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} \quad \text{FT}$$

Fourier Transform of aperiodic Signal $x(t)$

(Spectrum of Signal $x(t)$)

Linear Combination of different frequency components.

Decomposes $x(t)$ in terms of freq. components.

FT is equally valid for signals of infinite duration provided the integral ~~well defined~~ converges to finite

Convergence of Fourier Transforms

Does $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

represent ~~converge~~ ~~to~~ $x(t)$?

is it equal to $x(t)$ ~~our~~ ~~original~~

or when can you obtain a Fourier transform.

The answer is yes provided following Dirichlet conditions are satisfied by $x(t)$ [Sufficient !]

(i) $x(t)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

(ii). $x(t)$ have a finite no. of maxima & minima in any finite interval.

(iii) $x(t)$ have finite no. of discontinuities in ~~a~~ any finite interval.

- We will ~~a~~ make an exception for the case of <u>Dirac-impulse</u>. (later)

Example.

$$x(t) = e^{-at} u(t), \quad a > 0.$$

Compute Fourier transform

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt$$

$$= \int_0^\infty \frac{-1}{a+j\omega} e^{-(a+j\omega)t} dt.$$

$$= -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^\infty$$

$$= -\frac{1}{a+j\omega}\left[0 - 1\right] \text{ since } a>0.$$

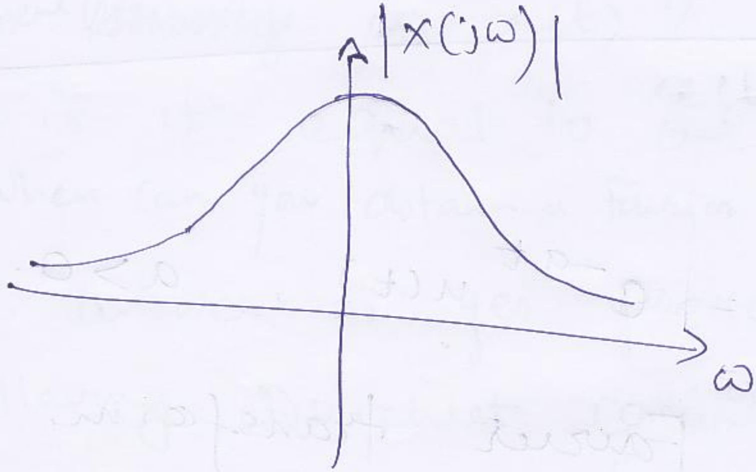
$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

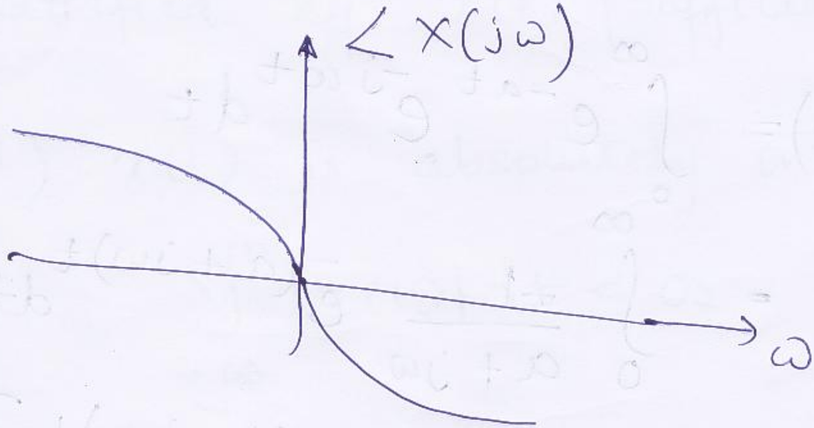
for $a<0$, not defined

Since e^{-at} is not integrable

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$





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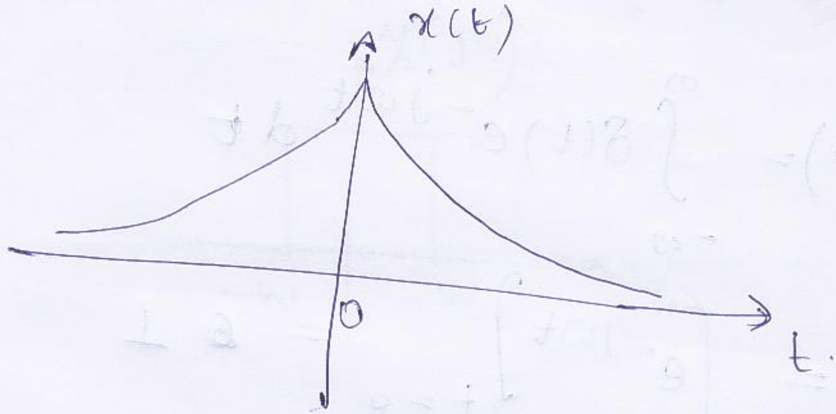
$x(t) = e^{-a|t|}$ (4)



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} + \left(\frac{-1}{a+j\omega}\right)\left[e^{-(a+j\omega)t}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega}\left[1 - e^{at} e^{-j\omega t}\right]_{-\infty}^{0}$$

$$+ \frac{1}{a+j\omega}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{1}{a^2+\omega^2}$$

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

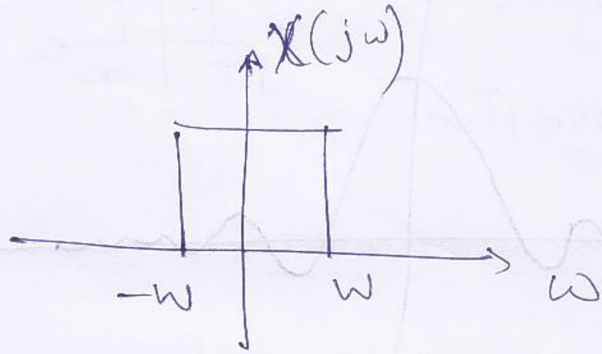
$$= \left[e^{-j\omega t}\right]_{t=0} = 1$$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2\,\frac{-e^{-j\omega T_1} + e^{j\omega T_1}}{2j\omega}$$

$$= 2\,\frac{\sin \omega T_1}{\omega}$$

Inverse Fourier Transform



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega$$

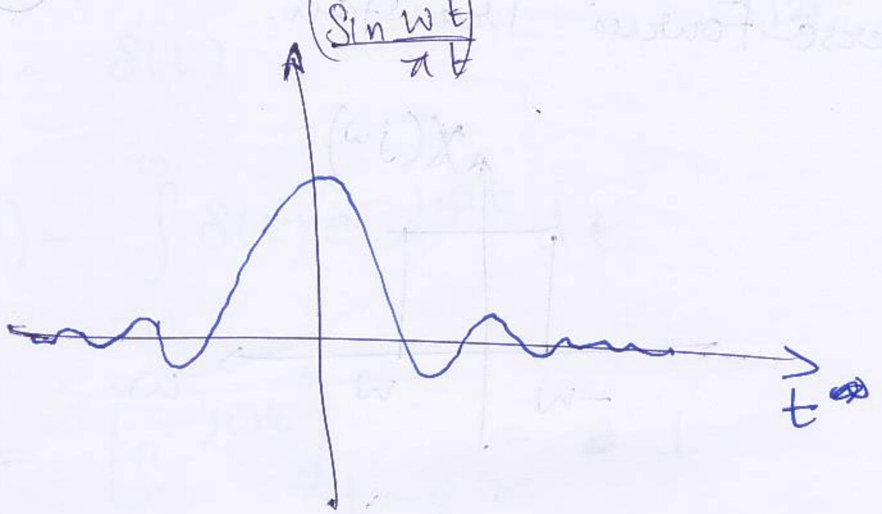
$$= \cancel{\frac{1}{2\pi j \omega}\,\frac{e^{jWt} - e^{-jWt}}{2j}}$$

$$= \cancel{\text{sinc}}\ \text{Sin}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W}$$

$$= \frac{1}{2\pi j t}\left[\frac{e^{jWt}}{\cancel{j\omega t}} - \frac{e^{-jWt}}{\cancel{+j\omega t}}\right]$$

$$= \frac{\sin Wt}{\pi t}$$



let $\operatorname{sinc}(\theta) = \dfrac{\sin(\pi\theta)}{\pi\theta}$

then.

$$\frac{2\sin\omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right).$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right).$$

Ideal low pass filter

Impulse response.

- Non-Causal. (anticipation reqd)
- Unstable. (not absolutely integrable)

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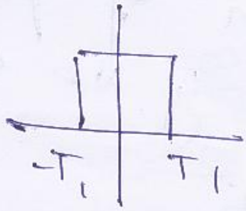
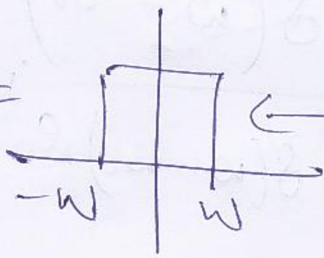
$x(t) =$  $\longleftrightarrow X(j\omega) = 2T_1 \,\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$

if

$X(j\omega) =$  $\longleftrightarrow \quad x(t) = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$

Now as $W \to \infty$

$X(j\omega) \to$ all 1 function

<u>all pass</u>

$x(t) \to \delta(t)$

$$\int_{-\pi/w}^{\pi/w} \left(\frac{\sin wt}{\pi t}\right) dt =$$

Fourier transform of periodic (16)
signal possible if we allow
Dirac - impulses.

Consider

$$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega$$

$$= e^{j\omega_0 t}$$

In general,

if $X(j\omega) = 2\pi\,\delta(\omega - k\omega_0)$

$$x(t) = e^{jk\omega_0 t}$$

$$X(j\omega) = 2\pi\,\delta(\omega).$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= 1$$

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then if

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

than $x(t) = \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{\text{Fourier series of } x(t)}$

In this way ~~of~~ Dirac-impulses can be used to represent Fourier Transform of Periodic Signals.

Unit step Fourier Transfor.

$$\lim_{a \to 0} e^{-at}u(t) \to u(t)$$

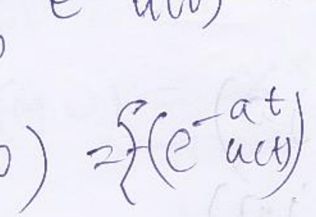
$$X(j\omega) = \mathcal{F}\left(e^{-at}u(t)\right) = \frac{1}{a+j\omega} = \frac{a}{(a^2+\omega^2)} - \frac{j\omega}{(a^2+\omega^2)}$$

as $\cancel{a} \to 0$ $\quad \frac{1}{\cancel{a}}\left(\frac{\cancel{a^2}}{1 + \frac{\omega^2}{a^2}}\right)$

as $a \to 0$ then $\downarrow$

$\pi\delta(\omega) - \frac{j}{\omega}$

$\delta(\omega)\pi + \frac{1}{j\omega}$



$$\int_{-\infty}^{\infty}\left(\frac{a}{a^2+\omega^2}\right)d\omega = \pi$$

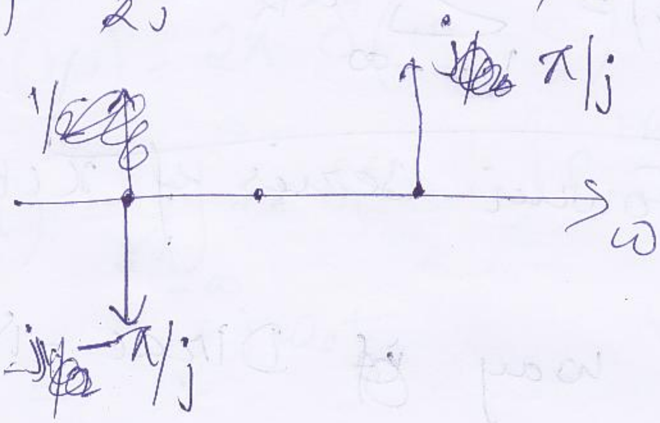
$= \tan^{-1}\left(\frac{\omega}{a}\right)\Big|_{-\infty}^{\infty}$

$= \frac{\pi}{\cancel{a}}$

$x(t) = \sin \omega_0 t$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$X(j\omega) = \frac{\cancel{2}\pi}{\cancel{2}j} \delta(\omega - \omega_0) - \frac{\cancel{2}\pi}{\cancel{2}j} \delta(\omega + \omega_0)$$



Properties.

$$\boxed{\begin{aligned} x(t) &\xleftrightarrow{F} X(j\omega) \\ e^{-at}u(t) &\xleftrightarrow{F} \frac{1}{a+j\omega} \end{aligned}}$$

Linearity.

$$x(t) \longleftrightarrow X(j\omega)$$

$$y(t) \longleftrightarrow Y(j\omega)$$

$$ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega)$$

Time Shifting

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)\, e^{-j\omega t_0}}\; e^{j\omega t}\, d\omega$$

$$F(x(t-t_0)) = e^{-j\omega t_0} X(j\omega)$$

Conjugation

$$x(t) \longleftrightarrow X(j\omega)$$

$$x^*(t) \longleftrightarrow X^*(-j\omega)$$

if $x(t)$ is real then

$X(j\omega) = X^*(-j\omega)$ [conjugate symmetry

$$x(t) \longleftrightarrow X(j\omega)$$

$$\text{Re}\, x(t) \longleftrightarrow \text{Re}\{X(j\omega)\}$$

$$\text{Od}(x(t)) \longleftrightarrow j\,\text{Im}\{X(j\omega)\}$$

- Differentiation.

$$x(t) \longleftrightarrow X(j\omega)$$

$$\frac{dx(t)}{dt} \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega X(j\omega)} e^{j\omega t}\, d\omega$$

$$\mathcal{F}\left(\frac{d}{dt}x(t)\right) \longleftrightarrow j\omega X(j\omega)$$

Time and Frequency Scaling

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

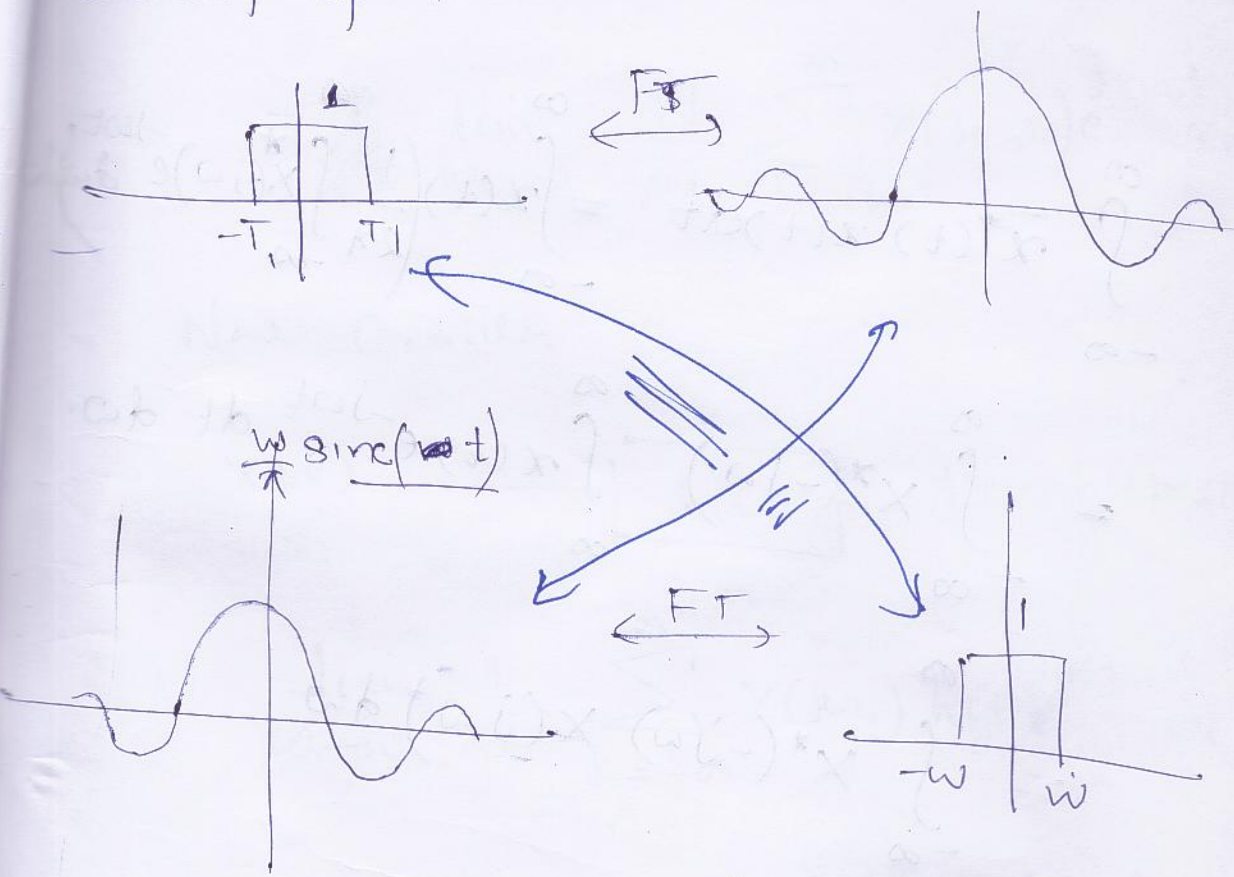
$$x(-t) \longleftrightarrow X(-j\omega)$$

$s = at$

$$\int x(at)\, e^{-j\omega t}\, dt = \frac{1}{a}\int x(s)\, e^{-j\left(\frac{\omega}{a}\right)s}\, ds$$

$$= X\left(j\frac{\omega}{a}\right)$$

Duality of FT



$$x(t) \longleftrightarrow X(j\omega)$$

$$X(t) \longleftrightarrow x(j\omega)$$

$$\frac{dX(j\omega)}{d\omega} \longleftrightarrow \int_{-\infty}^{\infty} -jt\,x(t)\,e^{j\omega t}\,dt$$

$$-jt\,x(t) \longleftrightarrow \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{\omega} x(\eta)\,d\eta \longleftrightarrow -\frac{1}{jt}x(t) + \pi x(0)\,\delta(t)$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega .$$

$$\int_{-\infty}^{\infty} x^*(t)\, x(t) dt = \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(-j\omega) e^{-j\omega t} d\omega \right) dt$$

$$= \int_{-\infty}^{\infty} X^*(-j\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\, d\omega .$$

$$= \int_{-\infty}^{\infty} X^*(-j\omega)\, X(j\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega .$$

Convolution

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

Now consider

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, h(t) * e^{+jk\omega_0 t}\, \omega_0$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) \int_{-\infty}^{\infty} h(\tau)\, e^{+jk\omega_0 (t-\tau)}\, d\tau\, \omega_0$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{kj\omega_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-jk\omega_0 \tau}\, d\tau\, \omega_0$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, H(jk\omega_0)\, e^{kj\omega_0 t}\, \omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, H(j\omega)\, e^{j\omega t}\, d\omega.$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega.$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt\, d\tau$$

$$r = t-\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(r)\, e^{-j\omega r}\, dr\right] e^{-j\omega \tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, H(j\omega)\, e^{-j\omega \tau}\, d\tau$$

$$= H(j\omega)\, X(j\omega).$$

FTS Multiplication Property.

$$r(t) = s(t)\,p(t)$$

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\, P(j(\omega-\theta))\, d\theta$$

Follows from duality.

LCCODE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

$$\frac{dy(t)}{dt} \longleftrightarrow j\omega\, Y(j\omega)$$

$$\frac{d^k y(t)}{dt^k} \longleftrightarrow (j\omega)^k\, Y(j\omega).$$

$$\sum_{k=0}^{N} a_k (j\omega)^k\, Y(j\omega) = \sum_{k=0}^{M} b_k X(j\omega)(j\omega)^k$$

$$Y(j\omega) = \left[\frac{\sum b_k (j\omega)^k}{\sum a_k (j\omega)^k}\right] X(j\omega) = \underbrace{H(j\omega)}_{\text{Fourier transform of Impulse response}} X(j\omega)$$