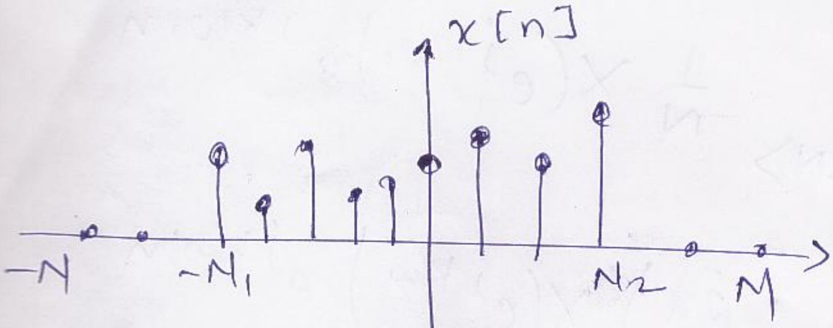
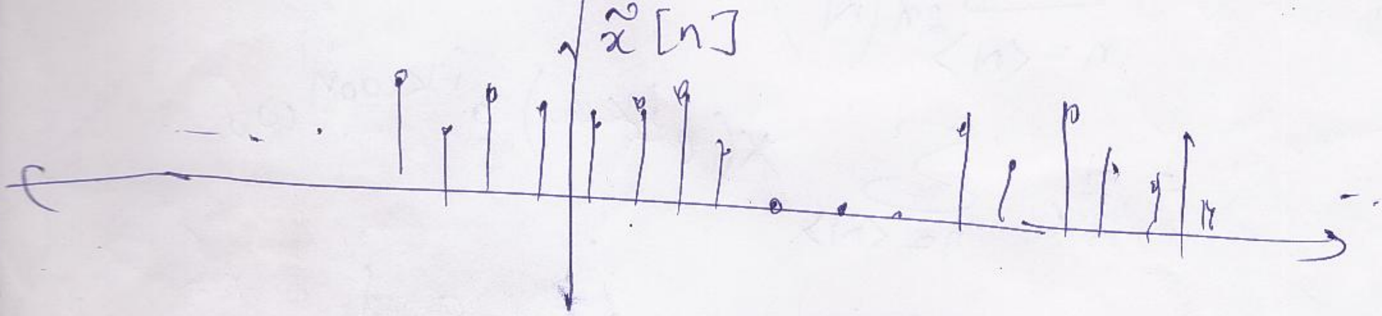
Discrete time Fourier Transform.

Consider $x[n]$ which is aperiodic signal and is non-zero in $-N_1 \le n \le N_2$

This signal can be made periodic.





$x[n] = \tilde{x}[n]$ On interval $-N \le n \le N$

Then

$$\tilde{x}[n] = \frac{1}{N} \sum_{n=\langle N \rangle} a_k e^{j\left(\frac{2\pi}{N}\right)kn}$$

Discrete time Fourier Series.

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Consider

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

Then $a_k = \frac{1}{N} X(e^{jk\omega_0})$

$$\tilde{x}[n] = \sum_{n=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$= \sum_{n=\langle N\rangle} \frac{1}{2\pi}\left(\frac{2\pi}{N}\right) X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

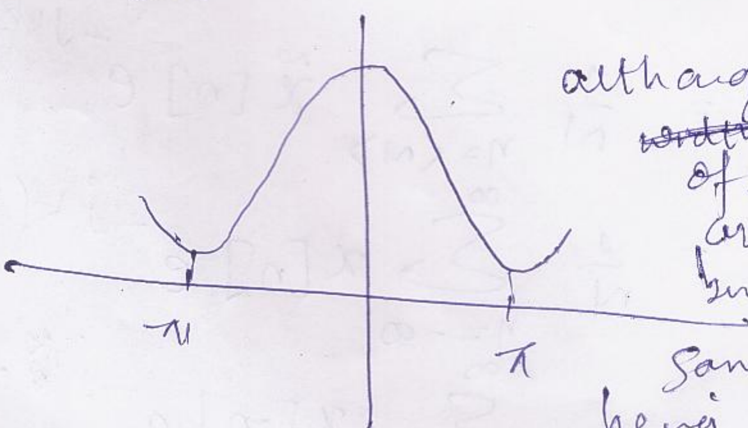
$$= \frac{1}{2\pi} \sum_{n=\langle N\rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

Now as $N \to \infty$, $\omega_0 \to 0$ and

~~the~~

$$\tilde{x}[n] \to x[n]$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$



although no. ~~width~~ ~~of~~ of samples are increasing but all the samples are being taken between $-\pi$ to π.

- DTFT is periodic with 2π as its period.
- Synthesis eq. is finite integral.

Example:

$$x[n] = a^n u[n], \quad |a| < 1$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

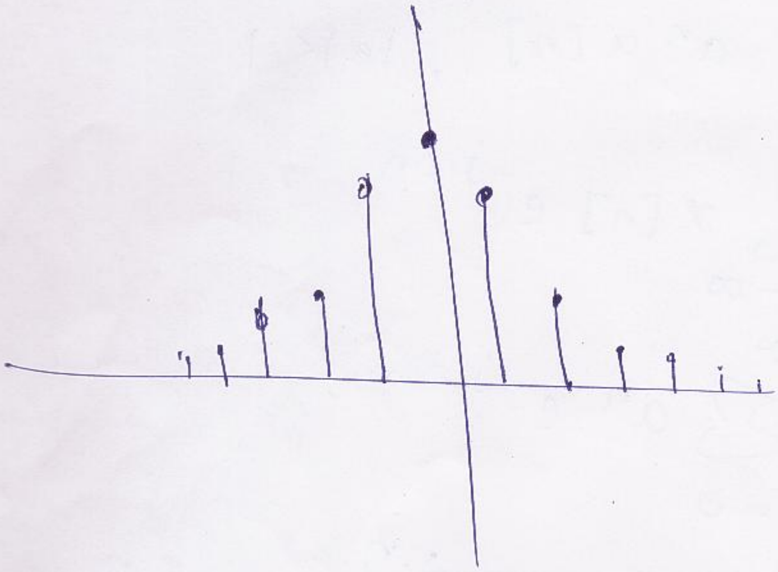
$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

Example :

$$x[n] = a^{|n|}, \quad |a| \leq 1$$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} \left(a e^{j\omega}\right)^{n} + \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^{n}$$

$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}}.$$

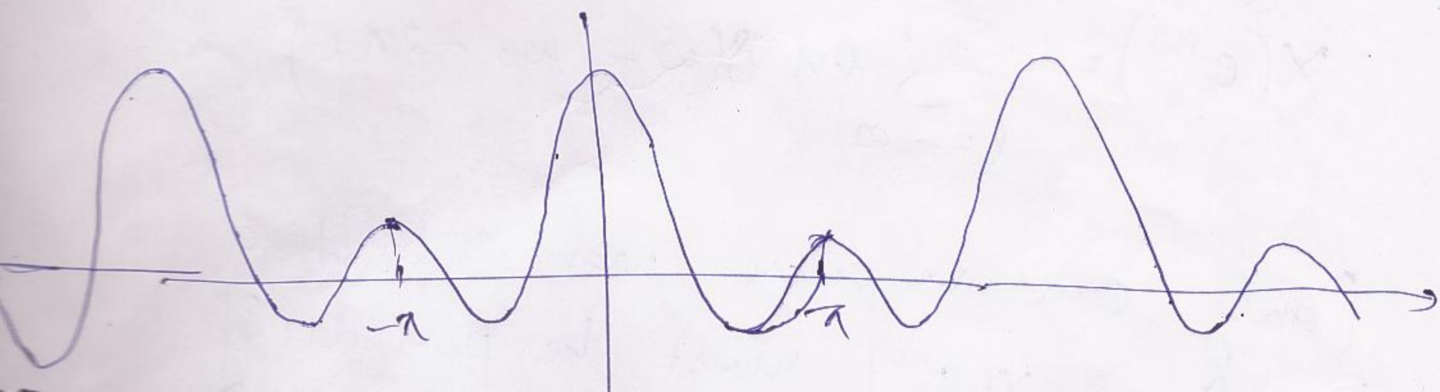
$$X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

Example.

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}.$$

$$= \frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$$



$x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

Note $\delta[n]$
$\delta(\omega)$ are different

What about

$x[n] = e^{j\omega_0 n}$? [Dirac impulse comes to the rescue]

~~Properties:~~

Consider,

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)$$

[only one impulse won't suffice since $X(e^{j\omega})$ must be periodic]

then

$$x[n] = \frac{1}{\cancel{2\pi}} \int_{-\pi}^{\pi} \sum_{l=-\infty}^{\infty} \cancel{2\pi}\, \delta(\omega - \omega_0 - 2\pi l) e^{j\omega n}\, d\omega$$

$$= e^{j(\omega_0 - 2\pi l)n} = e^{j\omega_0 n}.$$

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)$$

$$x[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta\left(\omega - \frac{2\pi k}{N} - 2\pi l\right)$$

Properties of DTFT

1) Periodicity $X\left(e^{j(\omega + 2\pi)}\right) = X(e^{j\omega})$

2) Linearity

$$x_1[n] \longleftrightarrow X_1(e^{j\omega})$$

$$x_2[n] \longleftrightarrow X_2(e^{j\omega})$$

$$a x_1[n] + b x_2[n] \longleftrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

3) $x[n] \longleftrightarrow X(e^{j\omega})$

the $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

- Conjugation & conjugate symmetry.

If $x[n] \xleftrightarrow{F} X(e^{j\omega})$

$$\overline{x^*[n]} \longleftrightarrow \overline{X\left(e^{-j\omega}\right)}$$

and for real signals.

$$X(e^{j\omega}) = \overline{X\left(e^{-j\omega}\right)}$$

- Differencing

$$x[n] - x[n-1] \xleftrightarrow{F} \left(1-e^{-j\omega}\right) X\left(e^{j\omega}\right)$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m] \longleftrightarrow \frac{1}{1-e^{-j\omega n}} X\left(e^{j\omega}\right)$$

$$+ \pi X(1) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- Time Reversal.

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \mid k \quad (k \text{ divides } n) \\ 0, & \text{if } n \nmid k. \quad (k \text{ does not divide } n) \end{cases}$$

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})$$

- Differentiation in Frequency.

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -j\, n x[n] e^{-j\omega n}$$

$$n x[n] \longleftrightarrow j \frac{dX(e^{j\omega})}{d\omega}.$$

- Parseval's Relation.

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$\sum_{K=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 \, d\omega.$$

- Convolution Property.

$$y[n] = x[n] * h[n].$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$