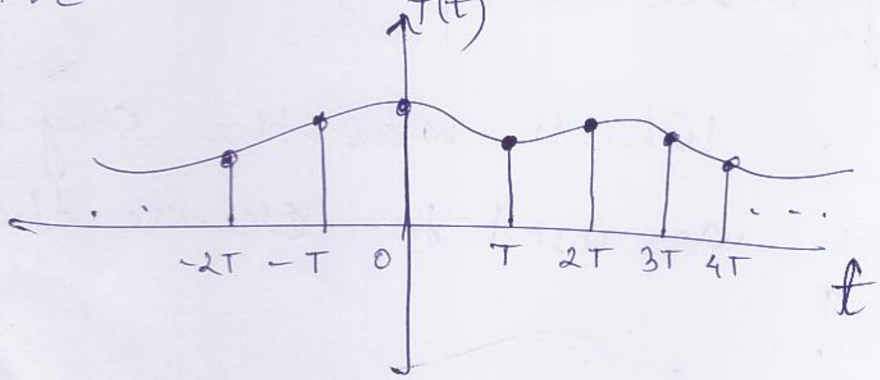
Sampling & interpolation.

- Sampling.

- Due to ease of doing digital operations on computers, harddisks, chips, etc it is convenient to represent a continuous (CT) time signal as a discrete (DT) time signal (which is obtained by sampling the respective continuous (CT) time signal) ~~at~~

- But Sampling a CT signal means we are loosing information about it ~~at~~ and only storing value of the CT signal at some specific instances of time



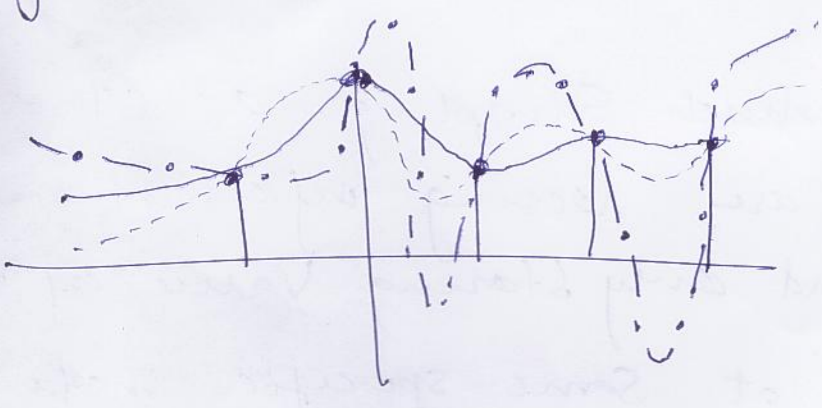
Question is if we ignore the information of signal at values not equal to KT for $K \in \mathbb{Z}$ are we loosing signal?

In otherwords from $f(KT)$ can we construct $f(t)$ back? ~~How~~

~~Is~~ When is $f(KT) \equiv f(t)$?

~~This answer~~

For any general signal, its samples may not represent it.



Many possible ways to connect the dots!. Which was the original one which was used to create these dots?

Turns out that if we know that the signal is band-limited and also know the range of band of frequencies in the signal then one can accurately reconstruct the signal back if sampled at a reasonable rate. What is this reasonable rate? The answer is in <u>Nyquist - Shannon Sampling theorem</u>:

~~Hypothetical~~

A Mathematical way to represent Sampled values is impulse train sampling.

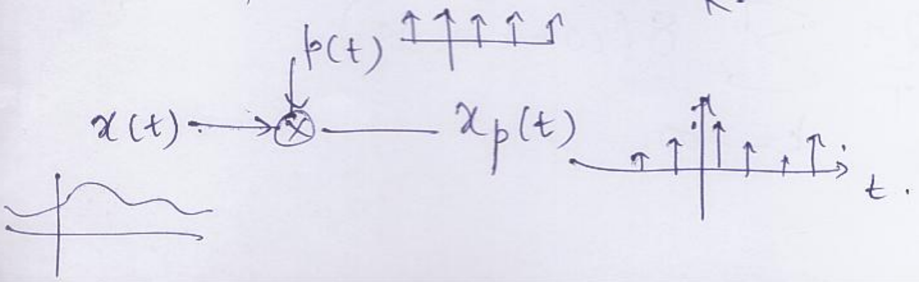
[Note that in practice this is not feasible since it is difficult ~~impossible~~ to even generate a narrow enough pulse of fixed area]

But for a pure representational value of it let us consider that we are able to generate perfect impulses. We can then consider sampling by multiplying a signal by a train of equispaced impulses.

* Impulse train Sampling.

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$ sampling interval of length T.

$$x(t)\,p(t) = x_p(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$$



multiplication. Convolution.

$$x(t)\,p(t) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, P(j(\omega-\theta))\, d\omega.$$

- Expressing $p(t)$ in fourier series

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt}.$$

Fundamental period T, $\omega_0 = \frac{2\pi}{T}$. (also samping frequ is ω_0)

Fourier Series Coefficients.

$$\underset{p(t)}{\cancel{\delta(t)}} \xleftrightarrow{FS} a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{jk\omega_0 t}\, dt = \frac{1}{T}$$

Fourier series cofficents are Constants

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\omega_0 kt}$$

Recall Fourier transform of

$$e^{jw_0 kt} \xleftrightarrow{FT} 2\pi\left(\delta(\omega - \omega_0 k)\right) \cancel{+\delta(\omega+\omega_0 k)})$$

$$p(t) \xleftrightarrow{FT} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 k)$$

$$X_p(j\omega) = X(j\omega) * P(j\omega)$$

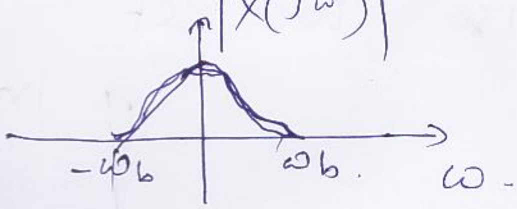
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, P(j(\omega-\theta))\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta) \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\, \delta(\omega-\theta-k\omega_0)\, d\theta$$

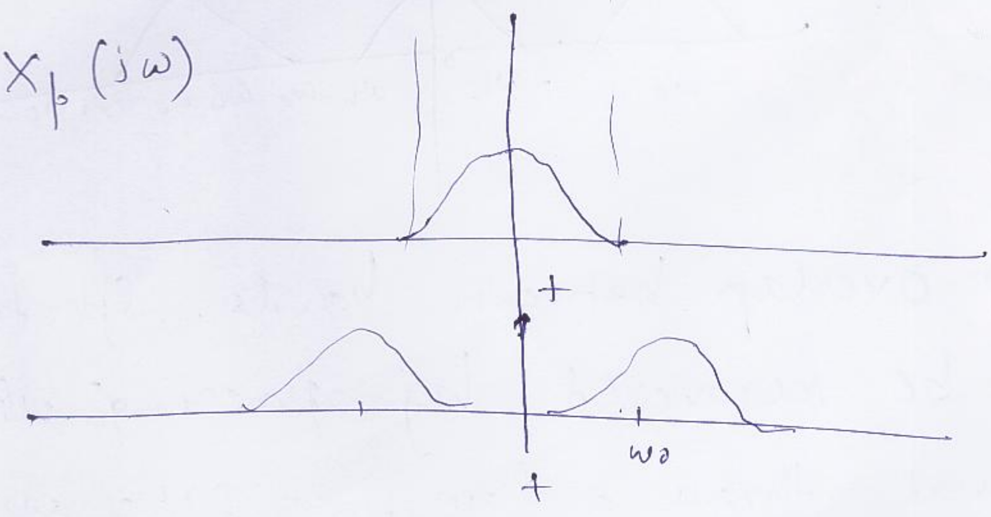
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\theta)\, \delta(\omega-\theta-k\omega_0)\, d\theta$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_0))\, d\theta$$

Thus. $X(j\omega)$

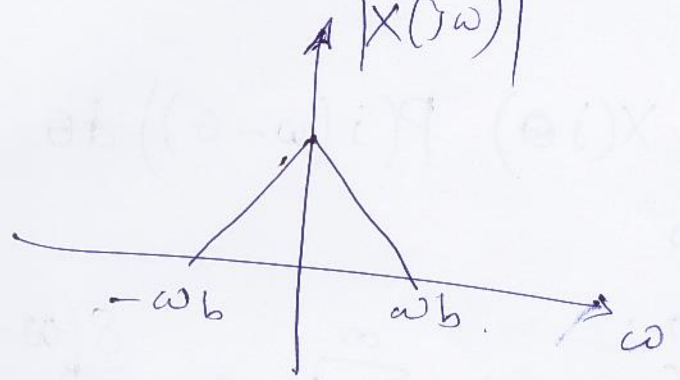


then $X_p(j\omega)$



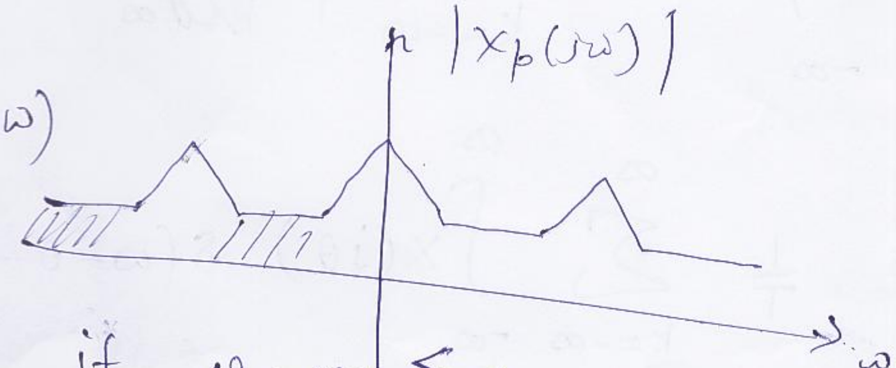
For simplicity assume that

$X(j\omega)$



then

$X_p(j\omega)$

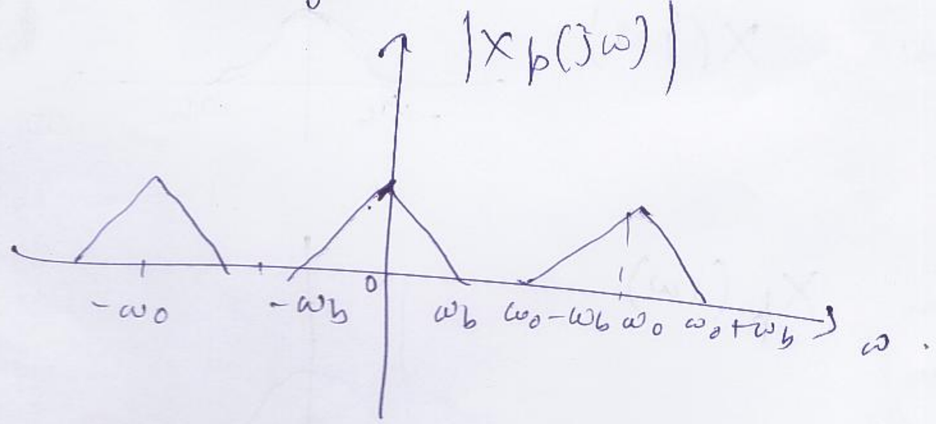


if $\omega_0 - \omega_b \leq \omega_b$

i.e $\omega_0 < 2\omega_b$.

There is slight corruption of spectrum

Otherwise we get



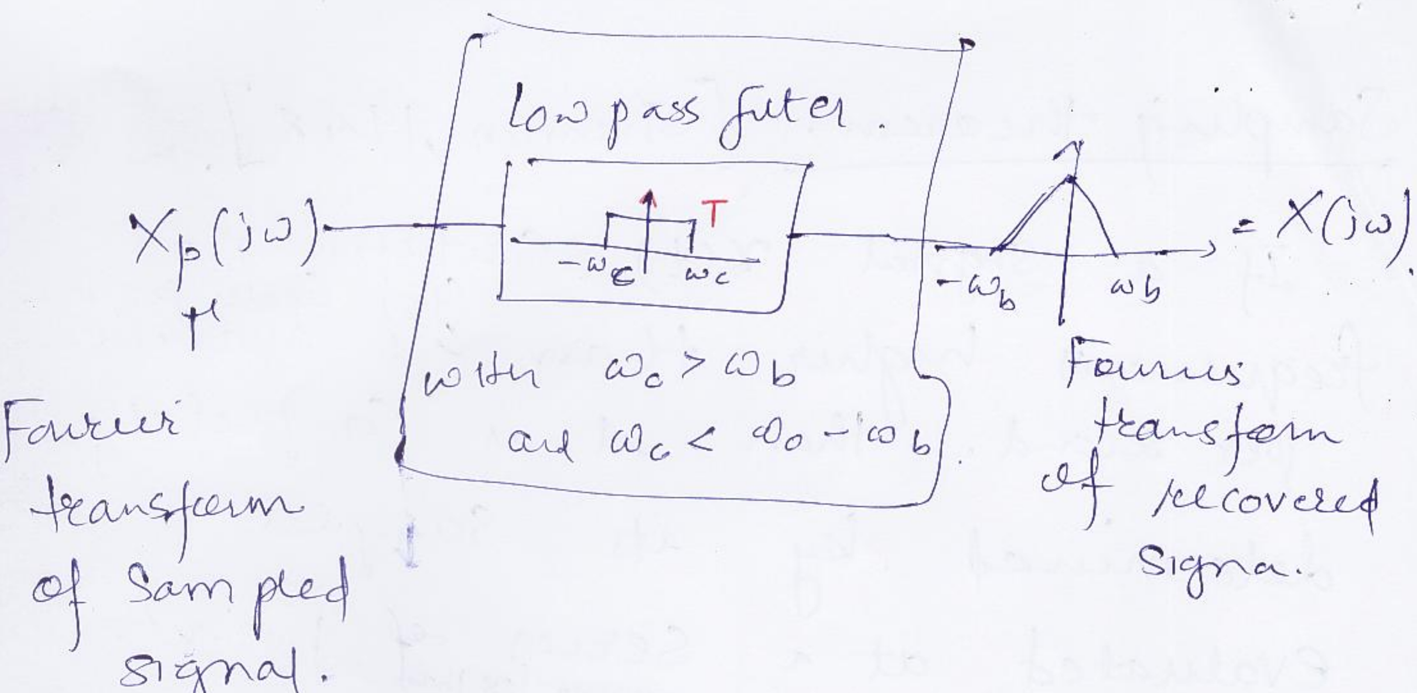
No overlap between bands. Spectrum can be recovered by passing the signal thraw a low pass filter with pass band cutoff freq. $> \omega_b$.

Sampling theorem. [Shannon, 1948].

If a signal $x(t)$ contains no frequencies higher than ω_b radians (f_b) cycles per sec. per second. then it is completely determined by its samples evaluated at a series of points which are spaced ~~atmost~~ less than $\frac{\pi}{\omega_b}$ seconds apart. $\left(\frac{1}{2f_b} \text{ seconds}\right)$.

i.e. $\boxed{\omega_0 > 2\omega_b}$. is sufficient condition.

for reconstructing exact signal back from sampled signal



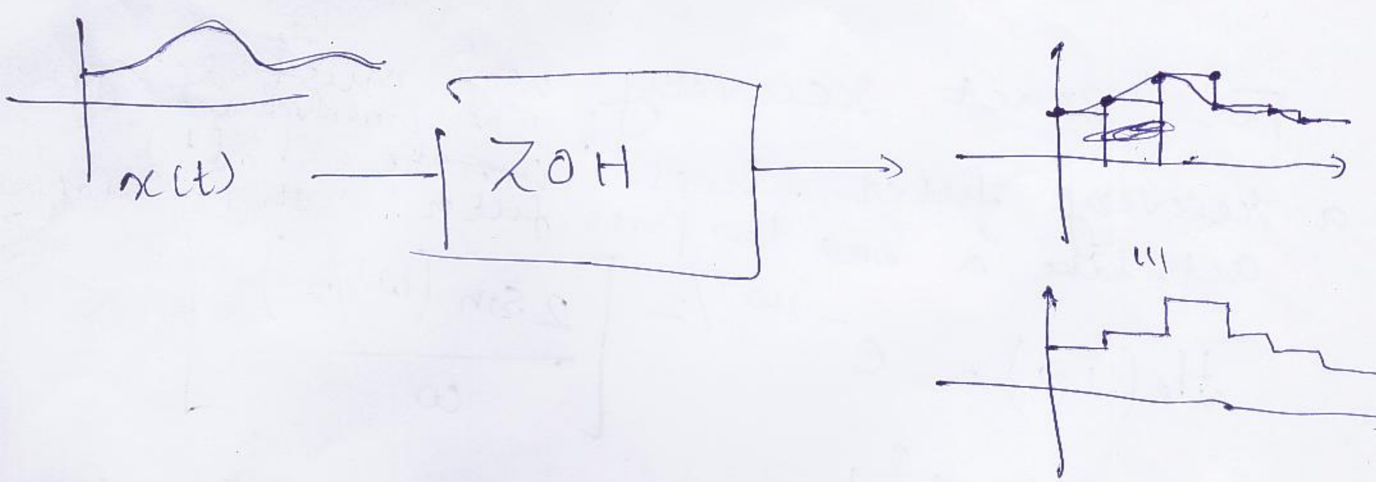
- This shows that if sampling frequency is chosen appropriately a band-limited signal can be be replaced by its samples without losing any information of the signal.

- Bandlimited already restricts the frequent changes in the signal.

- A well behaved signal can be accurately replaced by well-chosen samples.

This was ideal case. It is not possible to sample using ~~just~~ impulse train. Ideal impulse train generation is impractical. However the representation ~~of~~ is elegant.

In practice one can think of implementing Zero-order-hold. Sample a value of $x(t)$ at particular time and hold it for duration of sampling interval.

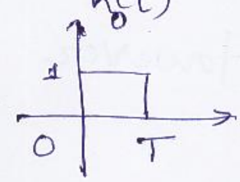


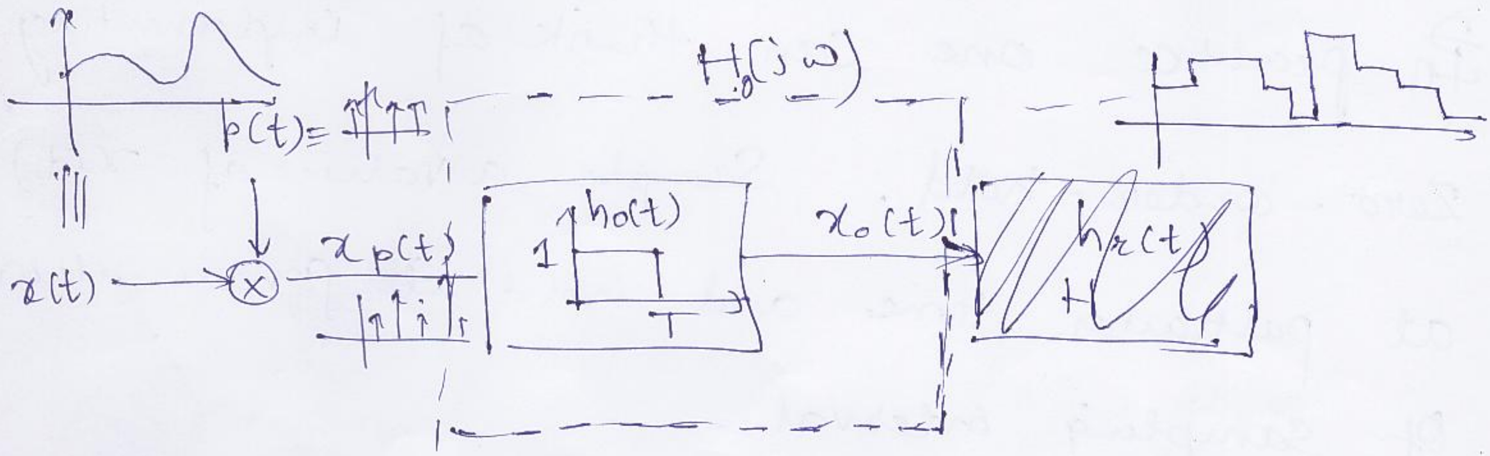
This can be done with ease ~~on~~ digitally.

This entire process can be described mathematically as follows.

(i) Sample by impulse train

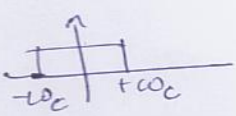
(ii) Convolve with a square pulse



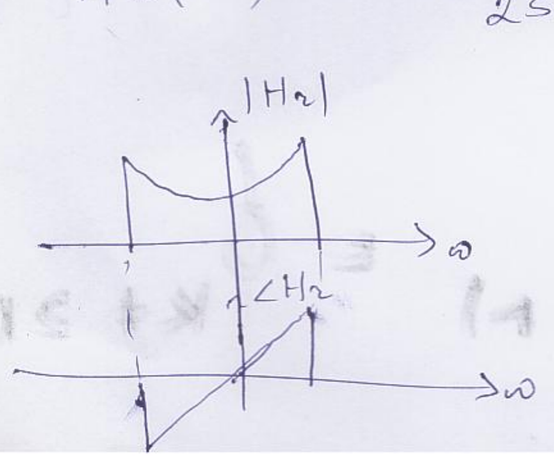


For exact recovery we must design a recovery filter which ~~inverse~~ when convolved with $h_0(t)$ is acts like a ~~ban~~ low pass filter with ~~pas~~ cutoff ω_c $> \omega_b$, $\omega_c < \omega_b - \omega_0$

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

Let $H(j\omega) =$ 

$$H_r(j\omega) = \frac{\omega e^{j\omega T/2} H(j\omega)}{2\sin(\omega T/2)}.$$



Remarks about this ideal process of Sampling & reconstruct.

- Bandpass filters like H_r, H are ~~no~~ not physically realizable and are also unstable.

- Therefore one does not use ideal filters for reconstruction.

- Usually ~~so~~ Some small errors can be tolerated ~~good~~ by use of non-ideal filters which are easy to implement practically

- One can use RC- low pass filter or any other recursive filter ~~can~~ is also possible. Reconstructed Signal will not be exact but a close replica depending upon how ~~close~~ good the filter is.

- Reconstruction using interpolation.

Let $h(t)$ be impulse response of the lowpass filter used in reconstruction.

Then,

$$x_r(t) = x_p(t) * h(t).$$

i.e, $$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT).$$

$$h(t-nT) = \frac{\omega_c T}{\pi} \frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}$$

For ideal filter.

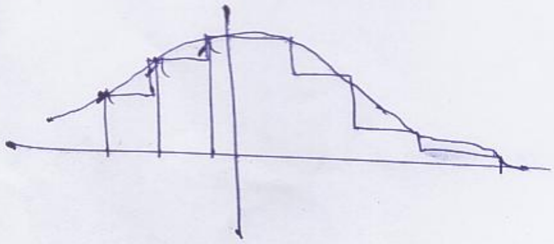
$h(t) =$  $= \frac{\omega_c T}{\pi} \frac{\sin \omega_c t}{\omega_c t}$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\left(\frac{\omega_c T}{\pi}\right)\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}}$$

~~Scaled & summed~~ ver

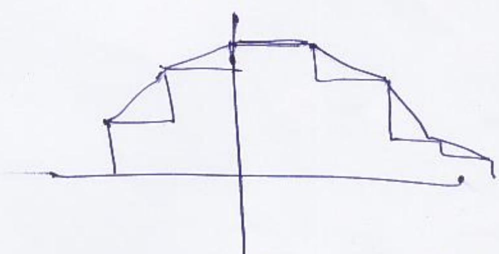
Sum of scaled & shifted gives back original signal. $x_r(t)$.

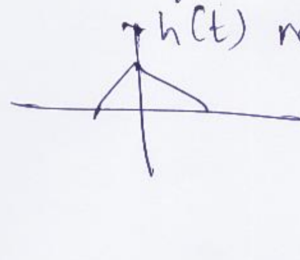
ideal case



or

non-ideal case





$h(t)$ non-ideal lowpass filter.

or

or

Aliasing

if $\omega_o < 2\omega_b$.

then there is overlapping of frequencies
and this creates aliasing effect.

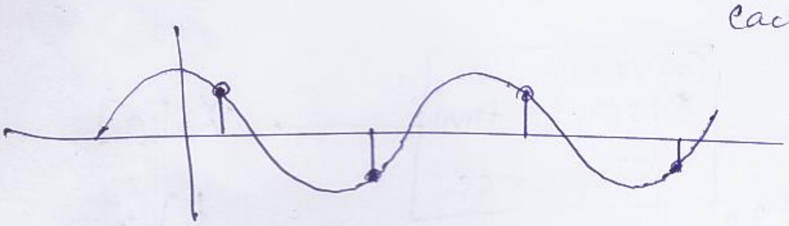
$x_r(t) \neq x(t)$

but $x_r(nT) = x(nT)$ for all n.

- Stroboscopic effect.

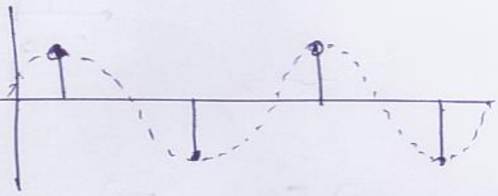
Consider $x_o(t) = \cos \omega_o t$

(i) $\omega_o < 2\omega_b$ eg: $\omega_o = \omega_b$
(sample once in each cycle)



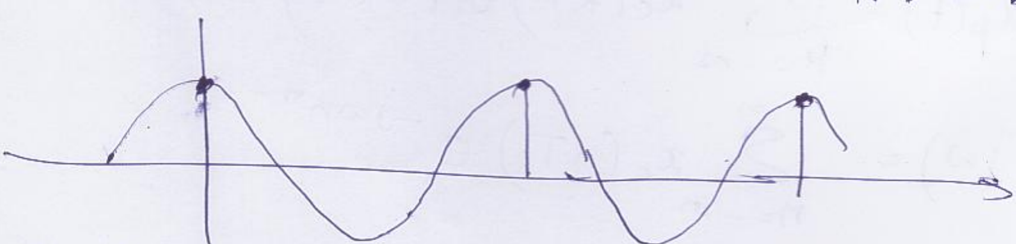
$\omega_o = \frac{\omega_b}{2}$

object assumes 2 positions.



$\frac{2\pi}{T_o} = \frac{2\pi}{2T_b}$

$T_o = 2T_b$.



Object still

$$x_d[n] \longleftrightarrow X_d(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega n}$$

$T\omega = \Omega$

$$= X_p\left(j\frac{\Omega}{T}\right) = X_d(e^{j\Omega})$$

$\omega = \frac{\Omega}{T}$

also. $$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_0))$$

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega}{T} - k\omega_0\right)\right)$$

$$\boxed{X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega - 2\pi k}{T}\right)\right)}$$

(i) Scaling of time axis by $\frac{1}{T}$ results in discrete signal $x_d[n]$. This results in scaling in frequency by T. (property) $\boxed{\Omega = \omega T}$

D/C

$y_d[n]$ → DT seq. to impulse → [T, $-\frac{\omega_s}{2}$, $\frac{\omega_s}{2}$, ω] → y_c

$y_c(t)$

$|\omega| \le \omega_c$

$|\omega| > \omega_c$.

$x_c(t)$ → $H_c(j\omega)$ → $y_c(t)$

$|\Omega| < \pi$

Suppose we were looking at $H_d(e^{j\Omega})$ as discrete time freq. response. What will be overall freq. resp. $H_c(j\omega)$.

ω.

$H_c(j\omega)$

$x_c(t)$ → ⊗ ($p(t)$) → Impulse train to seq. → $x_d[n]$ → H_d ($H_d(e^{j\Omega})$) → $y_d[n]$ → seq to imp. → $y_p(t)$ → Low pass filt. → $y_c(t)$

$$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})$$

$$= H_d(e^{j\omega T})X_c(j\omega)$$

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2} \\ 0 & |\omega| > \frac{\omega_s}{2} \end{cases}$$

1. **Sampling of DT signals**

$$x_p[n] = \begin{cases} x[n], & \text{if } n = \text{integer multiple of } N. \\ 0, & \text{otherwise.} \end{cases}$$

N → Sampling period.

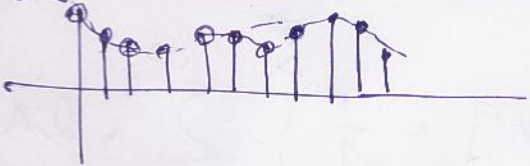
$$x_p[n] = x[n]\,p[n] = \sum_{k=-\infty}^{\infty} x[kN]\,\delta[n-kN].$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})\,X(e^{j(\omega-\theta)})\,d\theta$$

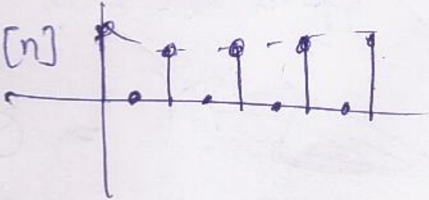
$$x[n] \longrightarrow \otimes \longrightarrow x_p[n]$$

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

$x[n]$



$x_p[n]$



$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{N}.$$

$$X_p(e^{j\omega}) = \frac{1}{N} \sum_{k=-\infty}^{\infty} X\left(e^{j(\omega - k\omega_0)}\right)$$

② Low pass filter

$$h[n] = N \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}$$

Scale

$$x_r[n] = x_p[n] * h[n].$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$