Laplace Transform.

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\, e^{st}$$

with $s \in \mathbb{C}$

$$\underbrace{H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt}_{\text{Laplace transform of } h(t)}$$

When s is restricted to be purely imaginary.

$$\underbrace{H(j\omega)}_{} = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt.$$

Fourier Transform [Provided integral exists and is finite].

Given a signal $x(t)$ we define

$$X(s) := \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$ as its

Laplace Transform.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s).$$

Setting $s = j\omega$ gives Fourier Transform.

$$X(s)\big|_{s=j\omega} = \mathcal{F}(x(t))$$

If we write $s = \sigma + j\omega$.

Laplace Transform can be considered as a Fourier Transform of $x(t)\, e^{-\sigma t}$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} e^{-\sigma t} e^{-j\omega t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

Example:

$$x(t) = e^{-at} u(t)$$

for $a > 0$ we can talk about Fourier Transform

$$X(j\omega) = \frac{1}{j\omega + a}.$$

For Laplace transform there is no such restriction

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-(s+a)t} u(t)\, dt$$

$$= \frac{-1}{s+a} \left[e^{-(s+a)t}\right]_0^{\infty}$$

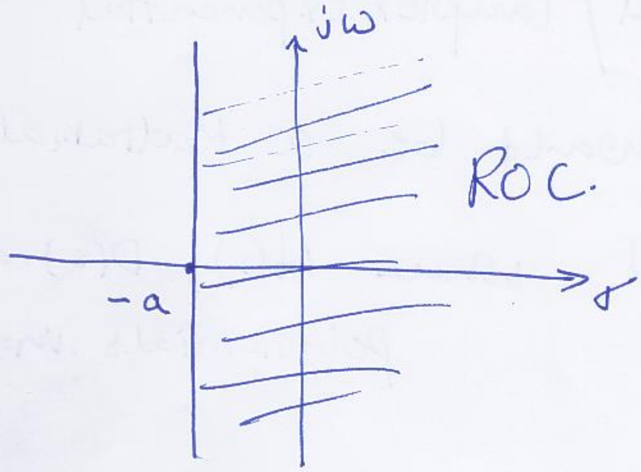
$$= \frac{-1}{(s+a)} \left[\underbrace{e^{-(s+a)\infty}}_{=\,0} - 1\right].$$

if $Re(s+a) > 0$.

$$X(s) = -\frac{1}{s+a}, \quad Re(s+a) > 0.$$

Re$(s+a) > 0$ defines a region in Complex plane in which Laplace transform integral ~~is well defined~~ converges to $\frac{1}{s+a}$

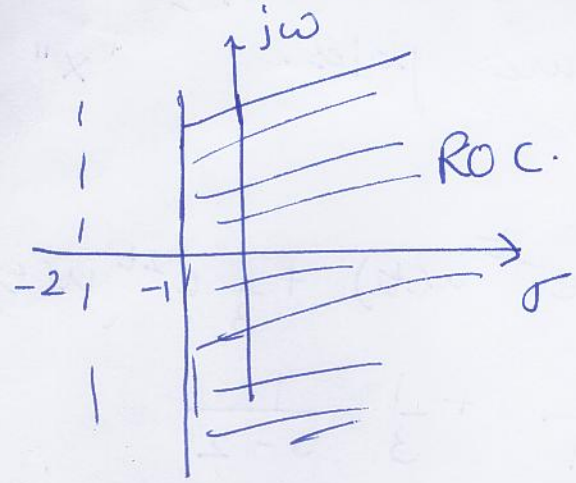
This is called Region of Convergence.



<u>Example.</u> $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{3s+3-2s+4}{(s+1)(s+2)}$$

$$= \frac{s-1}{(s+1)(s+2)}$$



$$x(t) = e^{-2t} u(t) + e^{-t} \cos 4t \, u(t)$$

$$X(s) = \frac{1}{(s+2)} + \frac{1}{(s+1)^2 + 4}$$

Typically for exponential / complex exponential. Laplace transform would be a Rational function. i.e. $\frac{N(s)}{D(s)}$ where $N(s)$, $D(s)$ are polynomials in s.

Then one can talk about the roots of $N(s)$, $D(s)$

The roots of $N(s)$ are zeros "o"

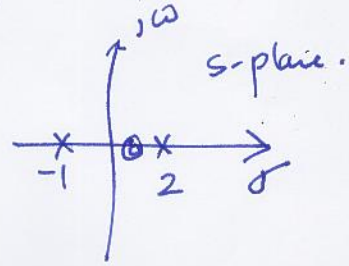
$D(s)$ are poles. "x"

Ex..

$$x(t) = \delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2}$$

$$= \frac{(s-1)^2}{(s+1)(s-2)} \quad [\text{for } Re(s) > 2]$$



Property 1

ROC of $X(s)$ consists of ~~regions which are~~ intersection of regions which are strips parallel to $j\omega$-axis.

ROC is those values of $s=\sigma+j\omega$ for which $x(t)e^{-\sigma t}$ has Fourier transform that converges to ~~finite~~ a function. $\Leftarrow \int_{-\infty}^{\infty} |x(t)| e^{-\sigma t}\, dt < \infty$.

Property 2

ROC does not contain any poles.

$X(s) \to \infty$ as $s \to$ pole.

Property 3

if $x(t)$ – finite duration along with

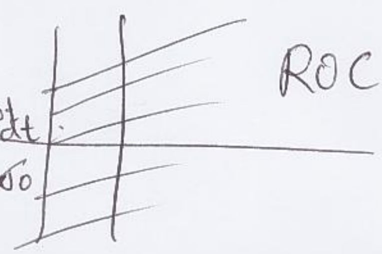
$\int_{-\infty}^{0} |x(t)|\, dt < \infty$

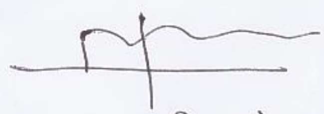
then ROC is entire s-plane.

Property 4

$x(t)$ right sided, ROC opens to right.

$\int_{T}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t}\, dt$





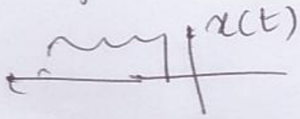
$e^{-2t}u(t)$

$\int_0^{\infty} \quad \frac{1}{s+2}$

$Re(s) > -2.0$

Property 5 Left sided

ROC opens to left



$\int_{-\infty}^{T} x(t) e^{-\sigma t} e^{-j\omega t}\, dt.$

$e^{2t}(u(-t))$

$\int_{-\infty}^{0} e^{2t} e^{-st}$

$2-\sigma > 0$, $\sigma < 2$.

Ex. 9.7

Property 6: Two sided. ROC is a finite strip in s-plane.

$x(t) = e^{-bt}$

$x(t) = e^{-bt} u(t) + e^{bt} u(-t)$

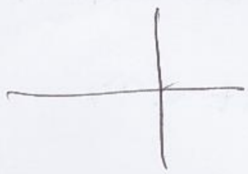
$= \frac{1}{s+b} + \frac{-1}{(s-b)}$

ROC
$\sigma > -b$ and. $\sigma < b$.

if $b \leq 0$ then. $b = -c$

$\sigma > c$ and. $\sigma < c$ always false.

no Laplace transf if $b \leq 0$.



Property 7: Ration

Inverse Laplace Transform.

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad ; \text{ ROC} = \text{Specified.}$$

given $X(s) \longleftrightarrow x(t) = ?$

Fourier Transform gives a Connection.

$$X(s) \longleftrightarrow \mathcal{F}\left(e^{-\sigma t} x(t)\right) = \int_{-\infty}^{\infty} e^{-\sigma t} x(t) e^{-j\omega t} dt$$

$$s = \sigma + j\omega.$$

Can use ~~the~~ Inverse Fourier Transform to define Inverse Laplace Transform

$$x(t) e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{+j\omega t} d\omega.$$

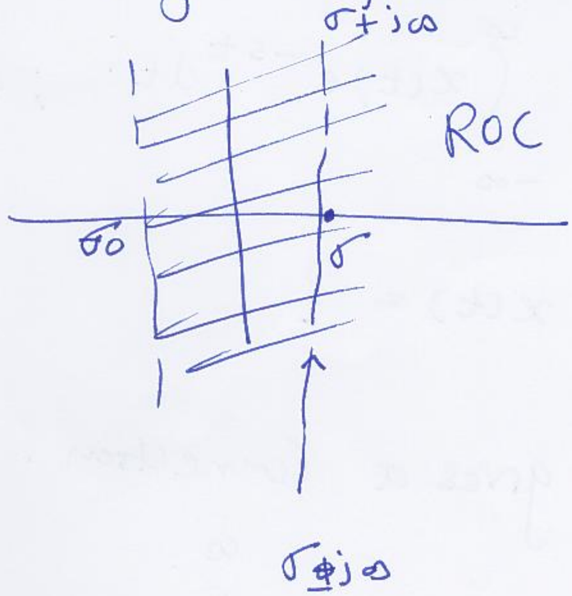
$$\Rightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega.$$

but $s = \sigma + j\omega$. σ is fixed to be in ~~ROC~~ s.t. line $\operatorname{Re}(s) = \sigma$ lies in ROC.

$ds = j\,d\omega.$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Contour Integral of $X(s)$ in s-plane.



It is more convenient to use standard tables while dealing with inverse Laplace transforms which are Rational functions.

One can ~~for~~ decompose a Rational function into its partial Fractions.
For example if there are no repeated poles.

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s+a_i}$$

$$x(t) = A_1 e^{-a_i t} + \cdots + A_m e^{-a_m t}$$

depending upon ROC.

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \lim_{s \to -1} (s+1) X(s) = +1$$

$$B = \lim_{s \to -2} (s+2) X(s) = -1$$

$$X(s) = \frac{1}{s+1} + \frac{-1}{s+2}$$

Now. if ROC $Re(s) > -1$

$$x(t) = e^{-t} u(t) - e^{-2t} u(t)$$

if ROC $Re(s) < -2$

$$x(t) = -e^{-t} u(-t) + e^{-2t} u(-t)$$

if ROC. $Re(s) > -2$ & $Re(s) < -1$

$$x(t) = -e^{-2t} u(t) - e^{-t} u(-t).$$

Properties of Laplace Transforms

(i) <u>Linearity</u> $x_1(t) \longleftrightarrow X_1(s)$ ROC_1

$x_2(t) \longleftrightarrow X_2(s)$ ROC_2

$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$

with $ROC = ROC_1 \cap ROC_2$.

(ii) <u>Time Shifting</u>

ex: $u(t) \Leftrightarrow \frac{1}{s}$

$u(t-1) \Leftrightarrow \frac{e^{-s}}{s}$

$x(t) \longleftrightarrow X(s)$ ROC.

$x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$ ROC unchanged.

(iii) $x(t) \longleftrightarrow X(s)$ $ROC = R$.

$e^{s_0 t} x(t) \longleftrightarrow X(s-s_0)$ $ROC = R + Re(s_0)$

(iv) Time Scaling $x(t) \longleftrightarrow X(s)$ $ROC = R_\phi$

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$ $ROC = aR_\phi$

~~$x(t) \longleftrightarrow X(s)$~~

$x(at) \longleftrightarrow X_n(s) = \int_{-\infty}^{\infty} e^{-st} x(at)\, dt$. $at = w$.

$= \int_{-\infty}^{\infty} e^{-\frac{s w}{a}} x(w) \frac{dw}{a} = \int_{-\infty}^{\infty} \frac{1}{a} e^{-\frac{st}{a}} x(w)\, dw$

now. if $a < 0$, put $w = -|a|t$

if $a > 0$ put $w = +|a|t$

$w \to t$, $\frac{w}{|a|}$

if $a<0$. $\int_{-\infty}^{\infty} \frac{1}{|a|} e^{-s\frac{t}{|a|}} x(-|a|t)\, dt = \frac{1}{a} X\left(\frac{s}{a}\right)$

$dw = |a|\, dt$.

Conjugation

$x(t) \longleftrightarrow X(s)$

$x^*(t) \longleftrightarrow X^*(s^*)$

Real $x(t)$

$X(s) = X^*(s^*)$

thus poles appear in complex conjugate pairs

Convolution.

$x_1(t) \longleftrightarrow X_1(s)$ ROC1

$x_2(t) \longleftrightarrow X_2(s)$ ROC2

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt\right] d\tau$$

$t-\tau = w. \quad dt = dw.$

$t = w+\tau$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(w)\, e^{-sw}\, e^{-s\tau}\, dw\right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[X_2(s)\right] e^{-s\tau}\, d\tau$$

$$= X_1(s)\, X_2(s) \qquad \underline{ROC1 \cap ROC2}$$

or Larger.

$X_1(s) = \frac{s+1}{s+3} \qquad X_2(s) = \frac{s+3}{s+1}$

$X_1(s) \cdot X_2(s) = 1$ ROC is entire s-plane.

- Differentiation in time domain.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$\frac{dx}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}\,ds.$$

$$x(t) \longleftrightarrow X(s) \qquad ROC = R$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) \qquad ROC \text{ contains } R.$$

Alternate approach more useful in considering ~~signals with jumps~~ / unilateral L.T.

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt = ?$$

$$\frac{d}{dt}\left(x(t)e^{-st}\right) = \frac{dx}{dt}e^{-st} + -s\,e^{-st}\,x(t)$$

$$\int_{-\infty}^{\infty} \frac{dx}{dt} e^{-st}\,dt = \int_{-\infty}^{\infty} \frac{d}{dt}\left[x(t)e^{-st}\right]dt + \int_{-\infty}^{\infty} s\,e^{-st}x(t)\,dt$$

$$= \left[x(t)e^{-st}\right]_{-\infty}^{\infty} + sX(s)$$

$x(-\infty) = 0$

$e^{-s\infty} = 0.$

$$= 0 + sX(s)$$

and if ~~initial condition for~~ $x(0^-) = 0$

then $$\int_{-\infty}^{\infty} \frac{dx}{dt} e^{-st}\,dt = sX(s)$$

ROC may expand or remain same

But will contain R as a subset.

- Integration.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \qquad ROC = R$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{X(s)}{s} \qquad ROC \text{ containing } R \cap \{Re\{s\} > 0\}$$

$$\parallel$$

$$x(t) * u(t) \longleftrightarrow \frac{X(s)}{s}$$

- Differentiation in s-domain

notice

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad ROC = R.$$

$$\Rightarrow \frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t)\,(-t)\, e^{-st}\, dt$$

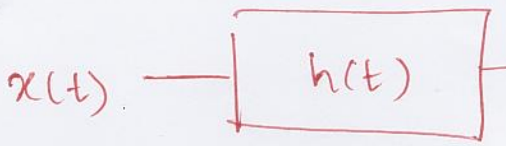
$$= \int_{-\infty}^{\infty} [-t\, x(t)]\, e^{-st}\, dt$$

$$\Rightarrow -t\, x(t) \longleftrightarrow \frac{d X(s)}{ds} \qquad ROC = R.$$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$t\, u(t) \leftrightarrow -\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$$

$$\frac{t^2}{2} u(t) \leftrightarrow \frac{1}{s^3} \quad \cdots \quad \frac{t^{n-1}}{(n-1)!} u(t) \longleftrightarrow \frac{1}{s^n}.$$



$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau$$

$$y(t) = h(t) * x(t)$$

$$Y(s) = \underbrace{H(s)}\,X(s)$$

H(s) is Laplace transform of the impulse response h(t) of LTI system

(i) ~~H(s)~~ Causal System if impulse response $h(t) = 0$ for $t < 0$.

$$H(s) = \int_{-\infty}^{\infty} h(t)\,e^{-st}\,dt = \int_{0}^{\infty} h(t)\,e^{-st}\,dt$$

$$= \int_{0}^{\infty} \left[h(t)e^{-\sigma t}\right] . \, e^{-j\omega t}\,dt$$

$\Rightarrow \sigma > b$ [

- ROC associated to H(s) must necessarily open towards Right half of the Complex plane.
- But it does not guarantee Causality.
- For Rational H(s) $\sigma >$ real part of Right most pole $\Rightarrow$ Causality.

example:

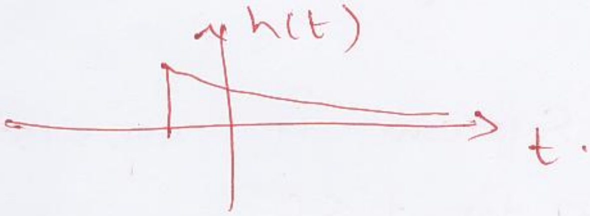
(i) $h(t) = e^{-t} u(t)$

$H(s) = \frac{1}{s+1}$ ROC $Re(s) > -1$

(ii) $H(s) = \frac{e^{s}}{s+1}$ ROC $Re(s) > -1$

but:

$h(t) = e^{-(t+1)} u(t+1)$



Not Causal.

Stability: h(t) absolutely integrable. [even with $\sigma = 0$ we must get Fourier transform]

~~For Rational H(s) we have following Rule~~

LTI-H(s) is stable iff ROC Contains entire $j\omega$-axis
system

$(\{s \mid Re(s) = 0\} \cap ROC)$.

example:

$H(s) = \frac{s-1}{(s+1)(s-2)}$ Causal? Stable?

~~LTI Causal~~

Rational H(s) Stable ≡ poles in OLHCP

LTI systems with LCCODE.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

with initial rest conditions.

Repeatedly applying differentiation property

$$y(t) \leftrightarrow Y(s)$$

$$\frac{dy(t)}{dt} \leftrightarrow sY(s)$$

$$\vdots$$

$$\frac{d^k y(t)}{dt^k} \leftrightarrow s^k Y(s)$$

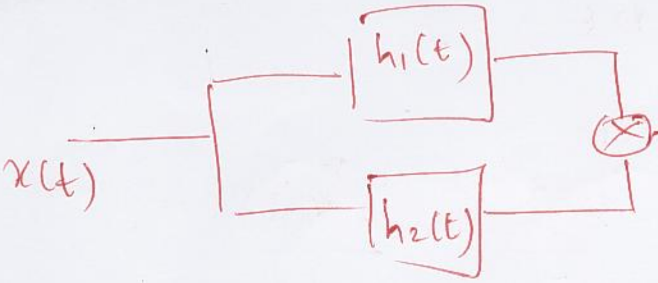
$$\left[\sum_{k=0}^{N} a_k s^k\right] Y(s) = \left[\sum_{k=0}^{M} b_k s^k\right] X(s)$$

$$Y(s) = H(s)\, X(s)$$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{N(s)}{D(s)}$$

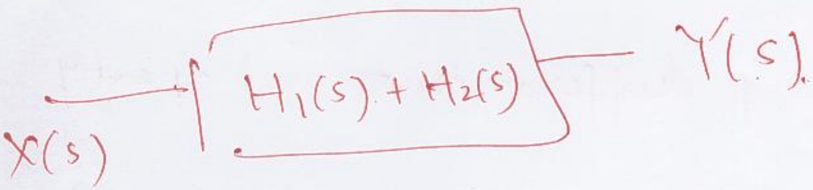
$\{s : N(s) = 0\}$:= Zeros of H

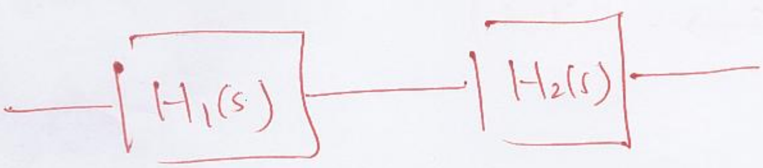
$\{s : D(s) = 0\}$:= Poles of H



$$y(t) = (h_1 + h_2) * x$$
$$= h_1 * x + h_2 * x$$
$$Y(s) = [H_1(s) + H_2(s)]X(s)$$

|||





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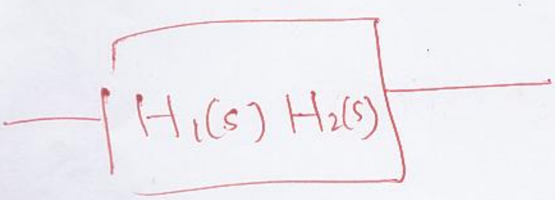

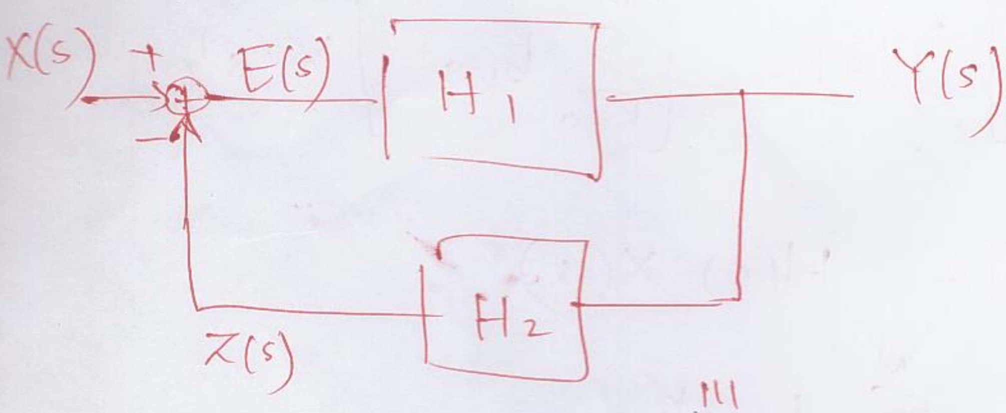

Unilateral Laplace transform defined for signals $x(t)$ s.t $x(t) = 0 \quad t < 0$.

$$\int_{0^-}^{\infty} e^{-st} x(t)\,dt = X(s)$$

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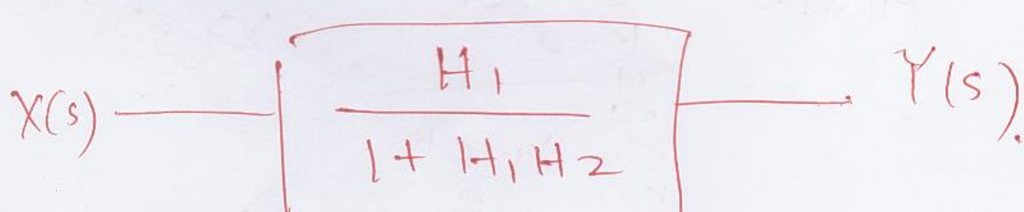

$$x(t) \longrightarrow \boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{dx}{dt} + x} \longrightarrow y(t)$$

$$H(s) = \frac{1}{s^2 + 3s + 2}$$

$$x(t) = a\,u(t)$$

$$X(s) = \frac{a}{s}$$

$$Y(s) = H(s)\,X(s) = \frac{1}{s^2 + 3s + 2}\,\frac{a}{s}$$

$$= \frac{a(1)}{s(s+1)(s+2)} = \frac{a}{s(s+2)(s+1)}$$

$$= \frac{1}{2}\left(\frac{a}{s} - \frac{a}{s+2}\right) = \frac{a}{2}\left[a\,u(t) - e^{-2t}u(t)\right]$$

$$= a\left[\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}\right]$$

$$= a\left[\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}\right]$$

Solve diff. eq. $y(0^-) = 1$, $y'(0^-) = 2$.

for $x(t) = a\,u(t)$

$$y(t) = a\left[\frac{u(t)}{2} + \frac{e^{-2t}}{2}u(t) - e^{-t}u(t)\right].$$

Let. $\frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = x(t)$

$y(0^-) = \beta \quad , \quad y'(0^-) = \gamma .$

$x(t) = a\,u(t)$

$$\frac{dy}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0^-)$$

$$\frac{d^2y}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0^-) - y'(0^-)$$

$$s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-)$$

$$+ 2Y(s) = X(s)$$

$$(s^2 + 3s + 2)\,Y(s) - s\beta - \gamma - 3\beta = X(s)$$

$$Y(s) = \underbrace{\frac{X(s)}{(s^2+3s+2)}}_{\text{Zero state response.}} + \underbrace{\frac{\beta(s+3)}{(s^2+3s+2)} + \frac{\gamma}{(s^2+3s+2)}}_{\text{Zero input Response.}}$$