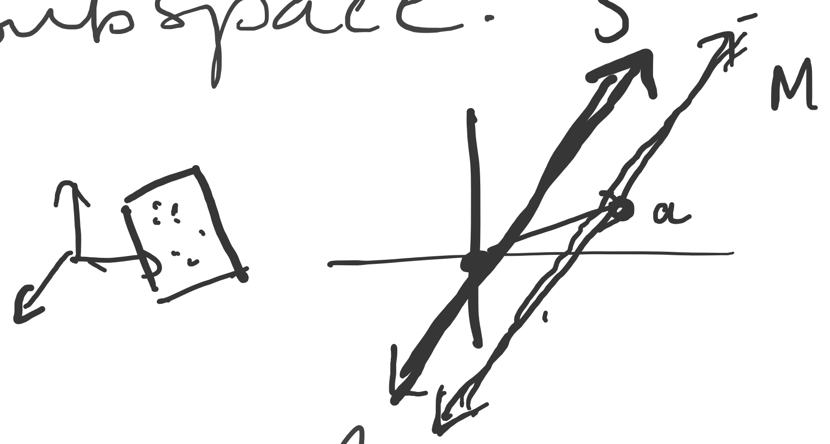
Affinely Spanning Set:

$\underline{M} = a + \mathcal{S}$ be an "affine" subspace.
$= \{ x + a \mid x \in S, a \notin S \}$

$\mathcal{S}$ is a "Linear" Subspace.



$\underline{Y} \subset \underline{M}$ is affinely Spanning M

if $M = \underline{\text{aff}(Y)}$ $\underline{\text{aff}\, Y} = \left\{ \sum_{i=1}^{m} \theta_i x_i \;\middle|\; \begin{array}{l} x_i \in Y \\ \theta_i \in \mathbb{R} \\ \boxed{\sum_{i=1}^{m} \theta_i = 1} \end{array} \right\}$

Affinely Independent Set.

$\{\underbrace{x_0, x_1, \ldots, x_k}_{k+1}\}$ are affinely Independent if

$x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0$ are

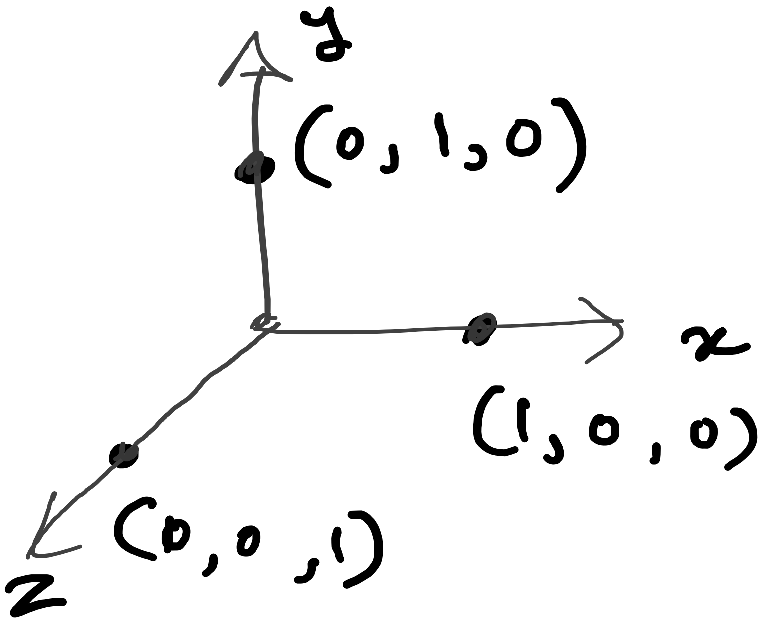
k —

Linearly Independent.

Simplex

$x_0, x_1, \dots, x_m \in \mathbb{R}^n$ be affinely Independe

$\mathrm{Conv}\{x_0, x_1, \dots, x_m\}$ is a simplex.

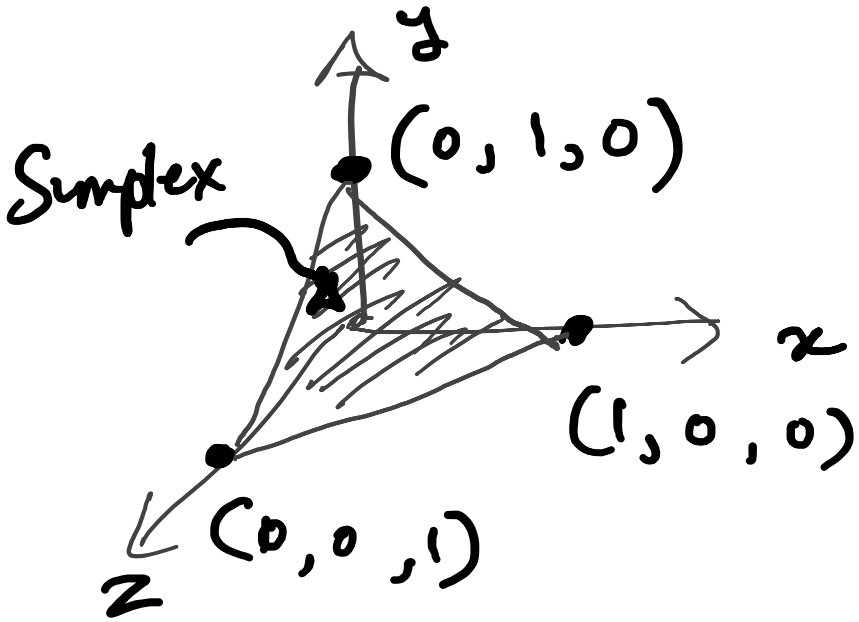


$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are affinely Independent

Simplex

$x_0, x_1, \cdots, x_m \in \mathbb{R}^n$ be affinely Independe

$\mathrm{Conv}\{x_0, x_1, \cdots, x_m\}$ is a Simplex.

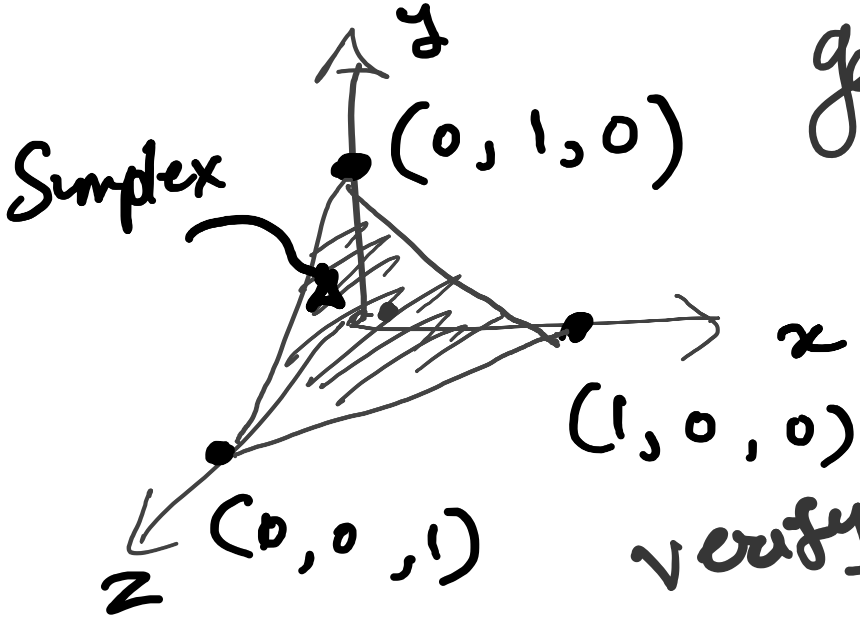


$$\mathrm{Conv}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

Simplex

$x_0, x_1, \dots, x_m \in \mathbb{R}^n$. be affinely Independe

$\mathrm{Conv}\{x_0, x_1, \dots, x_m\}$ is a simplex.



generate pts → Parametri

$$\mathrm{Conv}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

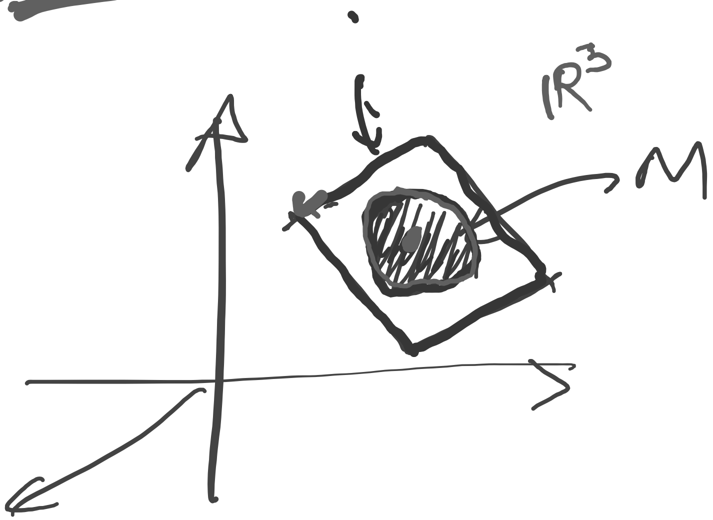
→ Impl.

verify →

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{array}{l} x + y + z = 1 \\ x \geq 0,\ y \geq 0,\ z \geq 0 \end{array} \right\}.$$

M is a set in $\mathbb{R}^n$.

$$\text{int}\, M = \{ x \in M \mid \exists\, \epsilon > 0,\ B_\epsilon(x) \subset M \}.$$

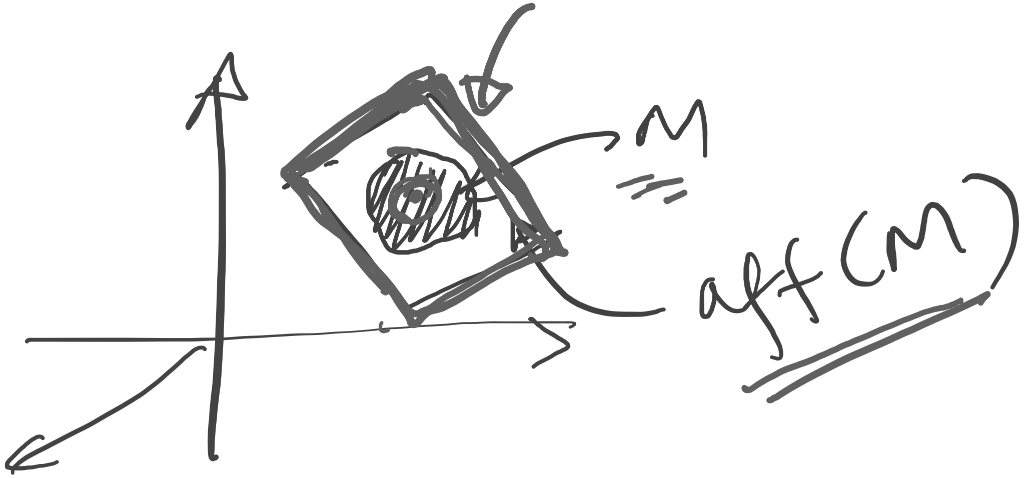


$$\text{int}(M) = ?$$

$$\text{int}\, M = \emptyset$$

M is a set in $\mathbb{R}^n$.

$$\text{int}\, M = \{ x \in M \mid \exists\, \epsilon > 0,\ B_\epsilon(x) \subset M \}.$$



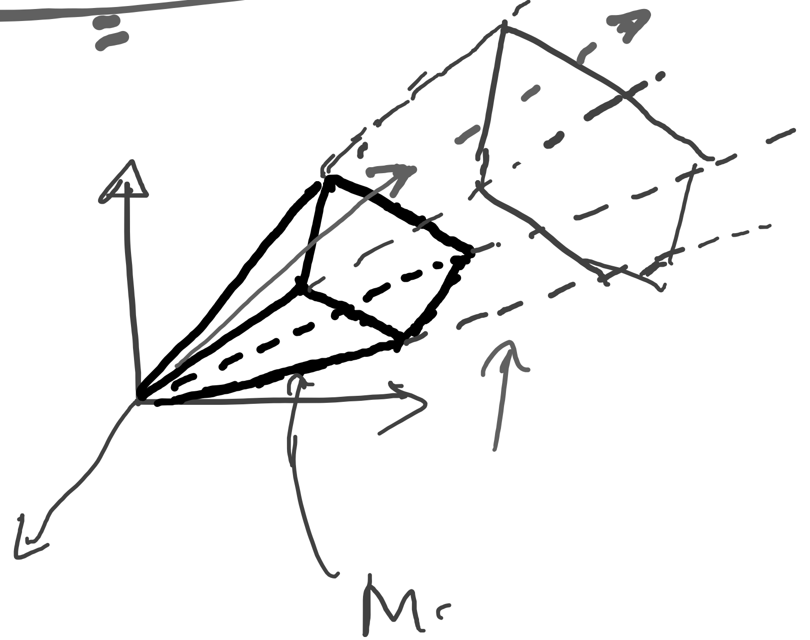
Relative Interior of M:

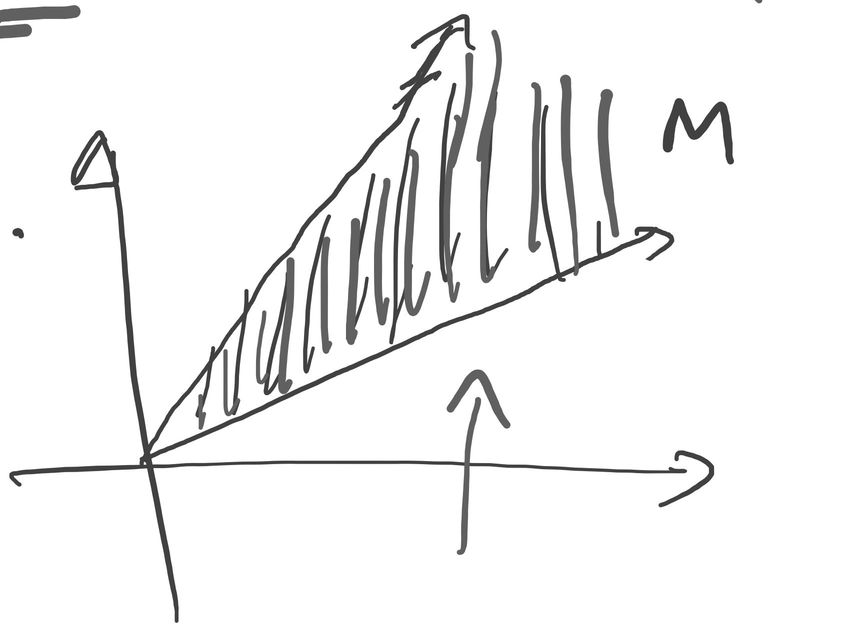
$$\boxed{\text{ri}\, M = \{ x \in M \mid \exists\, \epsilon > 0,\ B_\epsilon(x) \cap \text{aff}(M) \subset M \}}$$

Cones

$M \subseteq \mathbb{R}^n$ is conic if

$x \in M \Rightarrow \alpha x \in M \;\; \forall \; \alpha \geq 0.$

e.g. ① 

② . 

Cones

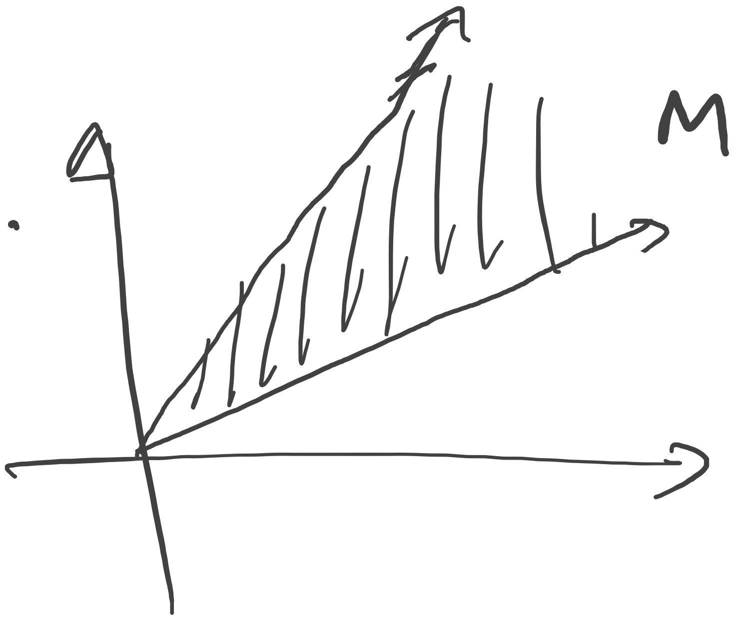
$M \subseteq \mathbb{R}^n$ is conic if

$$x \in M \implies \alpha x \in M \quad \forall\, \alpha \geq 0.$$

e.g. ③

Conic but not convex.

②.



Cone

Convex Conic set is Cone.

Thm: (b) $M \subseteq \mathbb{R}^n$ is a cone if and only if

if

(a) (i) $x \in M \Rightarrow \alpha x \in M \ \forall \alpha \geq 0$

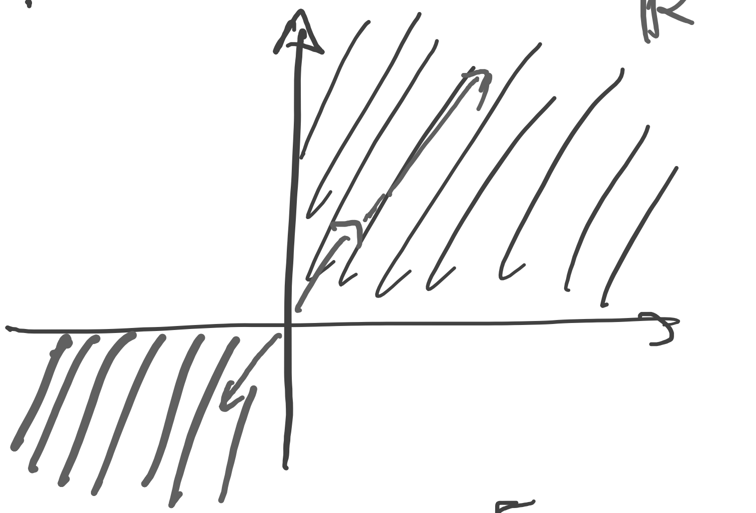
(ii) $x, y \in M \Rightarrow x + y \in M$

Proof: Excercise.

b $\Rightarrow$ a

a $\Rightarrow$ b

e.g.



$x \le 0$
$y \le 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le 0$$

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le 0 \right\}$$

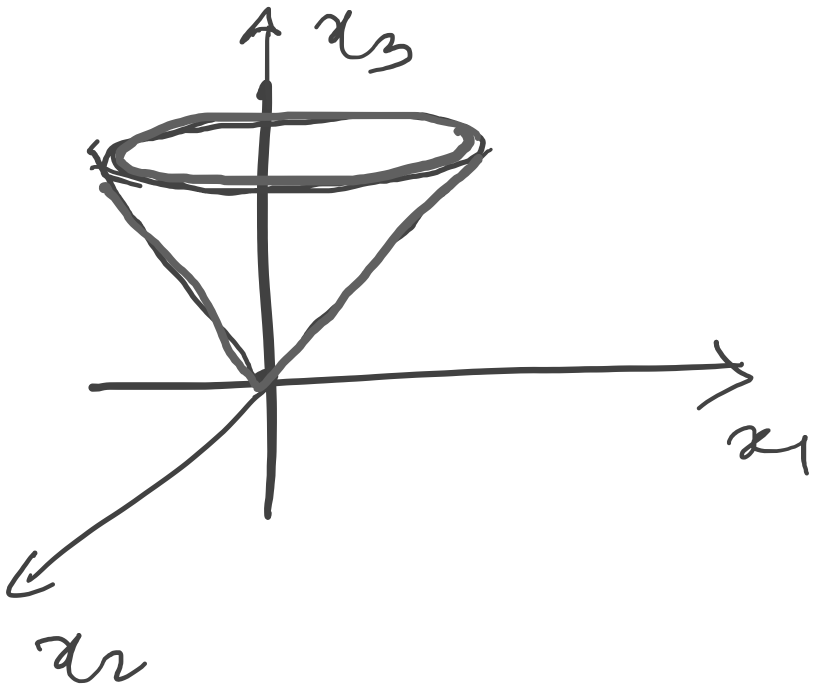
① Polyhedral Set.

Intersection of half spaces.

In general,

$$\{ \underset{\mathbb{R}^n}{x} \mid Ax \le 0 \}. \rightarrow \text{Polyhedral Cone.}$$

e.g. $\left\{ x \in \mathbb{R}^3 \;\middle|\; \cancel{\|x\|_2 \le x} \;\; x_1^2 + x_2^2 \le x_3, \;\; x_3 \geq 0 \right\}$



norm cone.

Thm $S^n_{\geq 0} = \{ \underline{A} \in \mathbb{R}^{n \times n} \mid A = A^T, \; x^T A x \geq 0 \;\; \forall x \in \mathbb{R}^n \}$,

Set of Symmetric p.s.d. matrices. is a cone.

Proof: $A \in S^n_{\geq 0}$

$\alpha A \in S^n_{\geq 0}$

$\forall A, B \in S^n_{\geq 0}$,

then $\underline{A} + \underline{B} \in S_{\geq 0}$.

Cone.

1. Polyhedral set →
2. affine subspace.
3. norm balls.
4. Cones
5. Simplex

Conic Combinations

$x_1, x_2, \dots, x_m$

$$\underline{x} = \sum_{i=1}^{m} \alpha_i x_i , \quad \alpha_i \geq 0 ,$$

is a conic combination of pts $x_1, \dots, x_m$

Conic Hull

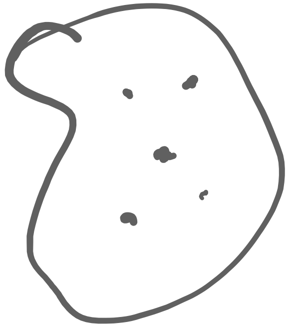
$$\underline{\underline{\text{Cone}}}(\underline{\underline{M}}) := \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; \alpha_i \geq 0 \quad x_i \in M \right\}.$$

Caratheodory Thm affine dimension.

$M \subset \mathbb{R}^n$. Let $\boxed{\dim \operatorname{conv} M = m}$

$\forall\, x \in \operatorname{conv} M$, $\exists\ \boxed{p \leq m+1}$, $x_1, x_2, \ldots, x_p \in M$

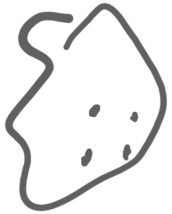
s.t. $x = \sum_{i=1}^{p} \theta_i x_i$, $\theta_i \geq 0$, $\sum_{i=1}^{p} \theta_i = 1$

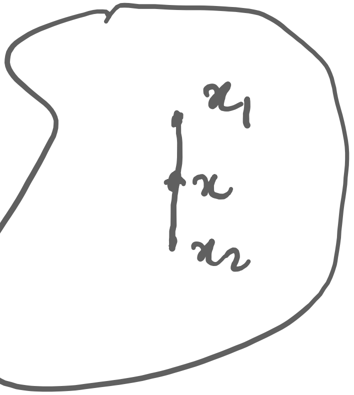


$\mathbb{R}^2$

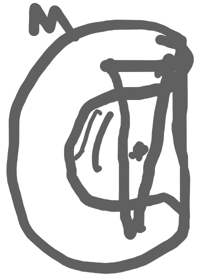
$\sum \theta_i = 1$

$x = \sum_{i=1}^{3} \theta_i x_i$



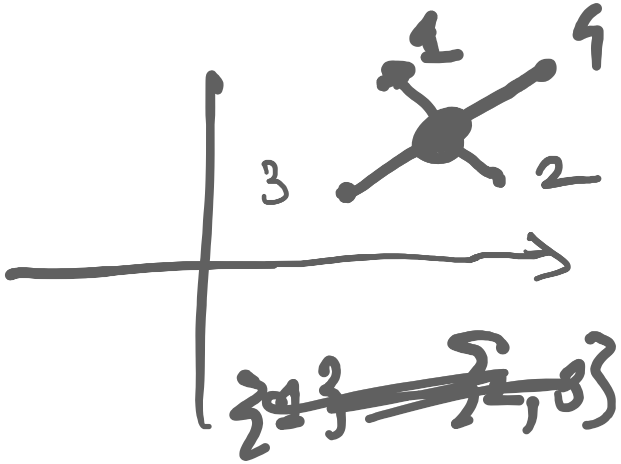


$x = \theta x_1 + (1-\theta) x_2$



Thm [Radon].

$\mathbb{R}^2$ =



$S = \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^n$

$N \geq n + 2$.

$\exists$ partitioning $I, J \subset \{1, 2, \dots, N\}$

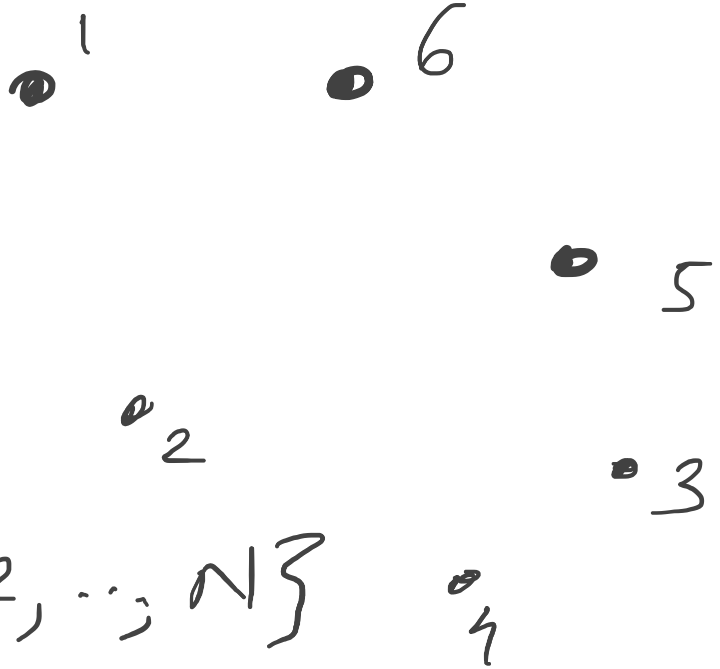
$\{1, 2\}$
$\{3, 4\}$

with $I \cup J = \{1, 2, \dots, N\}$, $I \cap J = \phi$

$\theta_i \geq 0, \theta_j \geq 0$

s.t. $\sum_{i \in I} x_i \theta_i = \sum_{j \in J} x_j \theta_j$, $\sum_{i \in I} \theta_i = \sum_{j \in J} \theta_j =$

<u>Thm</u> [Radon].



$S = \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^n$

$N \le n+2$.

$\exists$ partitioning $I, J \subset \{1, 2, \dots, N\}$

with $I \cup J = \{1, 2, \dots, N\}$, $I \cap J = \emptyset$

$\theta_i \ge 0, \theta_j \ge 0$

s.t. $\sum_{i \in I} x_i \theta_i = \sum_{j \in J} x_j \theta_j$, $\sum_{i \in I} \theta_i = \sum_{j \in J} \theta_j =$

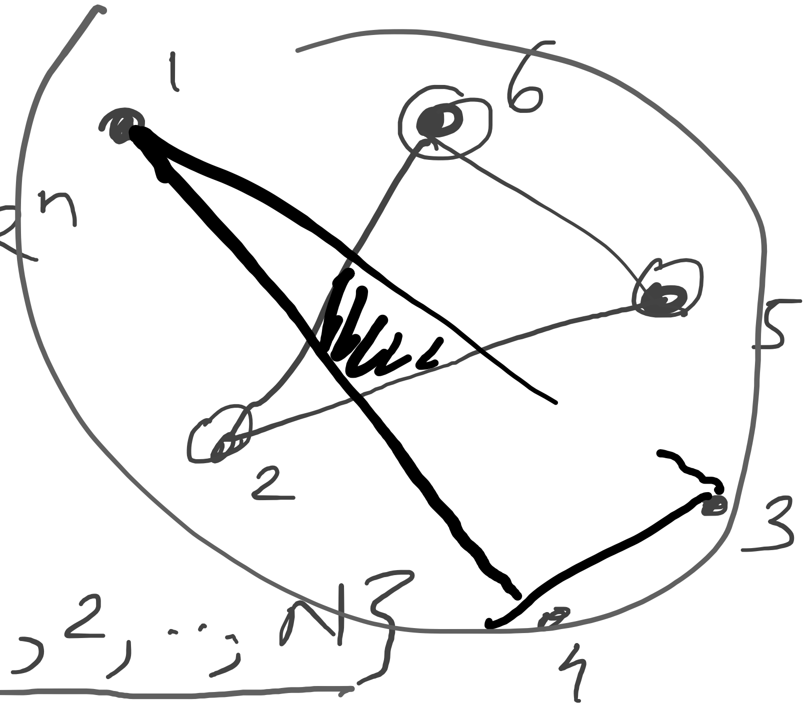
Thm [Radon].

$S = \{ x_1, x_2, \dots, x_N \} \subset \mathbb{R}^n$

$N \le n+2$.

$I = \{1, 4, 3\}$

$J = \{2, 5, 6\}$



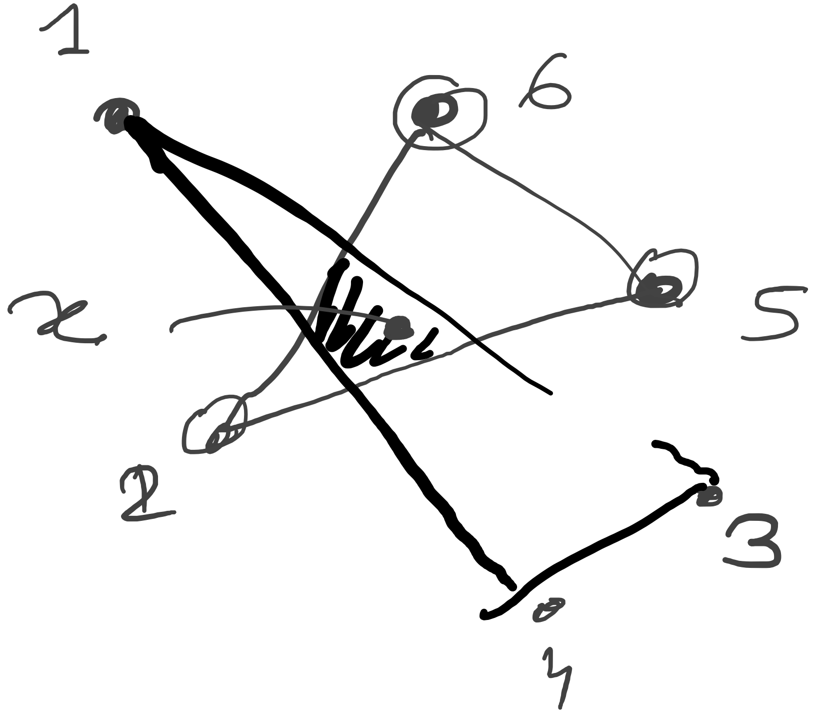
$\exists$ partitioning $I, J \subset \{1, 2, \dots, N\}$

with $I \cup J = \{1, 2, \dots, N\}$, $I \cap J = \phi$

$\theta_i \ge 0, \theta_j \ge 0$

s.t. $\sum_{i \in I} x_i \theta_i = \sum_{j \in J} x_j \theta_j$, $\sum_{i \in I} \theta_i = \sum_{j \in J} \theta_j =$

Thm [Radon].



$I = \{1, 4, 3\}$

$J = \{2, 5, 6\}.$

$I \cap J = \phi \quad , \quad I \cup J = \{1, 2, \cdots, 6\}.$

$$x = \sum_{i \in I} \theta_i x_i = \sum_{j \in J} \theta_j x_j$$

$\theta_i \geq 0, \ \theta_j \geq 0$

$\sum_{i \in I} \theta_i = \sum_{j \in J} \theta_j = 1.$

Helly's Thm.

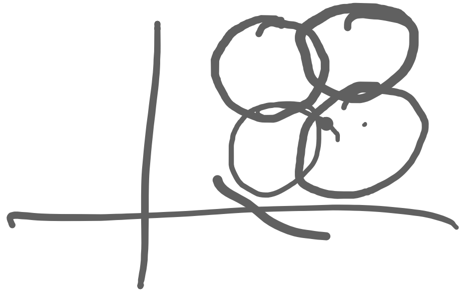
$\mathcal{F} = \{S_1, S_2, S_3, S_4\}$, ~~$S_{n+1}$~~ $\}$

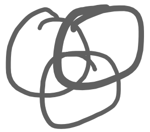
$\mathcal{F}$ be a family of finitely many convex sets in $\mathbb{R}^n$.

$S_1 \cap S_2 \cap S_3 \neq \phi$ $\quad$ $S_1 \cap S_2 \cap S_4 \neq \phi$

$S_2 \cap S_3 \cap S_4 \neq \phi$ $\quad$ $S_1 \cap S_3 \cap S_4 \neq \phi$

If every $n+1$ sets in $\mathcal{F}$ have non-empty intersection





then all sets in $\mathcal{F}$ have non-empty intersection.

$S_1 \cap S_2 \cap S_3 \cap S_4 \neq \phi$