Defn: Convex function

$f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x)$, let $X = \operatorname{dom} f \subseteq \mathbb{R}^n$ be non-empty.

f is Convex if

(i) X is Convex set.

(ii) $\forall\, x_1, x_2 \in X$ and $\theta \in [0, 1]$

$$\boxed{f(\theta x_1 + (1-\theta) x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)}$$

Defn: **Convex function**

$f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$ non-empty ,

f is strictly convex if

(i) X is convex set.

(ii) $\forall\ x_1, x_2 \in X$ and $\theta \in [0, 1]$

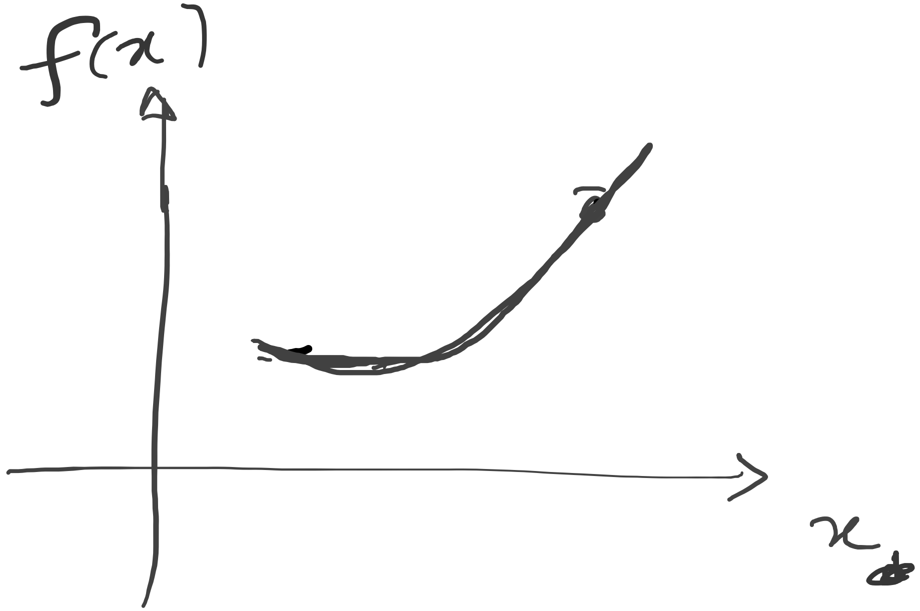
$$f(\theta x_1 + (1-\theta) x_2) < \theta f(x_1) + (1-\theta) f(x_2)$$

e.g.

$$f(\theta x_1 + (1-\theta) x_2) \le \theta f(x_1) + (1-\theta) f(x_2)$$

$$\Updownarrow$$

Convex f.



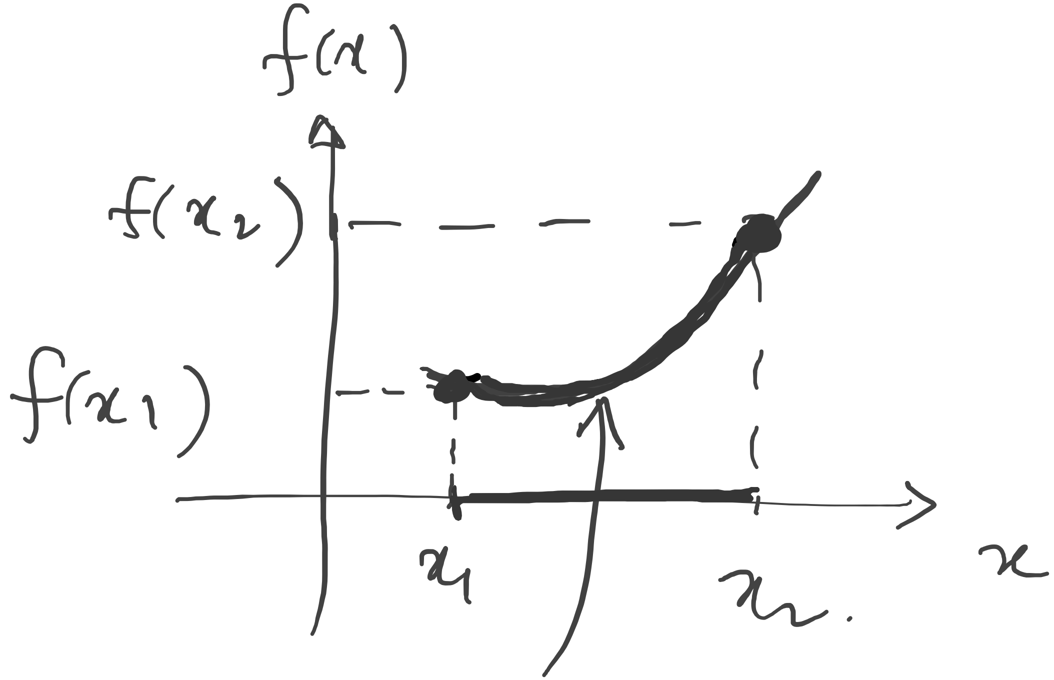
e.g.

$$f(\theta x_1 + (1-\theta) x_2)$$

$$\leq \theta f(x_1) + (1-\theta) f(x_2)$$

$$\Updownarrow$$

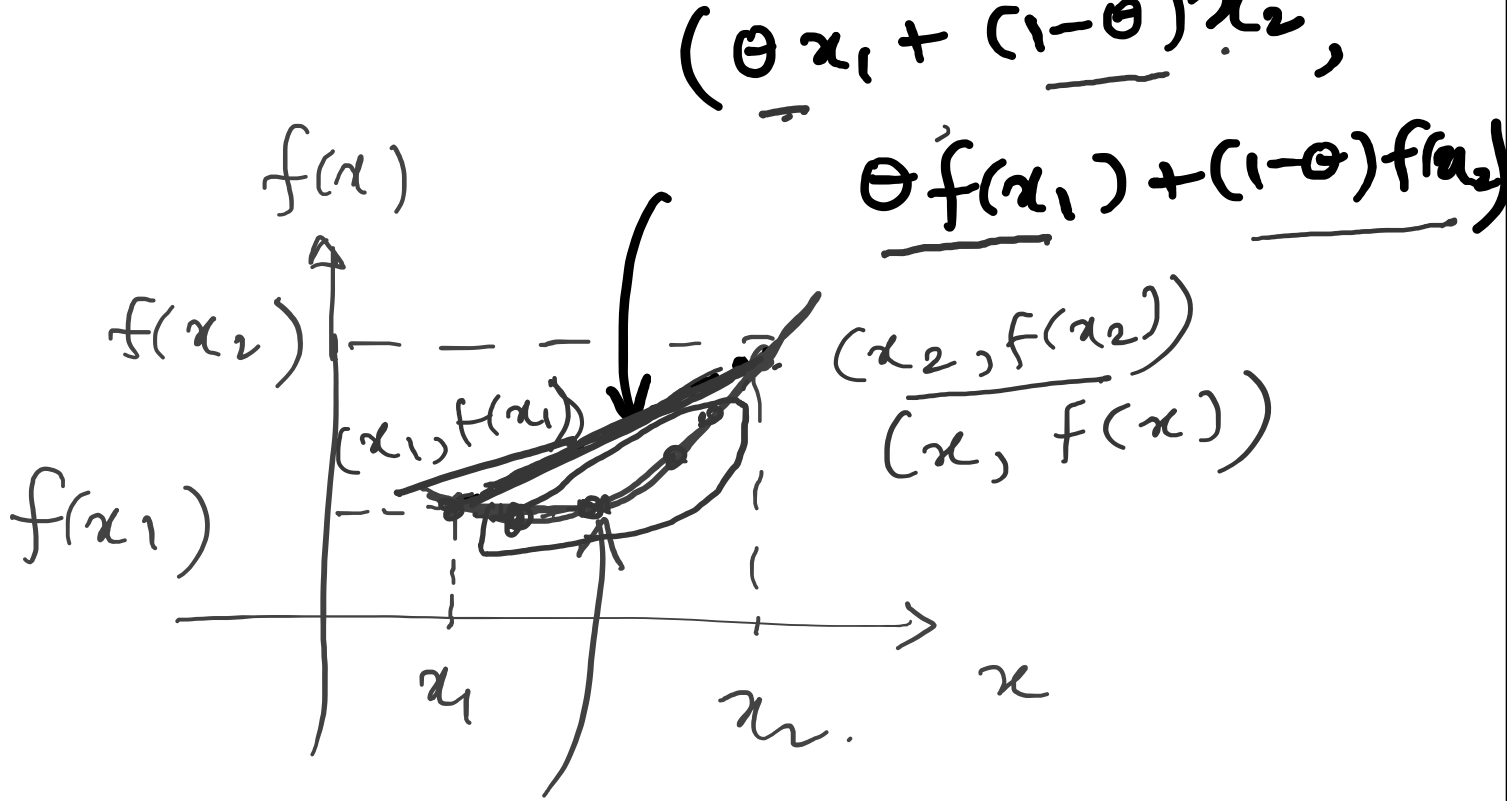
Convex f.



$f(\theta x_1 + (1-\theta) x_2)$

e.g.

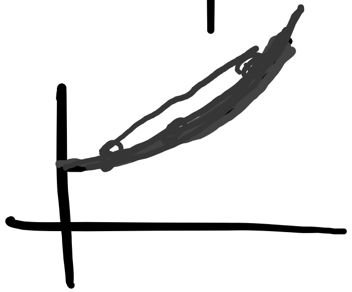
$$f(\theta x_1 + (1-\theta) x_2)$$
$$\leq \theta f(x_1) + (1-\theta) f(x_2)$$
$$\Updownarrow$$
Convex f.



$f(\theta x_1 + (1-\theta) x_2)$

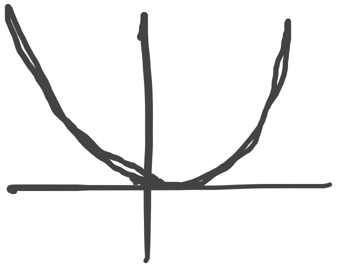
e.g.

① $f(x) = \underline{a^T x + b}$.



② $f(x) = e^x$

③ $f(x) = x^2$

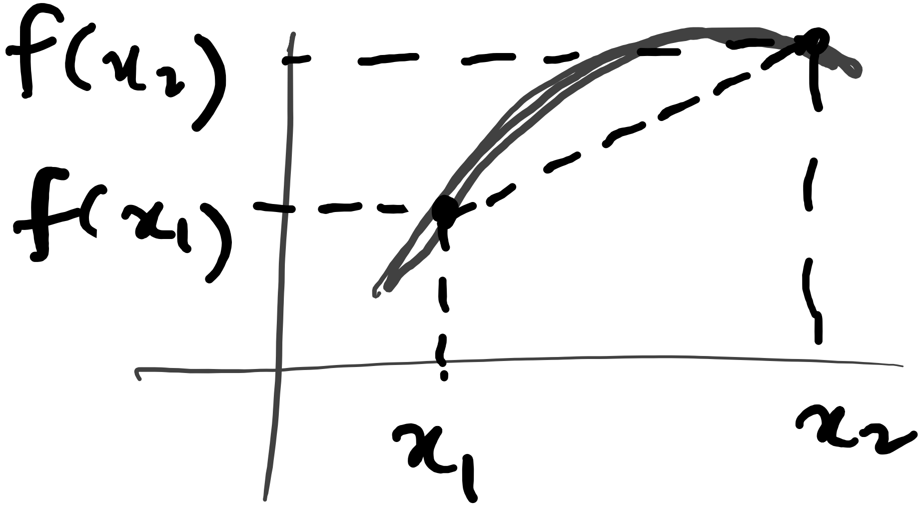


④ $\underline{f(x)} = \underline{x^T P x}$ with $P \in \underline{S^n_+}$

Quadratic forms.

If a function f is convex on X

then $-f$ is said to be "Concave".



$\underline{\underline{g}}$ $\quad$ $-g$

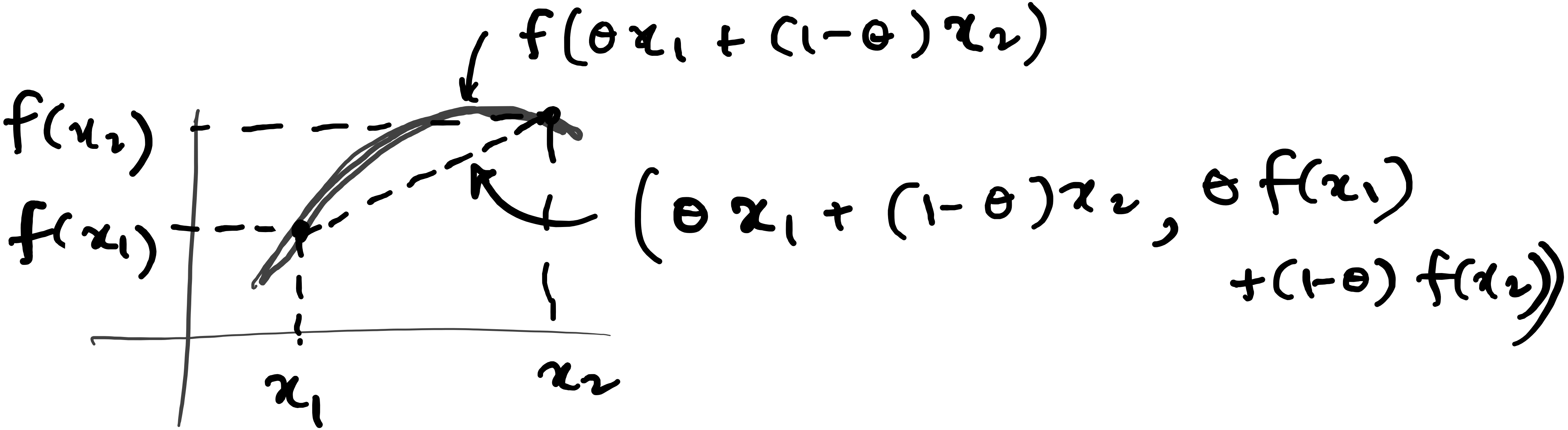
f $\quad$ $-f$

Concave iff:

$$f(\theta x_1 + (1-\theta) x_2) \geq \theta f(x_1) + (1-\theta) f(x_2)$$

$$\theta \in [0,1]$$

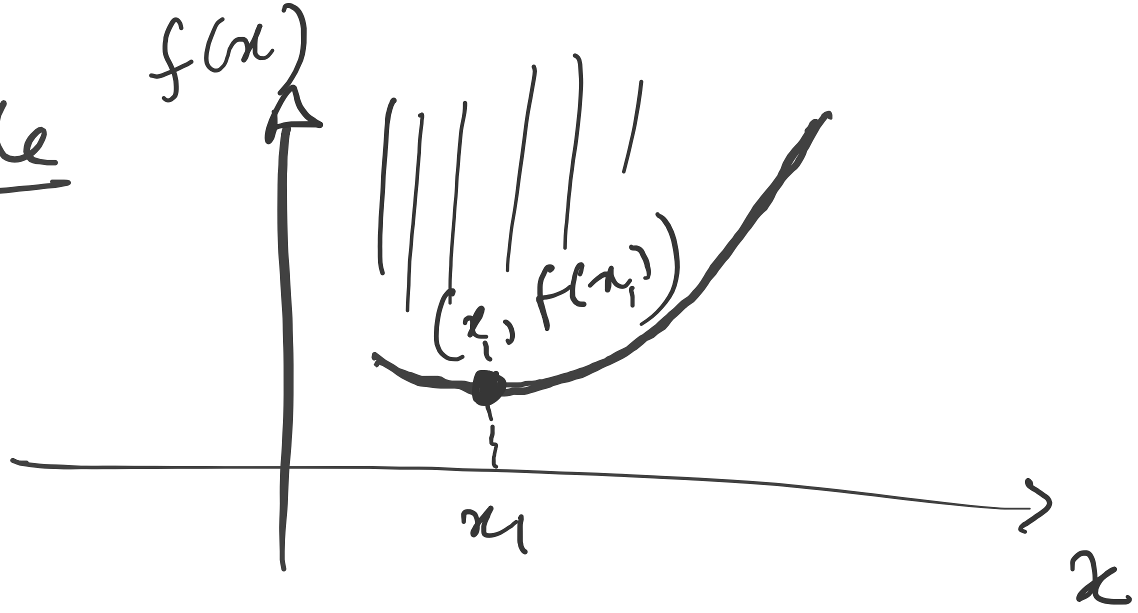
If a function f is convex on X then $-f$ is said to be "Concave".



"Epigraph" of function f.

$$\text{Epi}\, f := \{ (\alpha, x) \in \mathbb{R}^{n+1} \mid x \in \text{dom}\, f,\ f(x) \leq \alpha \}.$$

example

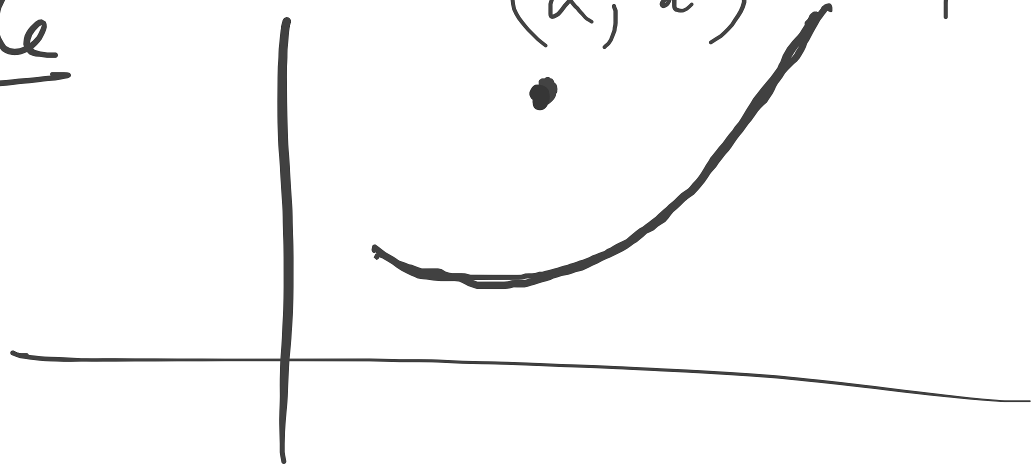


Epigraph of function f.

$$\mathrm{Epi}\, f := \{ (\alpha, x) \in \mathbb{R}^{n+1} \mid x \in \mathrm{dom}\, f,\ f(x) \le \alpha \}.$$

example

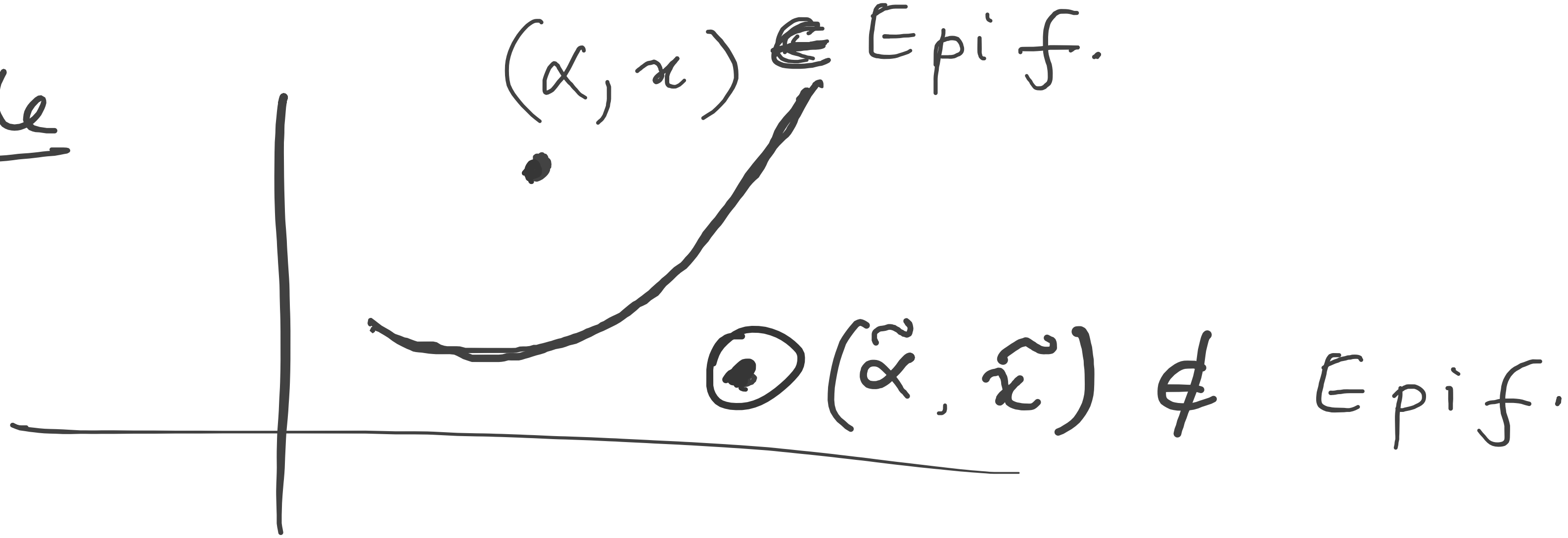
$(\alpha, x) \in \mathrm{Epi}\, f.$



Epigraph of function f.

$$\mathrm{Epi}\, f := \left\{ (\alpha, x) \in \mathbb{R}^{n+1} \;\middle|\; x \in \mathrm{dom}\, f,\ f(x) \leq \alpha \right\}.$$

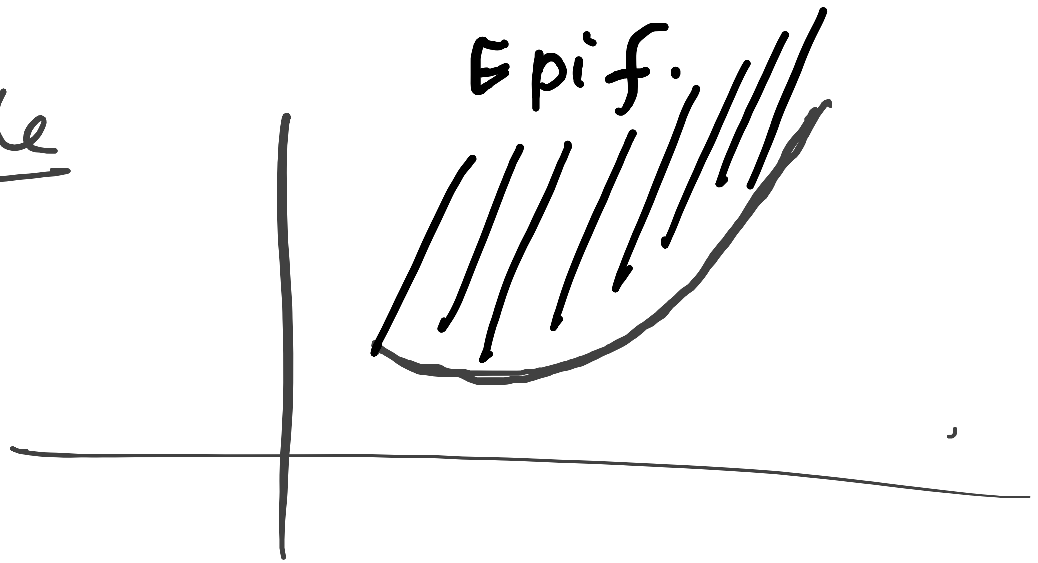
<u>example</u>



Epigraph of function f.

$$\mathrm{Epi}\, f := \{ (\alpha, x) \in \mathbb{R}^{n+1} \mid x \in \mathrm{dom}\, f,\ f(x) \le \alpha \}.$$

<u>example</u>



Thm: $f : \mathbb{R}^n \to \mathbb{R}$ is convex
$x \mapsto f(x)$

if and only if

Epi f is convex set.

Proof: $\text{Epi} f = \{(\alpha, x) \mid f(x) \le \alpha, x \in \text{dom} f\}$

Show: ① Epi f convex $\Rightarrow$ $\underline{f(\theta x_1 + (1-\theta) x_2)}$
$\underline{\le \theta f(x_1) + (1-\theta) f(x_2)}$

② f convex $\Rightarrow$ Epi f convex. [Exercise!]

Prop: Let $f : \mathbb{R}^n \to \mathbb{R}$ Satisfy
$x \mapsto f(x)$

[homogenous of degree 1] $f(tx) = t f(x)$ $\forall$ $x \in \mathbb{R}^n$ and $t \geq 0$.

$\boxed{\text{then}}$

f is convex if & only if

$$f(x_1 + x_2) \leq f(x_1) + f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

Proof: Exercise. Hint: Prove Epi f is cone.

Prop: Let $f : \mathbb{R}^n \to \mathbb{R}$ Satisfy
$x \mapsto f(x)$

$$\boxed{f(tx) = t\, f(x)} \quad \forall\ x \in \mathbb{R}^n \text{ and } t \geq 0.$$

Then

f is convex if & only if

[sub-additivity] $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ $\forall\ x_1, x_2 \in \mathbb{R}^n$.

Proof: Exercise. Hint: Prove Epi f is cone.

The previous proposition showed basically that all norms are convex functions!

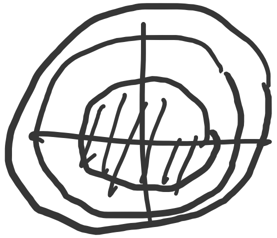
Thm: $f : R^n \to R$ be convex. [level sets of f]
$x \mapsto f(x)$

and $\text{lev}_\alpha f := \{ x \in R^n \mid f(x) \le \alpha \}$
$\alpha \in \mathbb{R}, \alpha \geq 0 \}$

$\text{lev}_\alpha f$ is convex.

$f(x_1, x_2) = x_1^2 + x_2^2$

$\text{lev}_\alpha f = \{ x_1^2 + x_2^2 \le \alpha \}$



Proof: Exercise!

$\boxed{\text{Is Converse true?}}$

Think!

Exercise!

Thm: [Jensen's Inequality].

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. $\sum_{i=1}^{m} \theta_i = 1$

$x_1, \dots, x_m \in X = \text{dom} f$. $\theta_i \in [0, 1]$

$$f\left(\sum_{i=1}^{m} \theta_i x_i\right) \le \sum_{i=1}^{m} \theta_i f(x_i).$$

Proof: Two ways to prove this.

(1) Induction (Exercise!)

(2) Use Epi f Convex. (Exercise!)