Checking Convexity.

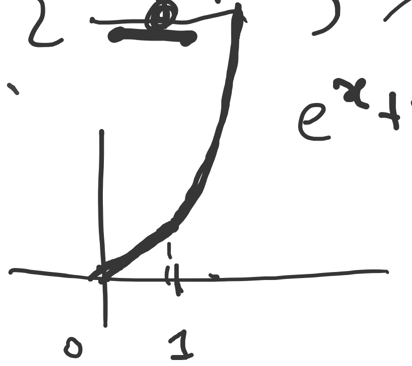
① f, g convex then $\alpha f + \beta g$ is also convex for $\alpha, \beta \in \mathbb{R}$.

② $g(x) \overset{?}{=} f(Ax + b)$ is convex if f is convex.

③ $\sup_i \{ f_i(x) \}$ where f_i are convex. $=$ is a convex function.

$\{ x + 2, \cancel{x \ldots + 3} \}$

$e^x + 2$, $x^2 + 2 \}$



Thm: Let $f:(a,b) \to \mathbb{R}$ be differentiable on (a,b). $\quad (a,b) \subset \mathbb{R}$

f is convex if & only if

$$\boxed{\frac{f(y)-f(x)}{y-x} \geq f'(x)} \quad \forall\, x,y \in (a,b)$$

$$\boxed{f(y) \geq f(x) + f'(x)(y-x)}$$

Proof: To show:

f convex $\Rightarrow$ $\boxed{\dfrac{f(y)-f(x)}{y-x} \geq f'(x)}$

Let $\theta \in [0, 1]$, $x, y \in [a, b)$

$$f(\theta y + (1-\theta) x) \leq \theta f(y) + (1-\theta) f(x)$$

$$\frac{f(x+\theta(y-x))}{\theta} \leq f(y) + \frac{1}{\theta} f(x) - f(x)$$

$$\frac{f(y)-f(x)}{y-x} \geq \left[\frac{f(x+\theta(y-x)) - f(x)}{\theta(y-x)}\right]$$

Proof: To show:

f convex $\Rightarrow \dfrac{f(y)-f(x)}{y-x} \geq f'(x)$

Let $\theta \in [0, 1]$

$$f(\theta y + (1-\theta) x) \leq \theta f(y) + (1-\theta) f(x)$$

$$\frac{f(x + \theta (y-x))}{\theta} \leq f(y) + \frac{1}{\theta} f(x) - f(x)$$

$$\frac{f(y) - f(x)}{y-x} \geq \lim_{\theta \to 0} \left[\frac{1}{\theta} \left(\frac{f(x + \theta(y-x)) - f(x)}{(y-x)} \right) \right]$$

Proof: To show:

$$f \text{ convex} \Rightarrow \frac{f(y)-f(x)}{y-x} \geq f'(x).$$

Let $\theta \in [0, 1]$

$$f(\theta y + (1-\theta)x) \leq \theta f(y) + (1-\theta) f(x)$$

$$\frac{f(x+\theta(y-x))}{\theta} \leq f(y) + \frac{1}{\theta} f(x) - f(x)$$

$$\frac{f(y)-f(x)}{y-x} \geq f'(x).$$

To show. $\dfrac{f(y)-f(x)}{y-x} \geq f'(x) \Rightarrow f$ convex.

Let $x, y \in (a, b)$. Let $z = \theta x + (1-\theta) y$

$\theta \in [0, 1]$.

$$\left[\frac{f(x)-f(z)}{x-z} \geq f'(z)\right] \text{ and } \left[\frac{f(y)-f(z)}{y-z} \geq f'(z)\right].$$

$f(x) \geq f(z) + f'(z)(x-z)$ and $f(y) \geq f(z) + f'(z)(y-z)$.

To show: $\dfrac{f(y) - f(x)}{y - x} \geq f'(x) \Rightarrow f$ convex.

Let $x, y \in (a, b)$. Let $z = \theta x + (1-\theta) y$, $\theta \in [0, 1]$.

$$\frac{f(x) - f(z)}{x - z} \geq f'(z) \text{ and } \frac{f(y) - f(z)}{y - z} \geq f'(z).$$

$$\theta f(x) \geq \theta f(z) + \theta f'(z)(x - z) \qquad (1-\theta) f(y) \geq f(z)(1-\theta) + f'(z)(1-\theta)(y - z).$$

To show. $$\frac{f(y)-f(x)}{y-x} \geq f'(x) \Rightarrow f \text{ convex.}$$

Let $x, y \in (a, b)$. Let $z = \theta x + (1-\theta) y$
$\theta \in [0, 1]$

$$\boxed{\theta f(x) + (1-\theta) f(y) \geq f(z) = f(\theta x + (1-\theta) y)} \quad \square$$

$\Uparrow$.

$\theta f(x) \geq \theta f(z) + \theta f'(z)(x-z) \qquad (1-\theta) f(y) \geq (1-\theta) f(z) + (1-\theta) f'(z)(y-z)$

Thm: $f : \mathbb{R}^n \to \mathbb{R}$, Let $\operatorname{dom} f = X \subseteq \mathbb{R}^n$

$x \mapsto f(x)$, and X is convex.

f is convex if and only if

$$\boxed{f(y) \geq f(x) + [\nabla f(x)]^T (y-x)'} \quad \forall\, x, y \in X$$

Proof: choose $x, y \in X$, $g(\theta) = f(\cancel{\theta y + (1-\theta)} x)$ z

$= \nabla f(z)^T dz/d\theta$

define $g(\theta) = \boxed{f(\theta y + (1-\theta x)}$ on $\theta \in [0, 1]$.

$$g'(\theta) = \nabla f(\theta y + (1-\theta)x)^T (y - x)$$

To show f is convex $\Rightarrow$ $f(y) \geq f(x) + \nabla f(x)^T (y-x)$

$\left[g(1) \geq g(0) + g'(0)_{(1-0)} \right]$ $\left[\because f \text{ convex} \Rightarrow g \text{ convex} \right]$

$\Downarrow$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

To show: If $f(y) \geq f(x) + \nabla f(x)^T (y-x)$

$\forall x, y \in X$ then f is convex.

Choose x, y s.t inequality holds.

take $z_1 = \theta y + (1-\theta) x$ $\quad \theta, \tilde{\theta} \in [0, 1]$

$z_2 = \tilde{\theta} y + (1-\tilde{\theta}) x$

$$f(z_1) \geq f(z_2) + \nabla f(z_2)^T (z_1 - z_2)$$

$$\underbrace{f(z_1)}_{g(\theta)} \geq \underbrace{f(z_2)}_{g(\tilde{\theta})} + \underbrace{\boxed{\nabla f(z_2)^T (y-x)} (\theta - \tilde{\theta})}_{g'(\tilde{\theta})(\theta - \tilde{\theta})}$$

$$\underline{g(\theta) \geq g(\tilde{\theta}) + g'(\tilde{\theta})(\theta - \tilde{\theta})}$$

$$\Downarrow$$

g is convex

$$\Downarrow$$

f is convex.

$\square$

Prop: $f: \mathbb{R}^n \to \mathbb{R}$ be "twice" differentiable
$x \mapsto f(x)$ on its domain

f is convex if and only

if $\boxed{\nabla^2 f(x) \succeq 0}$ $\forall\, x \in \text{dom} f$

Hessian is positive semi-definite

matrix.

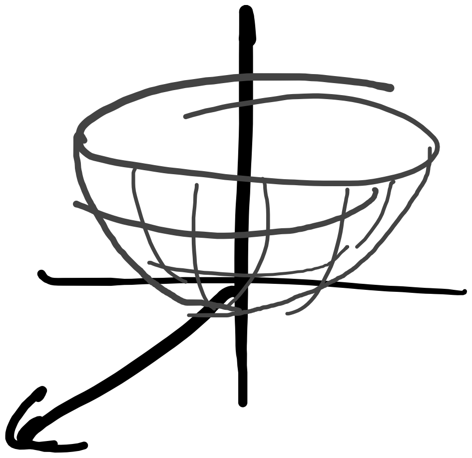
Proof: Exercise !

Example

$$\boxed{f(x) = x^T A x}$$

$$\nabla^2 f = A$$

f is convex iff $A \succeq 0$.



$\cancel{f(x_1 + x_2}$

$f(x_1, x_2)$

$$= [x_1 \ x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$