x^* is minimizer of function f

on some set $\Omega \subseteq \mathbb{R}^n$.

then

(i) $x^* \in \underline{\Omega}$.

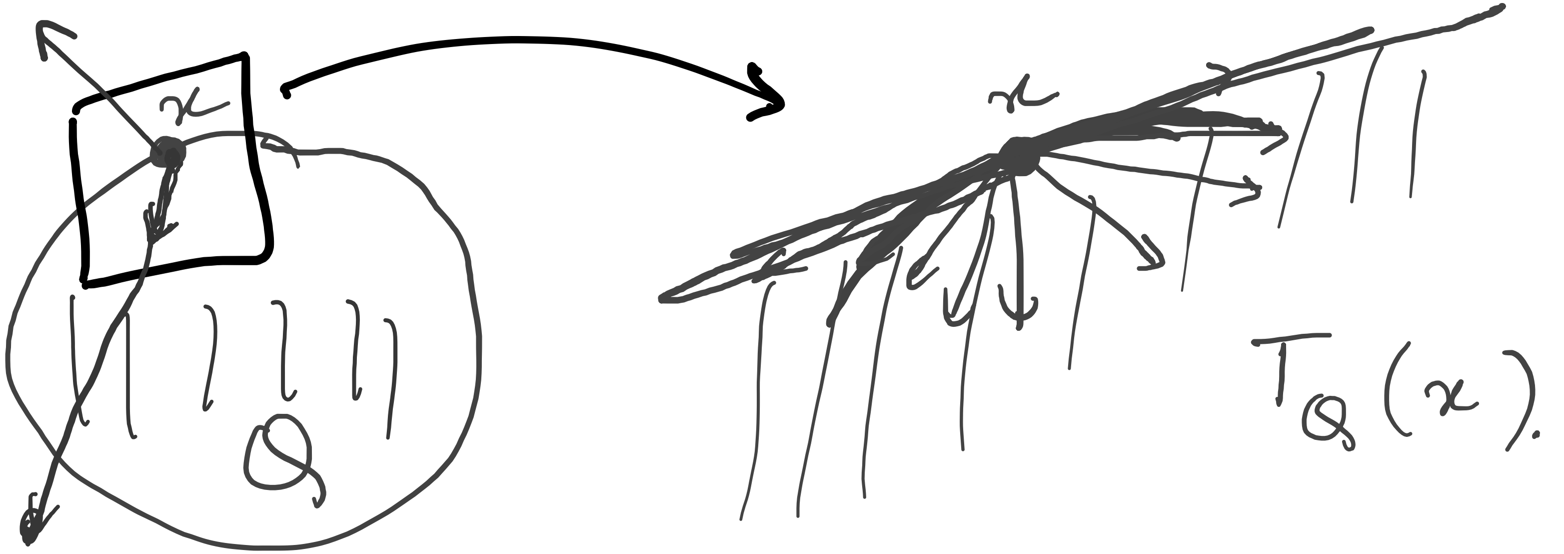
(ii) $f(x^* + \underline{\epsilon}\, \underline{h}) \geq f(x^*)$

$\forall\, \underline{h}$ s.t $\underbrace{\underline{\underline{x^* + \epsilon h}} \in \underline{\Omega}}$.

Perturbation needs to be along a feasible direction

$\boxed{\text{Radial cone}}$ of Q at $x \in Q$ is.

$$T_Q(x) := \left\{ h \in \mathbb{R}^n \;\middle|\; \exists\, \epsilon > 0, \text{ s.t } x + \epsilon h \in Q \right\}$$



SET OF All FEASIBLE DIRECTIONS AT x.

Thm: $f : Q \to \mathbb{R}$, $Q \subseteq \mathbb{R}^n$, convex set, f
$x \mapsto f(x)$

convex function on Q. f is differentiable

at $x^* \in Q$.

then $\boxed{x^* \text{ is minimizer}}$ if and only if

$$\boxed{\boxed{h^T \nabla f(x^*)} \geq 0} \quad \forall\, h \in T_Q(x^*)$$

Proof: (1) To show

$$\boxed{x^* \text{ minimizer} \Rightarrow h^T \nabla f(x^*) \geq 0 \quad \forall\, h \in T_{\Omega}(x^*)}$$

Let $g(\theta) := f(x^* + \theta h)$, $h \in T_{\Omega}(x^*)$

$g(0) = f(x^*)$, Thus 0 is minimizer of $g(\theta)$.

$$g'(\theta) = \nabla f(x^* + \theta h)^T h$$

$$g'(0) = \nabla f(x^*)^T h$$

We Claim. $g'(0) \geq 0$.

Suppose not. i.e., $\boxed{g'(0) < 0}$.

Then $g(0+\theta) = g(0) + g'(0)\theta + \bar{o}(\theta)$

$$g(\theta) = g(0) + \boxed{g'(0)\,\theta} + \boxed{\bar{o}(\theta)}$$

Since $\boxed{\lim_{\theta \to 0} \frac{\bar{o}(\theta)}{\theta} = 0}$

Choose $\theta \neq 0$ s.t. $\boxed{|\bar{o}(\theta)| < |g'(0)\,\theta|}$

Now $g(\theta) - g(0) < \boxed{g'(0)\,\theta + |g'(0)\,\theta|}$

Since $g'(0) < 0$

$\boxed{g(\theta) - g(0) < 0} \Rightarrow g(\theta) < g(0)$

But $\theta \neq 0$ cannot be minimizer.

Thus $g'(0) \geq 0 \Rightarrow \boxed{\nabla f(x^*)^T h \geq 0}$

[Note for this part we did not use convexity]

To show $\boxed{\nabla f(x^*)^T \underline{h} \geq 0} \quad \forall\, h \in T_Q(x^*)$

$\Downarrow$.

x^* is minimizer.

Since. $f(y) \geq f(x) + \nabla f(x)^T(\underline{y}-\underline{x})$

f is convex.

we have,

$$\boxed{f(x^* + \theta h) \geq f(x^*) + \theta\, \boxed{\nabla f(x^*)^T h}}.$$

$$\Rightarrow f(x^* + \theta \underline{h}) \geq f(x^*) \quad \forall\, h \in T_Q(x^*).$$

$\Rightarrow$ $\boxed{x^* \text{ is minimizer.}}$

$\square$

Note the Condition

$$\underline{\nabla f(x^*)^T} h \geq 0 \quad \forall\, h \in T_{\mathcal{S}}(x^*)$$

becomes

$$\boxed{\nabla f(x^*) = 0} \quad \text{if } x^* \in \text{int}\, \underline{\mathcal{S}}.$$

[why ?].

$x^* \in \underline{\text{in}}\text{t}\ \mathcal{S}$

$\nabla f(x^*)^T h \geq 0$

$-h \quad \underline{\nabla f(x)^T (-h) \geq 0}$

Thm: $f : \mathbb{R}^n \to \mathbb{R}$ is convex; twice differentiable.

$\Updownarrow$

$\boxed{\nabla^2 f(x) \succcurlyeq 0}$ $\forall x \in \mathbb{R}^n$.

Proof: To show $\boxed{f \text{ convex}} \Rightarrow \nabla^2 f(x) \succcurlyeq 0 \ \forall x \in \mathbb{R}^n$

[Taylor's Thm]. $\forall h \in \mathbb{R}^n$.

$$f(x + \epsilon h) = \boxed{f(x) + \epsilon \nabla f(x)^T h} + \frac{\epsilon^2}{2} h^T \nabla^2 f(x)\, h$$

$$+ \bar{o}(\epsilon^2 h^2)$$

Since $\underline{f \text{ is convex}}$ $\left[f(y) \geq f(x) + \nabla f(x)^T (y-x)\right]$

$$f(x+\epsilon h) \geq f(x) + \epsilon \nabla f(x)^T h.$$

$$\Rightarrow \boxed{\frac{\epsilon^2}{2} h^T \nabla^2 f(x) h + \bar{o}(\epsilon^2 h^2)} \geq 0.$$

$$\Rightarrow \boxed{\frac{1}{2} h^T \nabla^2 f(x) h} + \frac{\bar{o}(\epsilon^2 h^2)}{\epsilon^2} \geq 0.$$

$\Rightarrow$ Taking $\lim_{\epsilon \to 0}$

gives $h^T \nabla^2 f(x) h \geq 0 \quad \forall h \in \mathbb{R}^n.$

To show $\nabla^2 f(x) \succeq 0 \Rightarrow f$ is convex

Let $x, y \in \mathbb{R}^n$ and

$$g(\theta) := f(\theta y + (1-\theta)x).$$

$g : \mathbb{R} \to \mathbb{R}$

$\theta \mapsto g(\theta)$, $\theta \in [0,1]$

$$g(0) = f(x), \quad g(1) = f(y)$$

By $\boxed{\text{"Extended" Mean Value Theorem}}$.

$$\exists\, \bar{\theta} \in [0,1] \text{ s.t. } \boxed{g(1) = \boxed{g(0) + g'(0)} + \cancel{\frac{g''(\bar{\theta})}{2}}}$$

$\boxed{\text{Rolle's Thm.}}$

Since $\boxed{\nabla^2 f \succeq 0}$

$$g''(\bar\theta) = \frac{d^2}{d\theta^2} f(\theta y + (1-\theta)x)$$

$$= \frac{d}{d\theta}\left(\nabla f(\theta y + (1-\theta)x)^T (y-x)\right)$$

$$= (y-x)^T \boxed{\nabla^2 f(\theta y + (1-\theta)x)}(y-x)$$

$$\geq 0$$

Convexity

$$\Rightarrow g(1) \geq g(0) + g'(0) \Rightarrow \boxed{f(y) \geq f(x) + \nabla f(x)^T (y-x)}$$

$\square$

$f(x) = \cancel{\log} - \log x$ $\boxed{f(x) = x^3}$ $\frac{2x}{3} \neq 0$

$f''(x) = \frac{1}{x^2} \geq 0 \quad \forall x \in R$

$-\log x$ is a convex function

$\log x$ is concave.