[PRIMAL]

$$\left.\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0 \\ & x \in X \end{array}\right] \quad c^* \text{ (optimal objective)}$$

[DUAL]

$$d^* \text{ (optimal objective)} \quad \left[\max_{\underline{\lambda \geq 0}} \left[\inf_{x \in X} L(x, \lambda) \right] \right.$$

(Convex Program)

$$L(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

(i) $c^* \geq d^*$ $\longrightarrow$ $(c^* - d^*)$ duality gap.

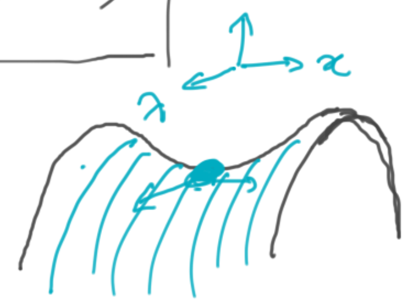
(ii) $c^* = d^*$ if Slater Condition
f, g_i Convex.

Def^n: ["Saddle point"]

$$L(x,\lambda) : \underset{=}{\overset{\mathbb{R}^n}{X}} \times \underset{=}{\overset{\mathbb{R}^n}{\Lambda}} \longrightarrow \mathbb{R}$$

(x^*, λ^*) is a saddle point of L if

$$\boxed{L(x,\lambda^*) \geq L(\underset{=}{x^*}, \underset{=}{\lambda^*}) \geq L(x^*, \lambda)}$$



$$\underbrace{\sup_{\lambda \geq 0} \inf_{x \in X} L(x,\lambda)}_{} \; \boxed{=} \; \underbrace{\inf_{x\in X} \sup_{\lambda \geq 0} L(x,\lambda)}_{\text{Primal problem.}} \qquad \boxed{f(x^*) \leq f(x)} \; x \in X$$

$$\boxed{\neq}$$

Thm: $\underline{\underline{x^* \in X}}$, $L(x,\lambda) \to$ Lagrangian

$$= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$\exists \lambda^* \geq 0$ s.t (x^*, λ^*) is saddle point of $L(x, \lambda)$, then x^* is minimizer of primal

Proof: $x^* \in X$, $\exists \lambda^* \geq 0$ s.t $L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda)$

$$\sup_{\lambda \geq 0} L(x^*, \lambda) = \sup_{\lambda \geq 0} f(x^*) + \sum_{i=1}^{m} \lambda_i \underline{g_i(x^*)}$$

$$\sup_{\lambda \geq 0} f(x^*) + \sum_{i=1}^{m} \lambda_i g_i(x^*) = \infty$$

if x^* is not feasible, $g_k(x^*) > 0$

$$\sup_{\lambda \geq 0} L(x^*, \lambda) = \infty$$

$$\underline{L(x, \lambda^*) \geq L(x^*, \lambda^*)} \geq L(x^*, \lambda) = \infty \quad \forall x, \lambda.$$

Cannot happen.

So, x^* must be feasible.

Since x^* is feasible $g_i(x^*) \le 0$

$$\sup_{\lambda \geq 0} \underline{L(x^*, \lambda)} = \sup_{\lambda \geq 0} f(x^*) + \sum_{i=1}^{m} \lambda_i \boxed{g_i(x^*)}$$

$$= f(x^*)$$

$$L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq \underline{L(x^*, \lambda)} \quad \forall (x, \lambda)$$

$$\underline{L(x^*, \lambda^*)} \geq \underline{L(x^*, \lambda)}$$

$$L(x^*, \lambda^*) \geq f(x^*)$$

$$f(x^*) \geq \left[f(x^*) + \sum_{i=1}^{m} \lambda^* g_i(x^*) \right] = L(x^*, \lambda^*)$$

$$L(x^*, \lambda^*) = f(x^*)$$

for all feasible x

$$f(x) \geq \underline{L(x, \lambda^*)} = f(x) + \sum_{i=1}^{m} \lambda^* \underline{g_i(x)}$$

$$f(x) \geq \underline{L(x, \lambda^*) \geq \underline{L(x^*, \lambda^*)}} \geq L(x^*, \lambda)$$

$$\underset{\displaystyle f(x^*)}{\shortparallel}$$

$f(x) \geq f(x^*)$ for all feasible x.

x^* is ~~the~~ minimizer of Primal Problem.

Thm: If f, g_i convex, g_i satisfy Slater condition.
$\exists\, x \in X$ s.t $g_i(x) < 0$, $i = 1, \dots, m$.

x^* is minimizer of primal

$\Downarrow \quad \Uparrow$

for x^*, $\exists\, \lambda^* \geq 0$ s.t (x^*, λ^*) is saddle point of $L(x, \lambda)$

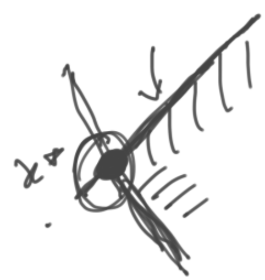
$$\min_{x} \left[\max_{\lambda} L(x, \lambda) \right] = \max_{\lambda} \left[\min_{x} L(x, \lambda) \right]$$

Primal   Dual.

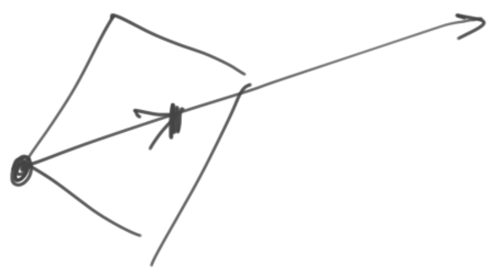
"Two player Zero games"

(x^*, λ^*) equilibrium of the game.

$$L(x, \lambda^*) \geq L(x^*, \lambda^*) \geq L(x^*, \lambda)$$



$$T_Q(x^*) = \{ h \mid x^* + \epsilon h \in Q \text{ for some } \epsilon > 0 \}$$



$$T_Q(x^*) = \mathbb{R}^n.$$

$$h^T \nabla f(x^*) \geq 0 \quad \forall h \in T_Q(x^*) = \mathbb{R}^n$$

$$(-h)^T \nabla f(x^*) \geq 0$$