$$\min \quad x^2 + 2x$$

$$\text{s.t.} \quad x^2 - 1 \leq 0$$

$$\underline{L(x, \lambda) := x^2 + 2x + \lambda(x^2 - 1)}$$

$$\ell(\lambda) = \inf_{x \in \mathbb{R}} L(x, \lambda) = \left(+\frac{1}{\underline{(1+\lambda)^2}}\right) + 2\left(-\frac{1}{1+\lambda}\right) + \lambda\left(\left(+\frac{1}{\underline{(1+\lambda)^2}}\right) - 1\right)$$

$$2x + 2 + 2x\lambda = 0$$

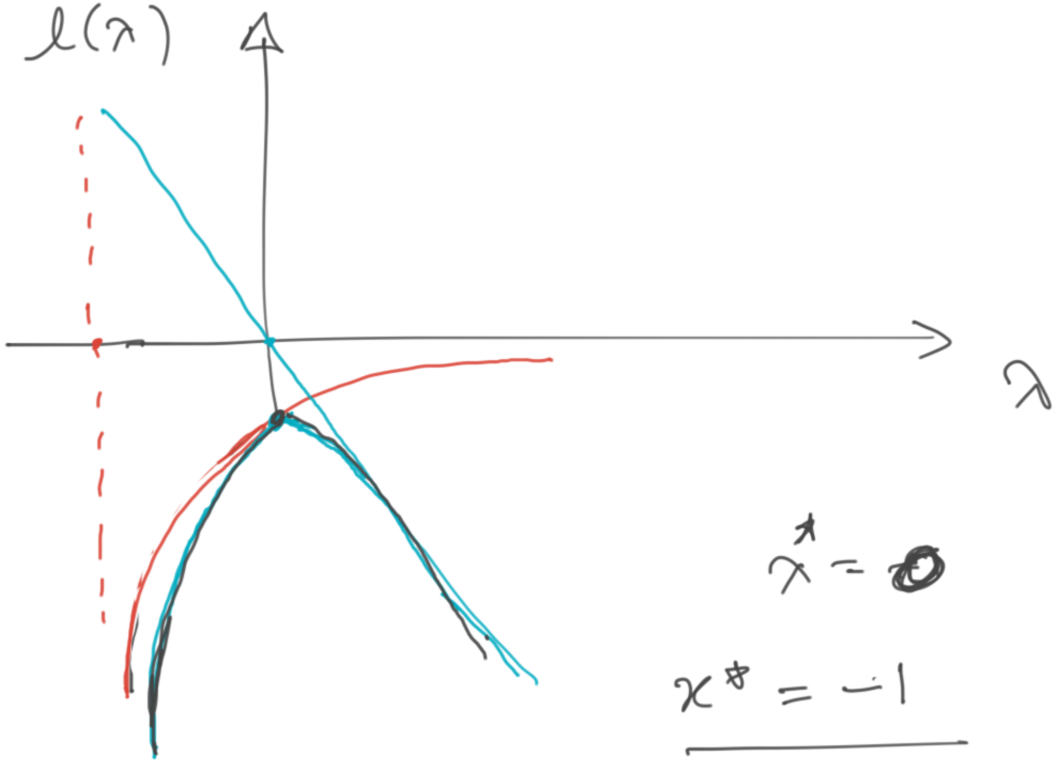
$$\underline{\left[\, x = -\frac{1}{1+\lambda} \,\right]}$$

$$\ell(\lambda) = \inf_{x \in \mathbb{R}} L(x, \lambda) = \frac{\cancel{1+\lambda}}{(1+\lambda)^2} - \frac{2}{1+\lambda} - \lambda = -\frac{1}{1+\lambda} - \lambda$$

$$\ell(\lambda) = \frac{-1}{1+\lambda} - \lambda \quad ,$$

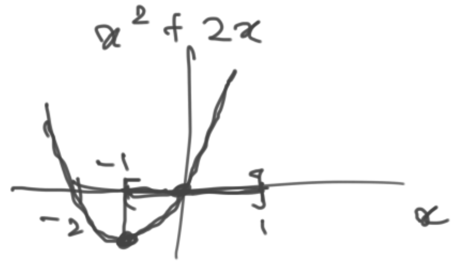
Dual Problem

$$\max_{\lambda \geq 0} \ell(\lambda) = -1$$



Primal Problem.

$$\min \; x^2 + 2x$$
$$\text{s.t} \quad x^2 - 1 \leq 0$$

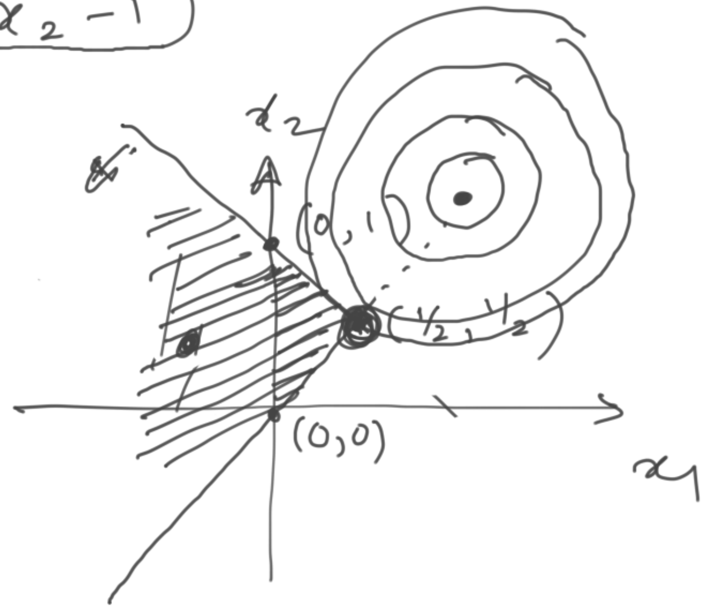


$$\lambda^* = 0$$

$$x^* = -1$$

Q. 2 minimize $(x_1 - 1)^2 + (x_2 - 1)^2$

s.t. $\begin{bmatrix} x_1 + x_2 \leq 1 \\ x_1 - x_2 \leq 0 \end{bmatrix}$



$L(x_1, x_2, \lambda_1, \lambda_2)$

$= (x_1 - 1)^2 + (x_2 - 1)^2$

$+ \lambda_1 (x_1 + x_2 - 1) + \lambda_2 (x_1 - x_2)$

$\inf_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2)$

$$\nabla_x \mathcal{L} = 0$$

$$\begin{bmatrix} 2(x_1 - 1) + \lambda_1 + \lambda_2 \\ 2(x_2 - 1) + \lambda_1 - \lambda_2 \end{bmatrix} = 0$$

$$x_1 = -\left(\frac{\lambda_1 + \lambda_2}{2}\right) + 1$$

$$x_2 = -\left(\frac{\lambda_1 - \lambda_2}{2}\right) + 1$$

$$\inf_{\substack{x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R}}} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = (x_1 - 1)^2 + (x_2 - 1)^2 + \lambda_1(x_1 + x_2 - 1) + \lambda_2(x_1 - x_2)$$

$$= \frac{(\lambda_1 + \lambda_2)^2}{4} + \frac{(\lambda_1 - \lambda_2)^2}{4} + \lambda_1\left(-\lambda_1 + 1\right) + \lambda_2(-\lambda_2)$$

$$= \frac{\lambda_1^2}{2} + \frac{\lambda_2^2}{2} - \lambda_1^2 - \lambda_2^2 + \lambda_1$$

$$= -\frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2 + \lambda_1$$

$$l(\lambda_1, \lambda_2) = \boxed{-\frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2 + \lambda_1}$$

Dual

$$\max_{\substack{\lambda_1 \geq 0 \\ \lambda_2 \geq 0}} l(\lambda_1, \lambda_2)$$

① $\lambda_1 > 0, \; \lambda_2 > 0$ $\qquad g_1(x) = 0, \; g_2(x) = 0$

② $\lambda_1 = 0, \; \lambda_2 > 0$ $\qquad g_2(x) = 0$

③ $\lambda_1 > 0, \; \lambda_2 = 0$ $\qquad g_1(x) = 0$

④ $\lambda_1 = 0, \; \lambda_2 = 0$

$$\boxed{\nabla_{\lambda} l(\lambda_1, \lambda_2) = 0}$$

$$\boxed{\begin{bmatrix} -\lambda_1 + 1 \\ -\lambda_2 \end{bmatrix} = 0} \quad , \quad \underbrace{\begin{matrix} \lambda_1^* = 1 \\ \lambda_2^* = 0 \end{matrix}}$$

$$\downarrow$$

$$\max_{\substack{\lambda_1 \geq 0 \\ \lambda_2 \geq 0}} l(\lambda_1, \lambda_2) = \frac{1}{2} \qquad \begin{bmatrix} x_1^* = \underline{1/2} \\ x_2^* = \underline{\underline{1/2}} \end{bmatrix}$$

$$\min (x_1 - 1)^2 + (x_2 - 1)^2 = \frac{1}{2}$$

strong duality