# Convex Thm on Alternatives

$$\left[\begin{array}{l} f(x) < c \\ g_i(x) \le 0, \quad i = 1, \dots, m \\ h_j(x) = 0 \quad \text{affine} \quad j = 1, \dots, n \\ \cancel{x \in X} \quad x \in \mathbb{R}^n \end{array}\right]$$

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$$\left[\begin{array}{l} \inf_{x \in X} L(x, \lambda, v) \geq c \\ \hline \lambda_i \geq 0 \\ v_j \ (\text{no constraints}) \end{array}\right]$$

$$L(x, \lambda, v) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{n} v_j h_j(x)$$

I feasible $\overset{\text{Slater's condition \& } f,\, g_i \text{ convex functions}}{\Longleftrightarrow}$ II infeasible.

Primal

$$\min f(x) = c^*$$

$$g_i(x) \leq 0 \quad , i = 1, \cdots, m$$

$$h_j(x) = 0 \quad j = 1, \cdots, n$$

Dual

$$d^* = \max_{\substack{\nu \\ \lambda \geq 0}} \min_{x} L(x, \lambda, \nu)$$

* Weak duality

$$c^* \geq d^*$$

* Strong duality

Slater's condition
$f$, $g_i$ convex
$h_j$ affine

$$c^* = d^*$$

* KKT Conditions (Karush-Kuhn-Tucker Conditions)

Suppose, $f$, $g_i$ convex, $h_j$ affine and Slater's Condition is satisfied.

If $x^*$, $\lambda^*$, $\nu^*$ are such that

$$g_i(x^*) \leq 0$$

$$h_i(x^*) = 0$$

$$\lambda_i^* \geq 0$$

$$\lambda_i^* g_i(x^*) = 0$$

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{n} \nu_j^* \nabla h_j(x^*) = 0$$

then $x^*$ is primal optimizer
$(\lambda^*, \nu^*)$ are dual optimizers.

with $c^* = d^*$.

$$\left[\begin{array}{l} \min \ \frac{1}{2} x^T \underline{P} x + q^T x + r \\ Ax = b \end{array}\right]$$

$$\underline{L(x, \nu)} = \frac{1}{2} x^T P x + q^T x + r + \nu^T (Ax - b)$$

$$\cancel{Px^*} + q + A^T \nu^* = 0$$

$$Ax^* = b$$

$\min f(x)$, $\quad f: \mathbb{R}^n \to \mathbb{R}$ convex

twice continuously differentiable

$x^*$ optimal if and only if $\nabla f(x^*) = 0$

$f(x^*) = p^*$

Typical iterative procedure will generate a sequence of $p^*$ $x^{(0)}, x^{(1)}, \dots, x^{(k)}, \dots \in \operatorname{dom} f$

s.t. $f(x^{(k)}) \longrightarrow p^*$ as $k \to \infty$.

$f(x^{(k)}) - p^* \leq \epsilon$

given $x \in \text{dom} f$ "Start"

repeat until Stopping criterion is met

Compute a direction $\Delta x$

choose step size in the direction $\Delta x$ $t > 0$

update $x \leftarrow x + t\,\Delta x$

$x^0$, $x^1$, $f(x^{(1)}) < f(x^{(0)})$

$x^{(0)}$, $S = \{ x \in \mathbb{R}^n \mid f(x) \leq f(x^{(0)}) \}$

$$\boxed{f(x) - p^*}$$

$x^T x$

$x^T x \le x_0^T x_0$

Assume on $S = \{ x \mid f(x) \le f(x^{(0)}) \}$

... $f$ is strongly convex.

$$\nabla^2 f(x) > 0 \quad \forall x \in S$$

or

$\forall x \in S$, $\exists$ "m" $> 0$

$$\boxed{\text{s.t } \nabla^2 f(x) \succeq m I}$$

for any $x, y \in S$

$$g(\theta) = f((1-\theta)x + \theta y) \qquad \theta \in [0, 1]$$

$$g'(\theta) = \nabla f((1-\theta)x + \theta y)^T (y - x)$$

$$g''(\theta) = (y-x)^T \nabla^2 f((1-\theta)x + \theta y)^T (y-x)$$

$\exists\ \tilde{\theta} \in [0, 1]$ s.t, $\underline{z} = (1-\tilde{\theta})x + \tilde{\theta} y$

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(\tilde{\theta}),$$

$$\boxed{f(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2} (y-x)^T \nabla^2 f(z)(y-x)}$$

$$\nabla^2 f(x) \succeq mI$$

$\forall\ x, y \in S$

$$\boxed{f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} \|(y - x)\|_2^2}$$

minimizer $\tilde{y}$ for RHS.

$$\frac{m}{\cancel{2}}(\tilde{y} - x) + \nabla f(x) = 0$$

$$\tilde{y} - x = -\frac{1}{m} \nabla f(x)$$

$$f(y) \geq f(x) - \frac{1}{m} \nabla f(x)^T \nabla f(x) + \frac{m}{2} \left(\frac{1}{m}\right)^2 \nabla f(x)^T \nabla f(x)$$

$$f(y) \geq f(x) - \frac{1}{2m} \|\nabla f(x)\|_2^2 \quad \forall\, x, y \in S$$

$y = x^*$

$$p^* \geq f(x) - \frac{1}{2m} \|\nabla f(x)\|_2^2$$

$$f(x) - p^* \leq \frac{1}{2m} \underline{\underline{\|\nabla f(x)\|_2^2}}$$

$\forall\ \epsilon > 0$ we have

$$\|\nabla f(x)\|_2^2 \le \underbrace{2m\epsilon}_{\delta} \Rightarrow f(x) - p^* \le \epsilon$$

$$\epsilon = \frac{\delta}{2m}$$

$$\boxed{\|x - x^*\|_2 \le \frac{2}{m}\|\nabla f(x)\|_2} \quad *$$

$$p^* = f(x^*) \ge f(x) + \nabla f(x)^T (x^* - x) + \frac{m}{2}\|x^* - x\|_2^2$$

$$\boxed{p^* - f(x)} \ge \nabla f(x)^T (x^* - x) + \frac{m}{2}\|x^* - x\|_2^2$$

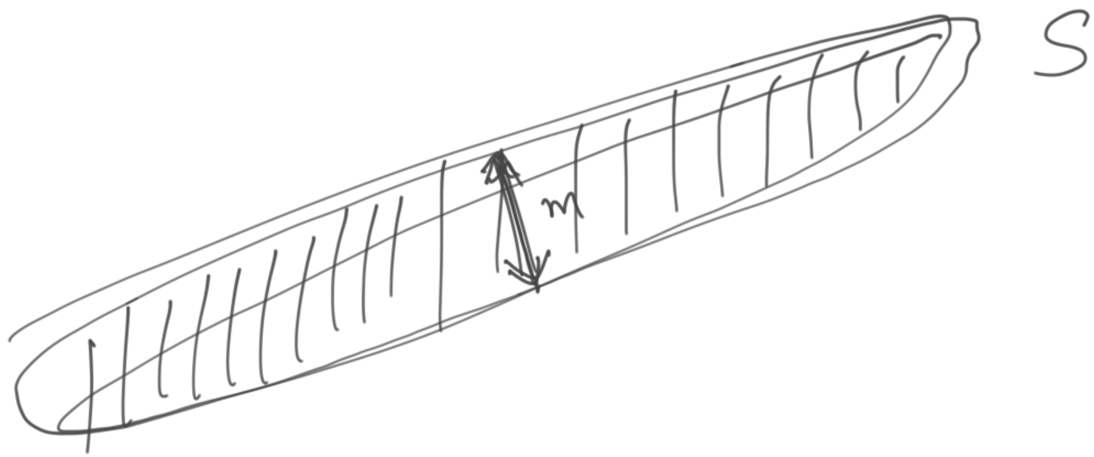
$$\nabla f(x)^T (x^* - x) + \frac{m}{2} \|x^* - x\|_2^2 \le 0$$

$$\left| \nabla f(x)^T (x^* - x) \right| \le \|\nabla f(x)\|_2 \, \|x^* - x\|_2$$

$$\boxed{-\|\nabla f(x)\|_2 \|x^* - x\|_2 \le \nabla f(x)^T (x^* - x)} \le \|\nabla f(x)\|_2 \|x^* - x\|_2$$

$$-\|\nabla f(x)\|_2 \, \|x^* - x\|_2 + \frac{m}{2} \|x^* - x\|_2^2 \le 0$$

$$\|x^* - x\|_2 \le \boxed{\frac{2}{m} \|\nabla f(x)\|_2}$$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