1. Initial guess $x^{(0)}$, $\epsilon \to$ tolerance

2. do the following until stopping criterion. $\|\nabla f(x)\|_2 \le \epsilon$

   1. $\Delta x^{(k)}$ chosen in s.t $f(x^{(k+1)}) < f(x^{(k)})$

      $\boxed{\nabla f(x^{(k)})^T \underline{\Delta x^{(k)}} < 0}$

   2. Choose an appropriate stepsize. t

      Back tracking line search.

   3. $x^{(k+1)} \leftarrow x^{(k)} + t \Delta x^{(k)}$

Back tracking line search.

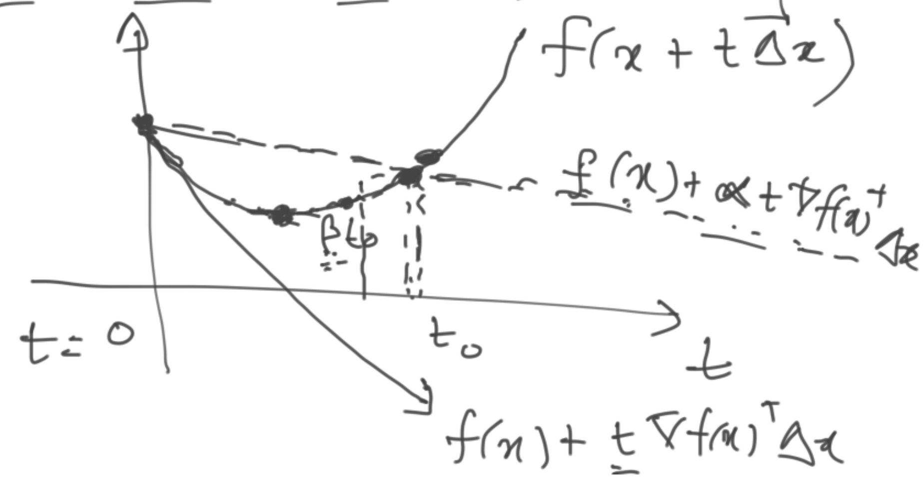
Δx , $f(x + t\Delta x) \approx f(x) + t\nabla f(x)^T \Delta x$

$$\boxed{f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x}$$

$t := 1$, $\alpha \in [0, 0.5]$, $\beta \in (0, 1)$

do until $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$

$t \leftarrow \beta t$



$t = 1$ or $t \in (\beta t_0, t_0)$

$$\boxed{t \geq \min\{1, \beta t_0\}}$$

Gradient descent

Initial guess x, ϵ

do until stopping order

(i) $\Delta x^{(k)} = -\nabla f(x^{(k)})$

(ii) $t^{(k)} := \underset{t}{\mathrm{Argmin}}\, f(x + t\Delta x)$

(iii) $x^{(k+1)} \leftarrow x^{(k)} + t^{(k)} \Delta x^{(k)}$

$$\Delta x = -\nabla f(x)$$

$$x^{+} = x + t\Delta x$$

$$= x - t\nabla f(x)$$

$$f\left(\underline{x - t\nabla f(x)}\right) = \tilde{f}(t)$$

$$\boxed{\tilde{f}(t) \leq f(x) - t\,\underline{\|\nabla f(x)\|_2^2} + \frac{M t^2}{2} \|\nabla f(x)\|_2^2}$$

from

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{(y-x)^T \nabla^2 f(z)(y-x)}{2}$$

$$\nabla^2 f(z) \preceq MI$$

$$\tilde{f}(t) \leq f(x) - t\,\|\nabla f(x)\|_2^2 + \frac{Mt^2}{2}\|\nabla f(x)\|_2^2$$

$$= f(x) + \|\nabla f(x)\|_2^2\left(-t + \frac{Mt^2}{2}\right)$$

$$t = \frac{1}{M}$$

$$\tilde{f}(t_{\text{exact}}) \leq f(x) + \|\nabla f(x)\|_2^2\left(-\frac{1}{M} + \frac{M \cdot 1}{2M^2}\right)$$

$$= f(x) + \|\nabla f(x)\|_2^2\left(-\frac{1}{2M}\right)$$

$$f(x^{(k+1)}) \leq f(x^{(k)}) - \frac{1}{2M}\|\nabla f(x^{(k)})\|_2^2$$

$$f(x^{(k+1)}) \le f(x^{(k)}) - \frac{1}{2M} \|\nabla f(x^{(k)})\|_2^2$$

$$f(x^{(k+1)}) - p^* \le f(x^{(k)}) - p^* - \frac{1}{2M} \underbrace{\|\nabla f(x^{(k)})\|_2^2}$$

$$\boxed{\frac{1}{\cancel{2m}} \|\nabla f(x^{(k)})\|_2^2 \ge 2m\, f(x^{(k)}) - p^*}$$

$$\left(f(x^{(k+1)}) - p^*\right) \le f(x^{(k)}) - p^* - \frac{1}{2M}\left(2m\left(f(x^{(k)}) - p^*\right)\right)$$

$$= \left(1 - \frac{m}{M}\right)\left[f(x^{(k)}) - p^*\right].$$

$$f(x^{(k+1)}) - p^* \leq \underline{\underline{c}}^k \left(f(x^{(0)}) - p^* \right)$$

$$c = \left(1 - \frac{m}{M}\right), \qquad c < 1$$

$$\frac{m}{M} < 1 \qquad \underline{\underline{m}}I \leq \nabla^2 f(x) \leq \underline{MI}.$$

$$\boxed{k \to \infty} \qquad f(x^{(k+1)}) - p^* \to 0$$

$$f(x^{(k+1)}) - p^* \leq \epsilon = c^k \left(f(x^{(0)}) - p^*\right)$$

$$\epsilon = c^k \left(f(x^{(0)}) - p^*\right)$$

$$\frac{1}{c^k} = \frac{f(x^{(0)}) - p^*}{\epsilon}$$

$$\boxed{k} = + \frac{\ln\left(\frac{f(x^{(0)}) - p^*}{\epsilon}\right)}{\boxed{\ln\left(\frac{1}{c}\right)}}$$

$$\ln\left(\frac{1}{c}\right) = -\ln c = -\ln\left(1 - \frac{m}{M}\right) \approx \left(\frac{m}{M}\right)$$

Backtracking line search (Convergence of Grad. Descent

$$f(x + t\Delta x) \leq f(x) + \alpha t \nabla f(x)^T \Delta x$$

$t := 1, \quad \alpha \in [0, 0.5), \beta \in (0, 1)$

do until

$t \leftarrow \beta t$

$$\Delta x = -\nabla f(x)$$

$$t \geq \min\{1, \beta t_0\}$$

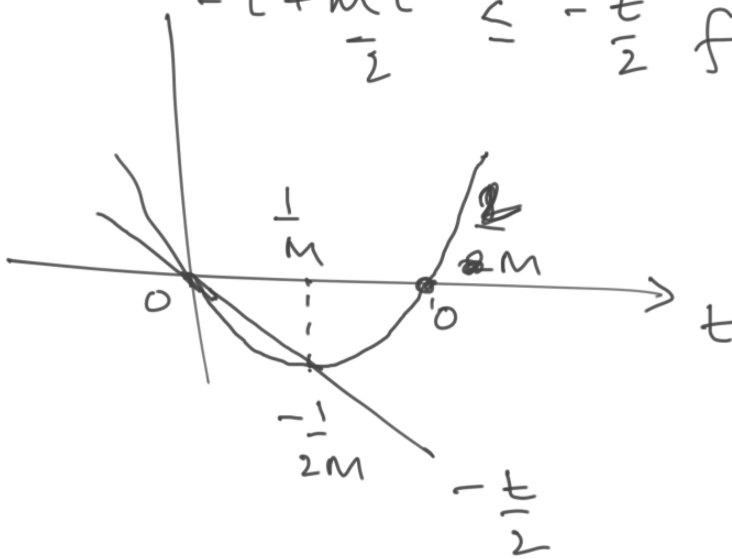
$$f(x - t \nabla f(x)) \leq f(x) - \alpha t \|\nabla f(x)\|_2^2$$

is Satisfied whenever $\boxed{0 \leq t \leq \frac{1}{M}}$

If $0 \leq t \leq \frac{1}{M}$

$$\tilde{f}(t) \leq f(x) + \|\nabla f(x)\|_2^2 \left(\underbrace{-t + \frac{Mt^2}{2}} \right)$$

$$-t + \frac{Mt^2}{2} \leq -\frac{t}{2} \text{ for } 0 \leq t \leq \frac{1}{M}.$$



$$\tilde{f}(t) \le f(x) - \frac{t}{2} \|\nabla f(x)\|_2^2$$

$$\alpha \le 0.5$$

$$\tilde{f}(t) \le f(x) - \alpha t \|\nabla f(x)\|_2^2$$

$$t \ge \min\left\{1, \frac{\beta}{M}\right\}$$

$$\tilde{f}(t) \le f(x) - \frac{\beta\alpha}{M} \|\nabla f(x)\|_2^2 \qquad t \ge \frac{\beta}{M}$$

or $\tilde{f}(t) \le f(x) - \alpha \|\nabla f(x)\|_2^2 \qquad t = 1.$

$$\underbrace{\tilde{f}(t)} \leq f(x) - \min\left\{\alpha, \frac{\beta\alpha}{M}\right\} \|\nabla f(x)\|_2^2$$

$$f(x^{(k+1)}) \leq f(x^{(k)}) - \min\left\{\alpha, \frac{\beta\alpha}{M}\right\} \underline{\underline{\|\nabla f(x)\|_2^2}}$$

$$f(x^{(k+1)}) - p^* \leq f(x^{(k)}) - p^* - \min\left\{\alpha, \frac{\beta\alpha}{M}\right\} \left(2m\left(f(x^{(k)}) - p^*\right)\right)$$

$$f(x^{(k+1)}) - p^* \leq \left(1 - \underset{\substack{!! \\ c}}{\min}\left\{2m\alpha, \frac{2\beta\alpha m}{M}\right\}\right)\left(f(x^{(k)}) - p^*\right)$$

$$c := 1 - \min\left\{2m\alpha,\ \underbrace{2\beta\alpha \frac{m}{M}}\right\} < 1$$

$$c^k \to 0 \quad \text{as } k \to \infty$$

$$f(x^{(k+1)}) - p^* \longrightarrow 0 \quad \text{as } k \to \infty.$$

(i) f is strongly convex on S