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* Set of all symmetric matrices.

* $\frac{n(n+1)}{2}$ dimensional subspace of $\mathbb{R}^{n \times n}$.

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examples

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 + 2x_1x_2$$

$$= (x_1 + x_2)^2 \geq 0$$

$$\forall (x_1, x_2) \in \mathbb{R}^2$$

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example

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + x_2^2 + 2x_1x_2$$

$$= (x_1 + x_2)^2 + 2x_1^2 > 0$$

Example

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\left(x_1^2 + x_2^2\right) + 2x_1x_2$$

For $x_1 = 0$, $x_2 = -1$

it evaluates to $\frac{1}{2} - 2 = -\frac{3}{2} < 0$

Not a positive definite matrix!

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$$S_{++}^n = \operatorname{int} S_+^n \quad [\text{interior of } S_+^n]$$

Eigenvalues & Eigenvectors

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$\{\lambda \mid \det(\lambda I - A) = 0\} =: \Lambda(A)$ Spectrum of A.

Eigenvalues & Eigenvectors Of Symmetric Matrices

Thm $A \in S^n$.

1. $\Lambda(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$
2. Eigenvectors $\{q_1, q_2, \cdots, q_n\}$ orthonormal.
3. $A = Q^T D Q$, $Q = [q_1, q_2, \cdots, q_n]$,
   $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Characterization of p.s.d. matrices

Thm: TFAE.

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2. $A \in S^n$, Eigenvalues of A are non-negative
3. $\exists\ R \in \mathbb{R}^{n\times n}$ s.t $A = R^T R$
4. $A \in \mathbb{R}^{n\times n}$, $\det A_{ii} = A_{(1:i,\, 1:i)}$

$\det A_{ii} \geq 0 \quad \forall\ i = 1, 2, \ldots, n$

Characterization of p. d. matrices

Thm: TFAE.

1. $A \in S^n_{++}$
2. $A \in S^n$, Eigenvalues of A are positive
3. $\exists\, R \in \mathbb{R}^{n\times n}$ s.t $A = R^T R$
   non-singular.
4. $A \in \mathbb{R}^{n\times n}$, $\det A_{ii} = A(1:i, 1:i)$

$\det A_{ii} > 0 \quad \forall\, i = 1, 2, \ldots, n$

Matrix Norms

$f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is a matrix norm.

$A \longmapsto f(A) = \|A\|$

if (i) $\|A\| \geq 0 \quad \forall A \in \mathbb{R}^{n\times n}$

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(iv) $\|A + B\| \leq \|A\| + \|B\|, \ \forall A, B \in \mathbb{R}^{n\times n}$

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(v) $\|AB\| \leq \|A\| \|B\| \quad \forall A, B \in \mathbb{R}^{n\times n}$.

Examples

1. Frobenius norm.

$$\|A\|_F := \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

[Compare with vector norm!].

* Induced Norms

* p- norm

* Matrix a linear map. $A : \mathbb{R}^n \to \mathbb{R}^n$

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$$\|A\|_p := \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

* Induced Norms

* p- norm
* Matrix a linear map.

$$A : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

$$\|A\|_p := \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Calculates maximum magnification.

2-norm:

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$$= \max_{j \in \{1, \dots, n\}} \sum_{i=1}^{n} |a_{ij}|$$

2-norm:

$$\|A\|_2 := \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2$$

1-norm:

$$\|A\|_1 := \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 = 1.} \|Ax\|_1$$

$$= \max_{j \in \{1, \dots, n\}} \sum_{i=1}^{n} |a_{ij}| \quad \left[\text{max. col. sum}\right]$$

∞ - norm :

$$\|A\|_\infty := \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

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max. row sum.