# Some Basics of Multi-Variable Calculus !

## Continuous function on $X \subset \mathbb{R}^n$

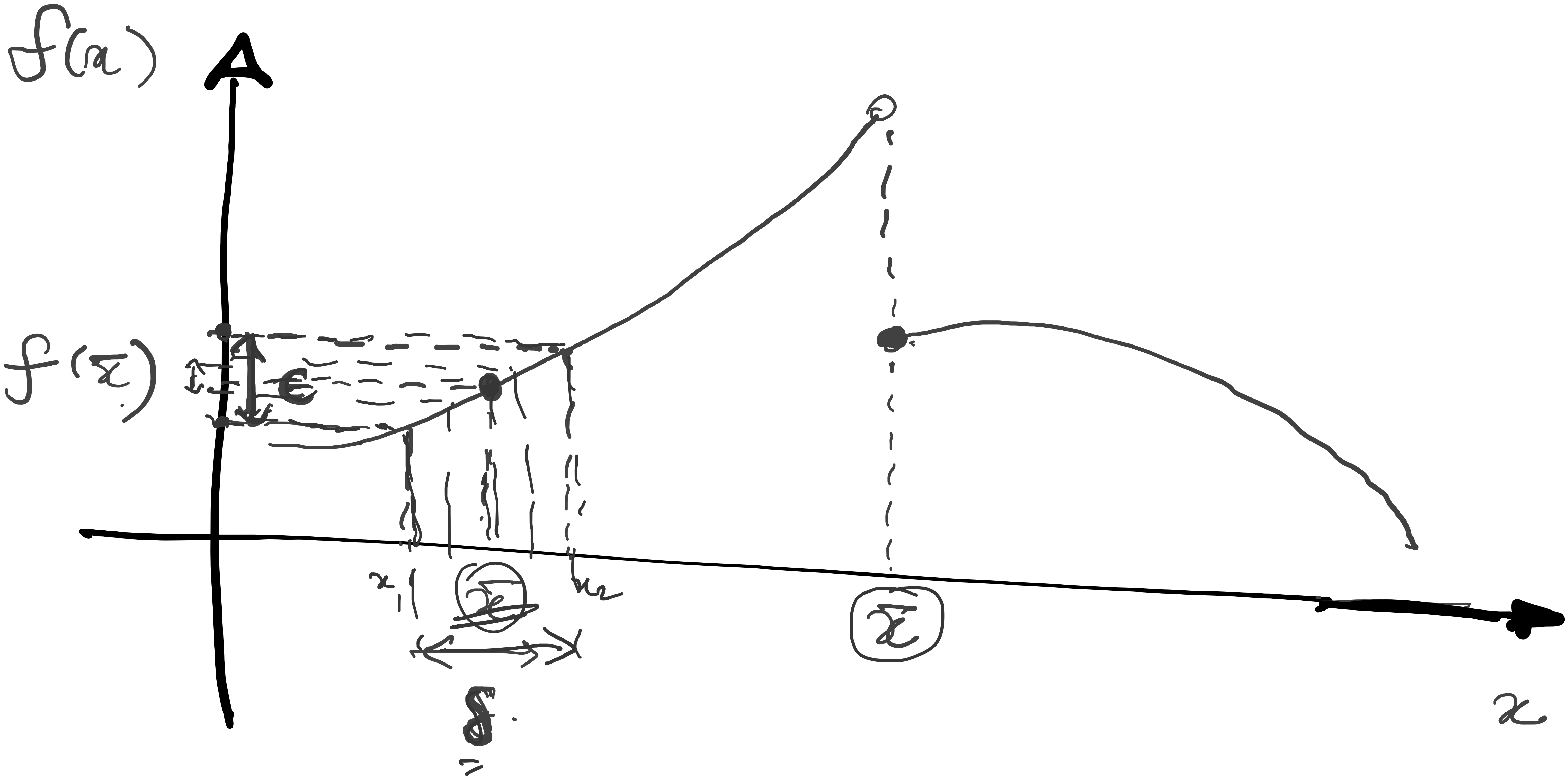
$$f : X \to \mathbb{R}^m \quad \text{is continuos at } \bar{x} \in X$$
$$x \mapsto f(x)$$

if

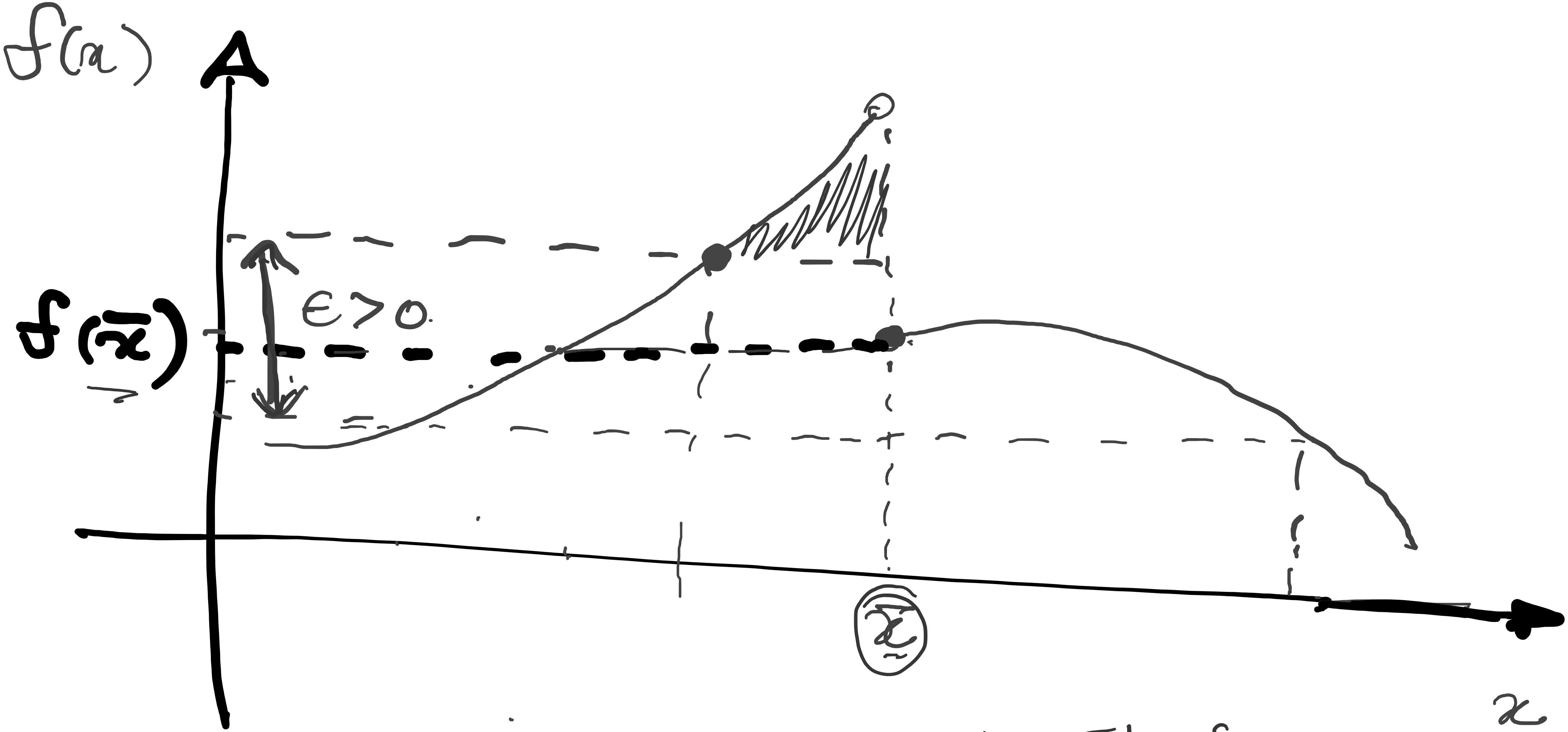
$$\forall \overbrace{\epsilon > 0}^{\text{Choice.}}, \quad \exists \overbrace{\delta > 0}^{\text{one should find}} \quad \text{s. to}$$

$$\boxed{\|x - \bar{x}\| < \delta \implies \boxed{\|f(x) - f(\bar{x})\| < \epsilon}}$$

For arbitrary bds on co-domain, $\exists$ bound on domain

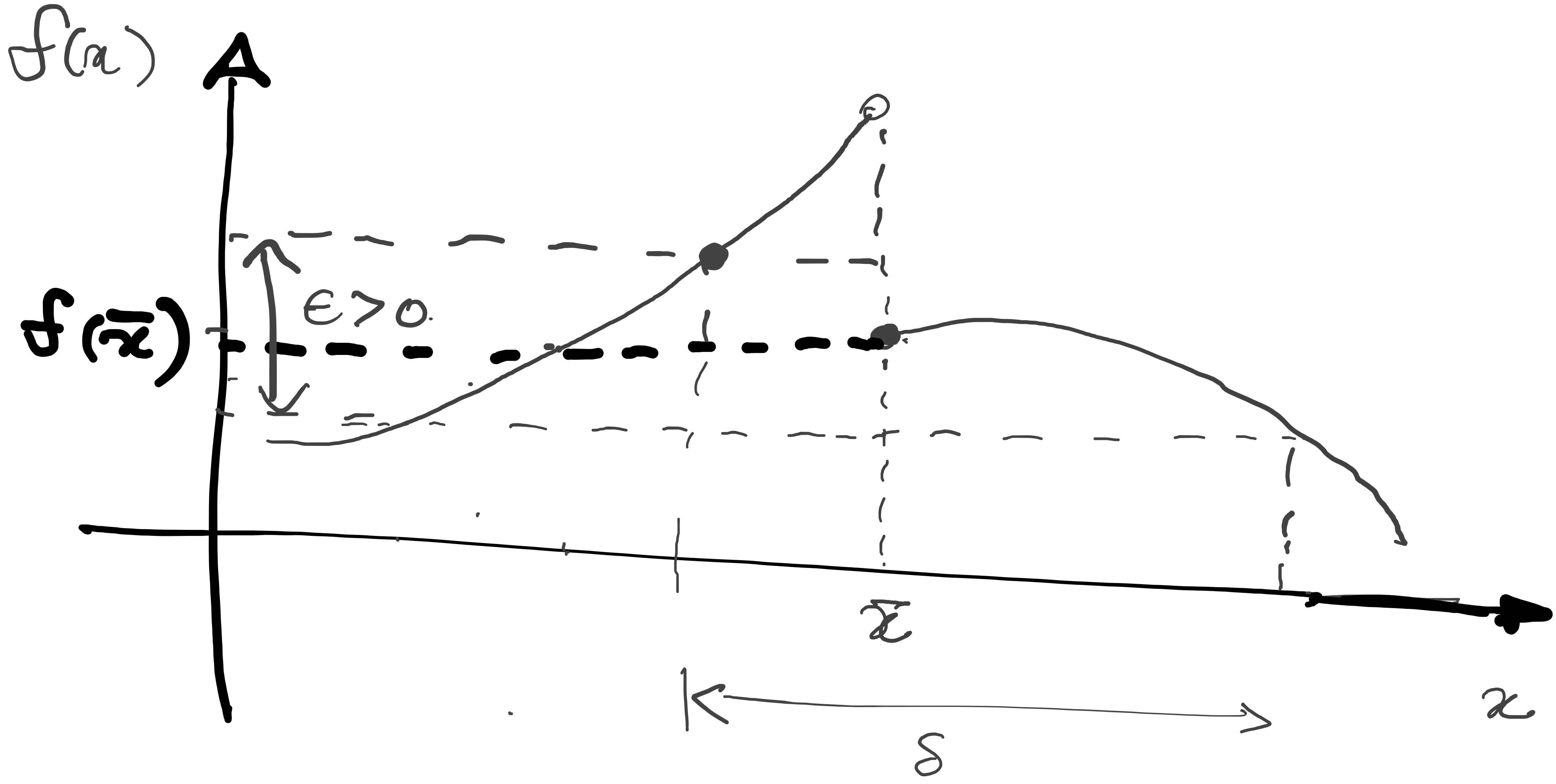



$f(x)$ is continuous at $\bar{x}$

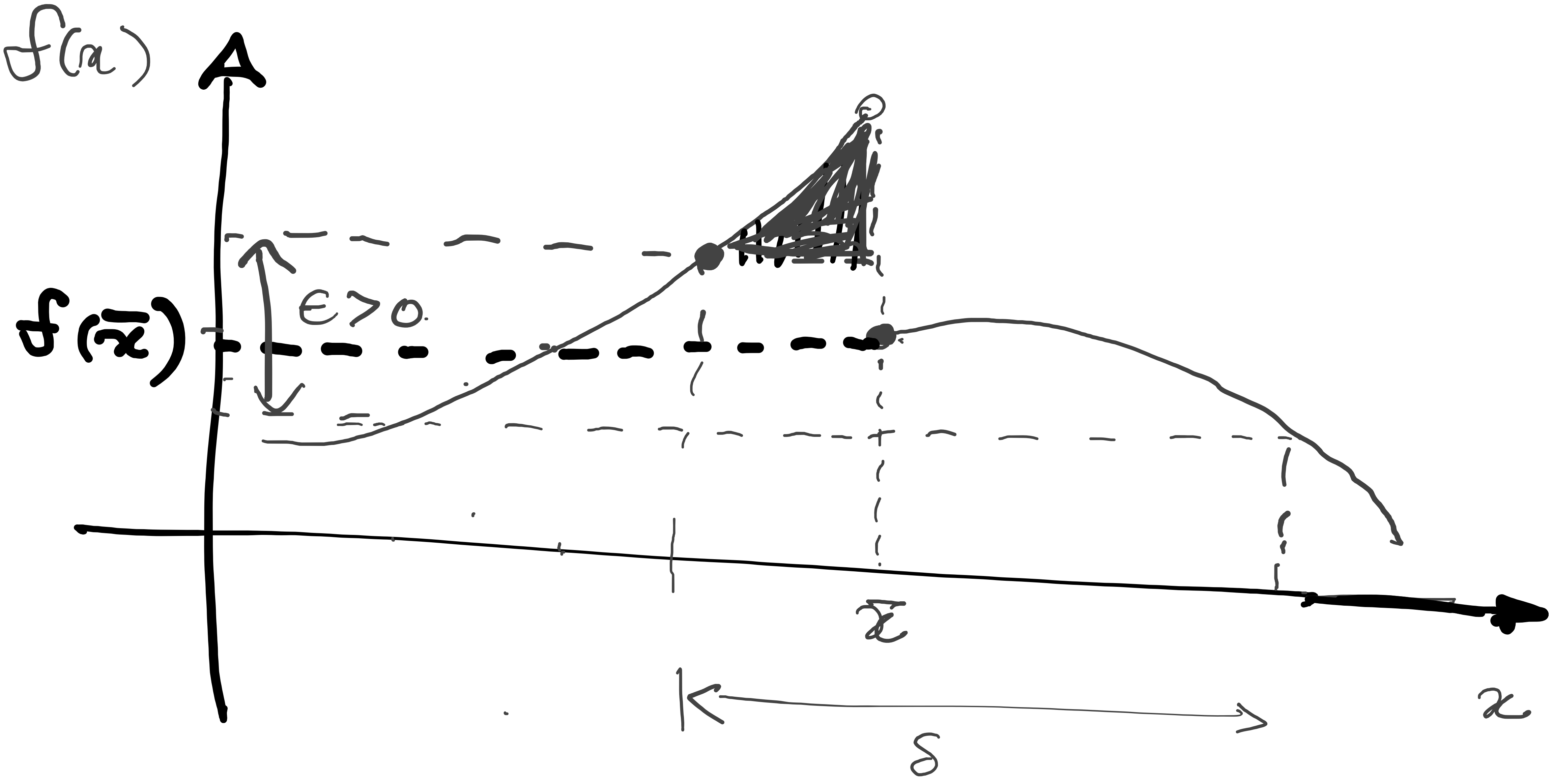



there is no $\delta$ s-t $\|x - \bar{x}\| \leq \delta$

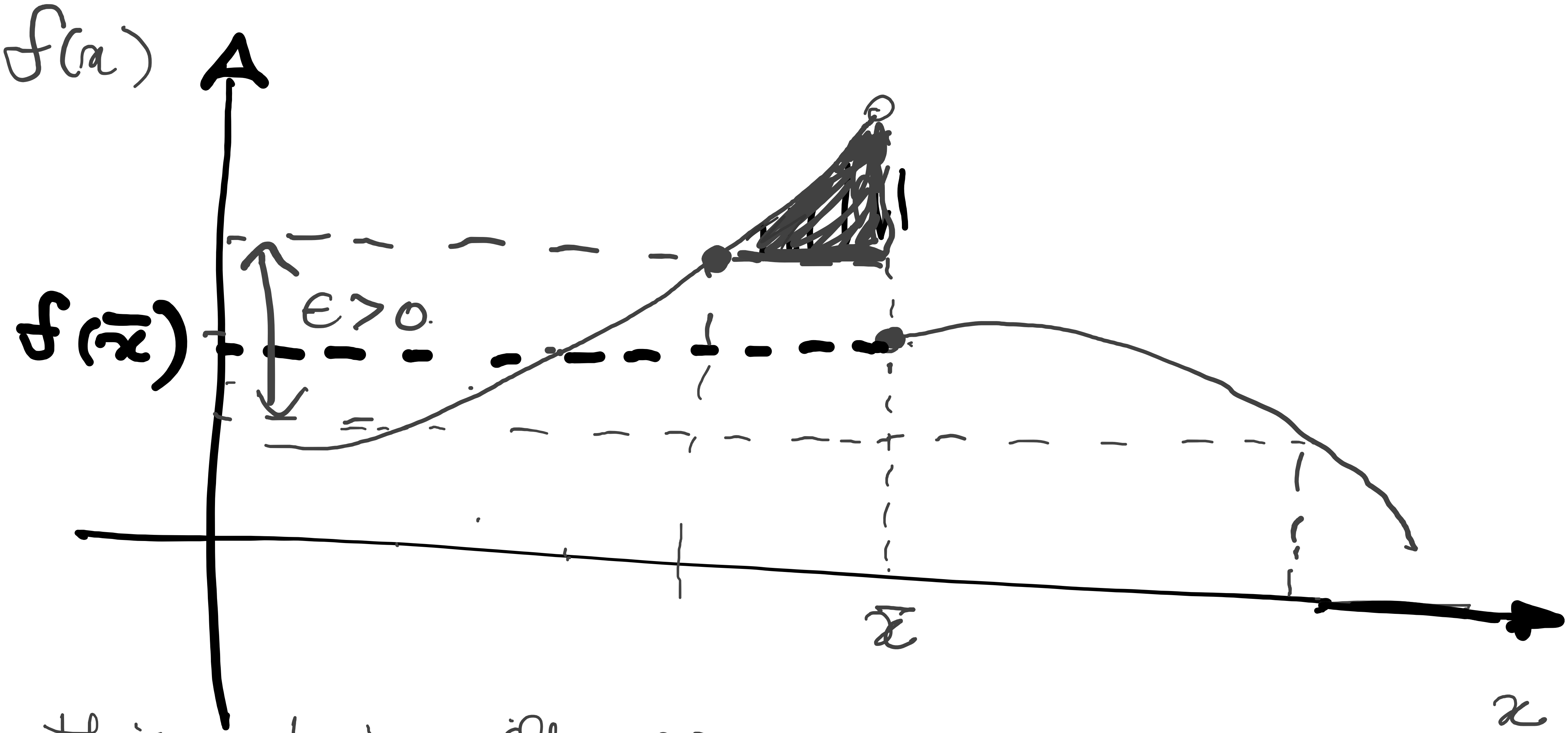
$\Rightarrow \|f(x) - f(\bar{x})\| < \epsilon$



f(x)
ε>0
f(x̄)
x̄
δ
x

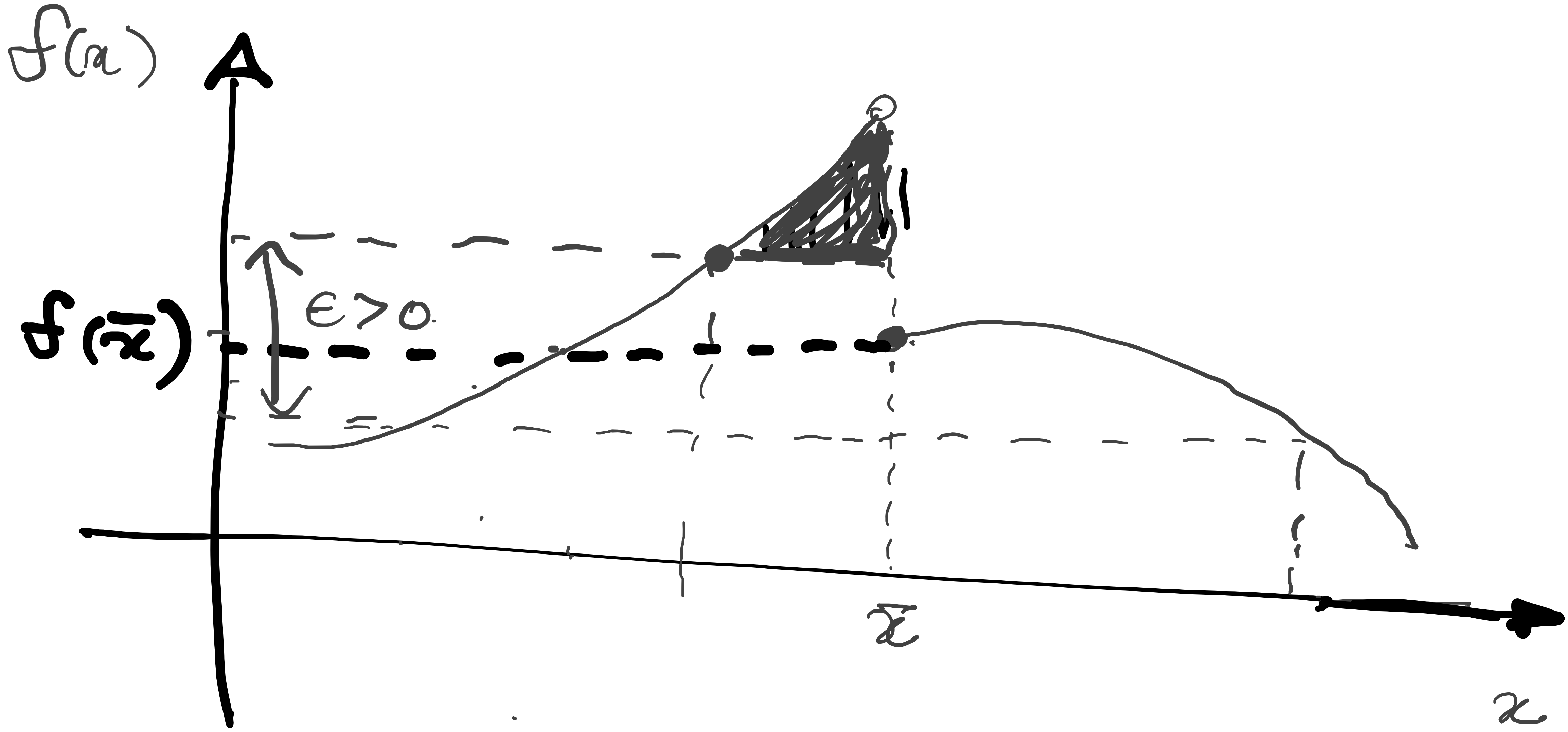


f(x)
ϵ>0
f(x̄)
x̄
x
δ




For this choice of $\epsilon > 0$,

No $\delta$ s.t. $\|x - \bar{x}\| < \delta \Rightarrow \|f(x) - f(\bar{x})\| < \epsilon$.




Not continuous at $\bar{x}$.

## Another defn.

$$\boxed{\lim_{x \to \bar{x} \text{ left}} f(x) = \lim_{\substack{x \to \bar{x} \\ \text{right}}} f(x)}$$

$f : X \to \mathbb{R}^n$ Continuous at $\bar{x}$

$X \subseteq \mathbb{R}^n$

if

$$\forall \{x_n\} \longrightarrow \bar{x},$$

$$f(x_n) \longrightarrow f(\bar{x}).$$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x)$$

The Derivative:

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

What if $f : \underset{\underset{x}{=}}{\mathbb{R}^n} \to \underset{\sim}{\mathbb{R}^m}$ ?

# Directional Derivative:

of $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ in direction $\underline{v}$.

$x \mapsto f(x)$.

$$f'(\underline{x}, \underline{v}) = \boxed{\lim_{\underline{t} \to 0^+} \frac{f(\underline{x} + \underline{t}\,\underline{v}) - f(x)}{t}}$$

$\underline{x} + t\underline{v}$ $\qquad f : \underline{\mathbb{R}} \to \mathbb{R}^m$

## Directional Derivative:

of $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ in direction $v$.

$x \mapsto f(x)$.

$$f'(x) = \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$$

## Partial Derivative:

$$f'(x, e_i) = \lim_{t \to 0^+} \frac{f(x + te_i) - f(x)}{t}$$

$x_i$

$$= \lim_{t \to 0^-} \frac{f(x + te_i) - f(x)}{t} = \left(\frac{\partial f}{\partial x_i}\right)$$

# Gateaux Differentiable at $x$

if

$$\boxed{f'(x, v)} = \lim_{t \to 0^+} \frac{f(x+tv) - f(x)}{t}$$

exists for all $v \in X$

$f'(x, v)$

i.e. Directional derivative exists in all directions $v \in X$ at $x \in X$

# Frechet Differentiability, Let $X \subset \mathbb{R}^n$

$f : X \to \mathbb{R}^m$ is Differentiable
$\quad x \mapsto f(x)$

at $x \in X$, if $\exists$ $\boxed{\text{Linear}}$ function.

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \approx f'(x)$$

$$\|f(x+\Delta x) - f(x) - f'(x)\Delta x\| \leq o(\|\Delta x\|)$$

$$D_x f : \mathbb{R}^n \to \mathbb{R}^m$$
$$v \mapsto D_x f\,[v]$$

s.t.

as, $v \to 0$, we have:

$$\boxed{\|f(x+v) - f(x) - D_x f\,[v]\|} \leq \boxed{o(\|v\|)}$$

# Frechet Differentiability, let $X \subset \mathbb{R}^n$

$$f : X \to \mathbb{R}^m \qquad \text{is Differentiable}$$
$$x \mapsto f(x)$$

at $x$, if $\exists$ "Linear" function.

$$D_x f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \mapsto D_x f[v]$$

Frechet Derivative of $f$.

s.t.

as $\cancel{\Delta x}\, v \to 0$, we have.

$$\| f(x+v) - f(x) - D_x f [v] \| \leq o(\|v\|)$$

$o(\|v\|_2)$ is a class of functions $\underline{\varphi}(\underline{v})$

s.t $$\lim_{\|v\| \to 0} \frac{\varphi(v)}{\|v\|_2} \to 0$$

e.g. $\varphi(v) = \|v\|_2^2$ is in $\underline{o}(\|v\|_2)$

$$\lim_{\|v\|_2 \to 0} \frac{\varphi(v)}{\|v\|_2} = 0$$

$$\| f(x+v) - f(x) - D_x f[v] \|_2 \leq o(\|v\|_2)$$

as $\|v\|_2 \to 0$

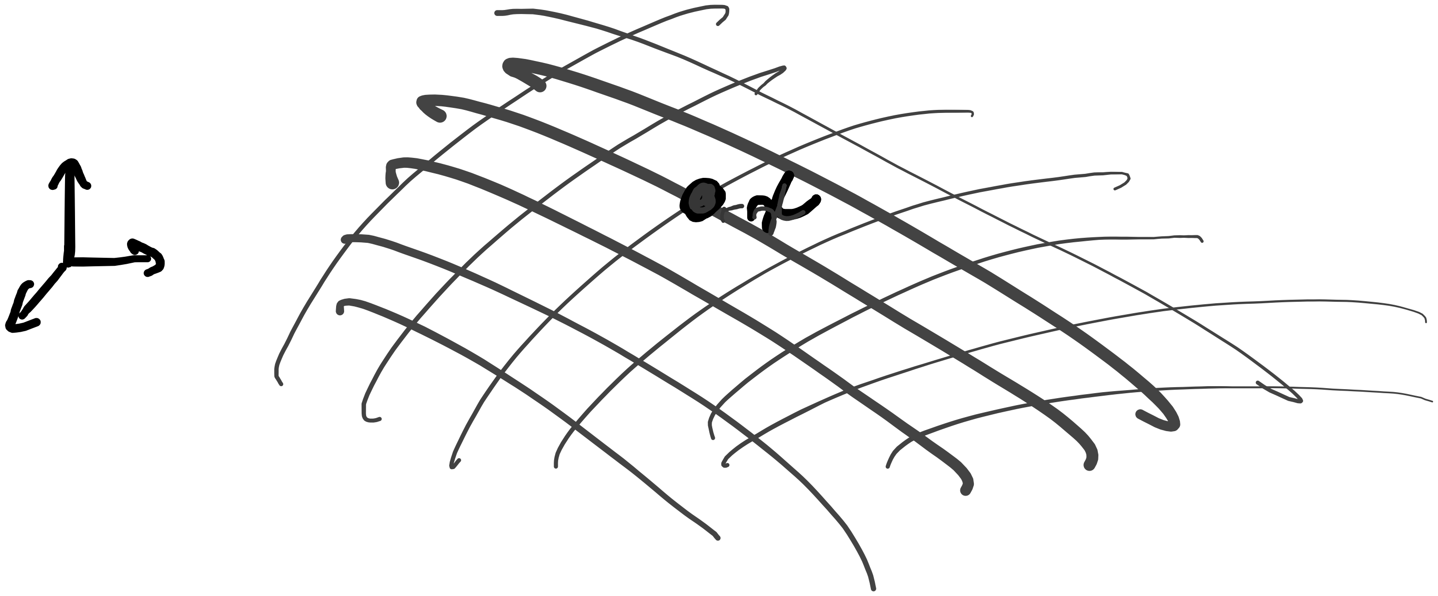
$$f(x+v) - f(x) \Rightarrow D_x f[v].$$

$$f(x+v) \approx \boxed{f(x) + D_x f [v]}$$

First order approximation $\max_x$ gives the Derivative of f at $x$.

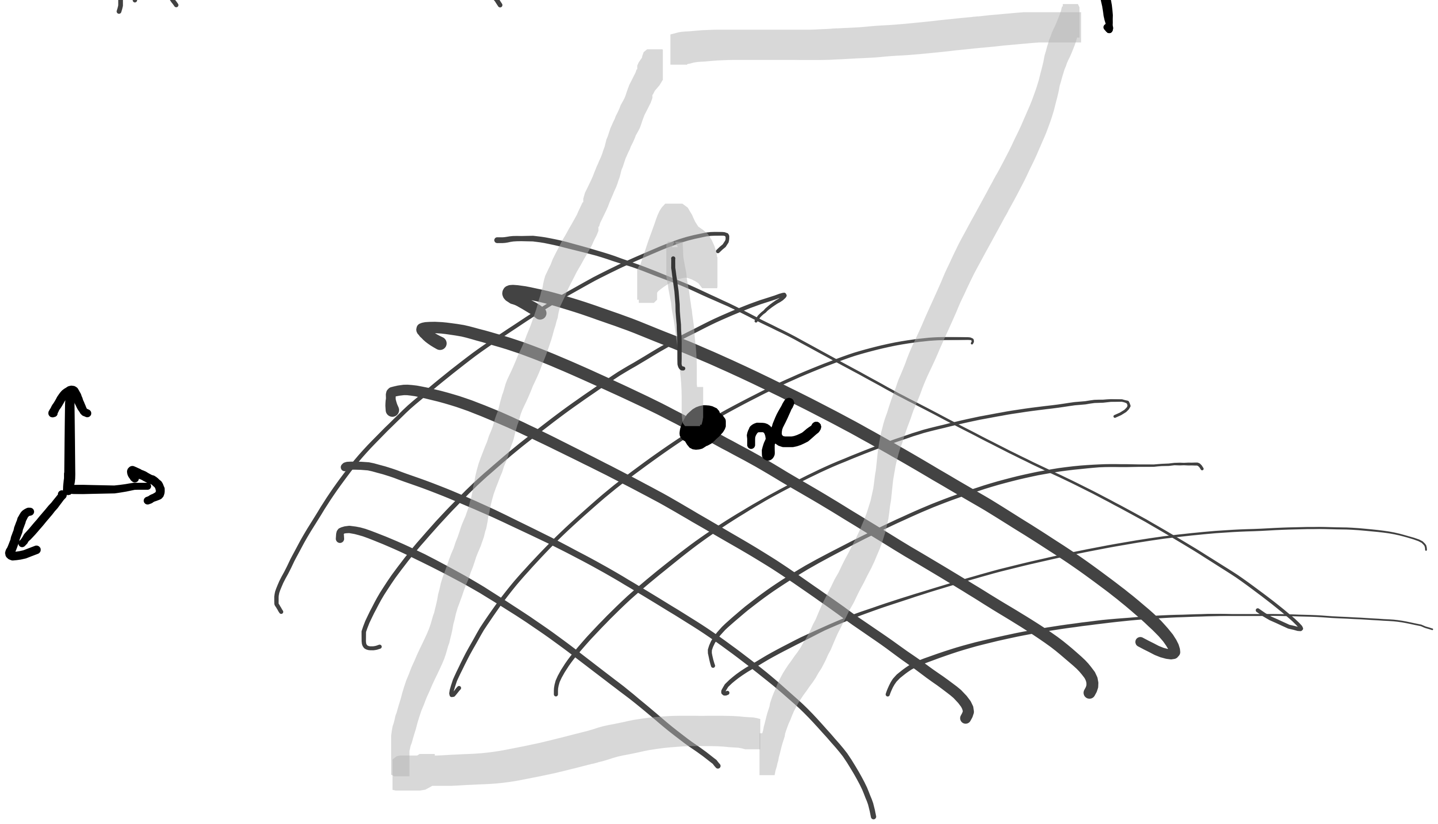
$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x_1, x_2) \to f(x_1, x_2)$$

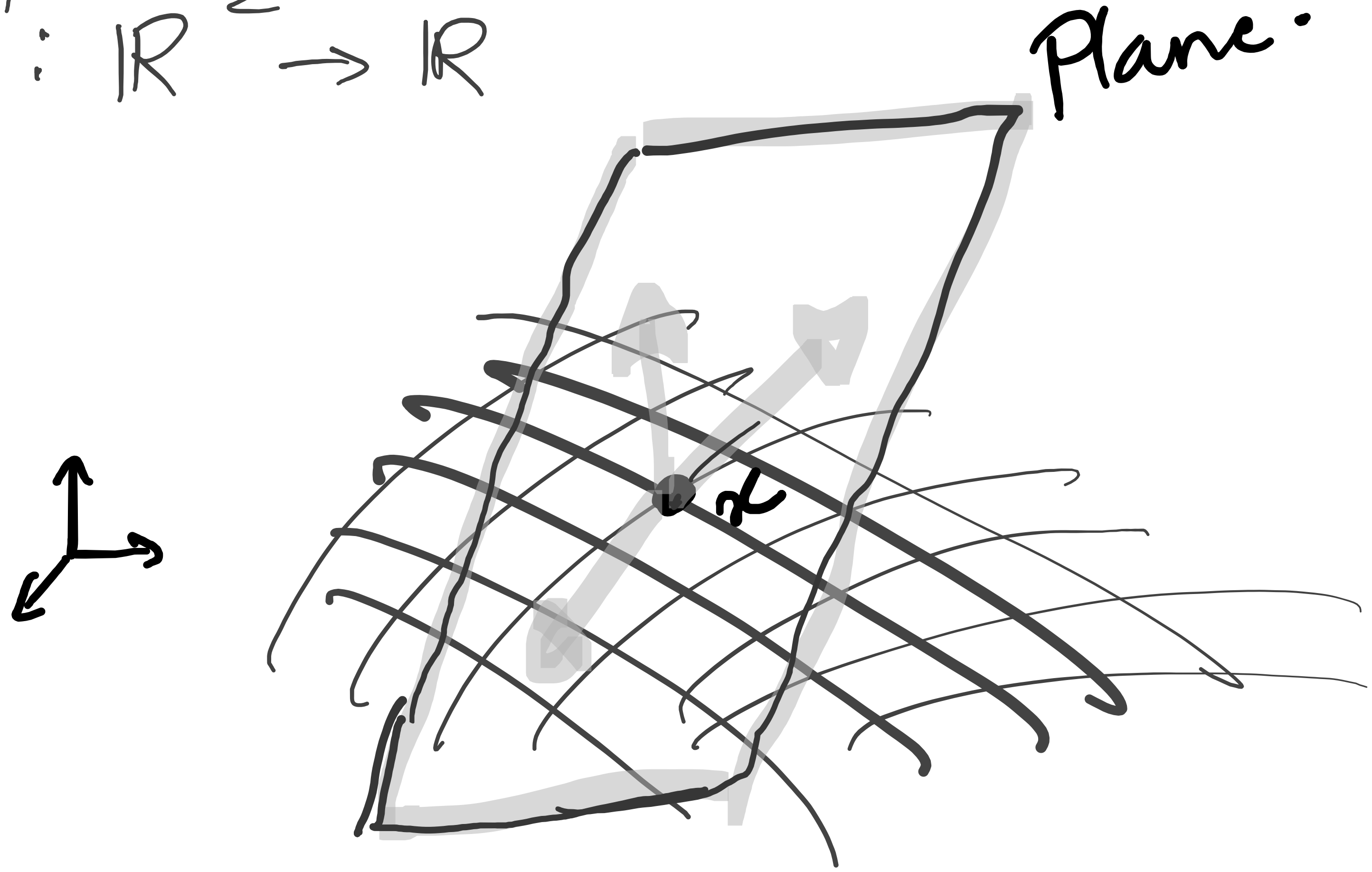


$f: \mathbb{R}^2 \to \mathbb{R}$

Plane.



$f : \mathbb{R}^2 \to \mathbb{R}$



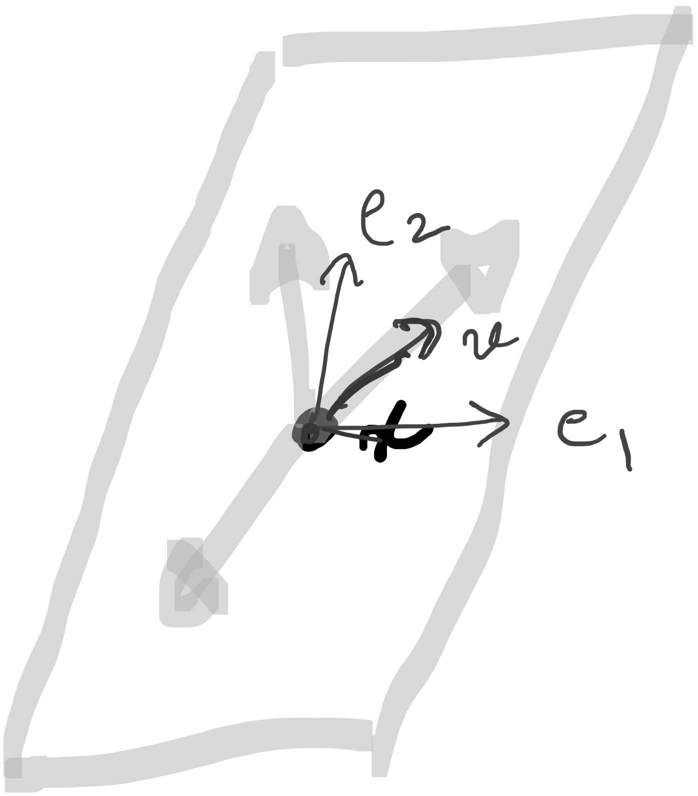

$$f(\underline{x + v}) - f(x) \approx \underbrace{D_x f}\,[\underline{\underline{v}}].$$

How to obtain representation?

$f : \mathbb{R}^2 \to \mathbb{R}$

Plane.

$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix}$.




$v = \alpha e_1 + \beta e_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$\left.\begin{aligned} D_x f[e_1] &= \frac{\partial f}{\partial x_1} \\ D_x f[e_2] &= \frac{\partial f}{\partial x_2} \end{aligned}\right] \to$$

$$f(x+v) - f(x) \approx D_x f[v] = \alpha \frac{\partial f}{\partial x_1} + \beta \frac{\partial f}{\partial x_2}$$

$$= \nabla f^T v.$$

$$\boxed{f : \mathbb{R}^{n \times m} \longrightarrow \mathbb{R}}$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x_1, x_2, \cdots, x_n) \mapsto f(x_1, x_2, \cdots, x_n)$$

$$(\nabla f)^T v + f(x) \approx f(\underline{x + v})$$

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix},$$

Think of ~~it~~ as

$$\boxed{f(x) + \langle \nabla f, v \rangle}$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$(x_1, x_2, \ldots, x_n) \longmapsto \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Jacobian of $f$ at $x$.

$$D_x f := \left[ \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_2}{\partial x_n} \end{bmatrix} \cdots \begin{bmatrix} \frac{\partial f_m}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{bmatrix} \right]$$

$n \times m$ matrix

# What about Second derivative?

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

$f : \mathbb{R}^n \to \mathbb{R}$

$\to$ take its Jacobian.

$G : \mathbb{R}^n \to \mathbb{R}^n$

:

$$D \nabla f = \begin{bmatrix} \partial^2 f / \partial x_1 & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \partial^2 f / \partial x_1 \partial x_2 & \frac{\partial^2 f}{\partial x^2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \partial^2 f / \partial x_1 \partial x_n & \partial^2 f / \partial x_2 \partial x_n & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Hessian.
of f.

$f : \mathbb{R}^n \to \mathbb{R}$

(1) $f(\underline{x}) = \underline{a} + \underline{g}^T x$

$\nabla f = \underline{g}$

$\| f(\underline{x+v}) - f(\underline{x}) - D_{xf}(v) \|$

$= \| \underline{a} + g^T (\underline{x+v}) - (a + g^T x)$

$- D_{xf}(v) \|$

$= \| \underline{g}^T v - D_{xf}(v) \|$

Choose $D_f(v) = g^T v$

$\swarrow$

$D_{xf}(v) = \langle g, v \rangle$

$\nearrow$

$D_{xf}$

① $f(x) = a + g^T x$

$\nabla f = g$

② $f : \mathbb{R}^n \to \mathbb{R}$

$f(x) = x^T A x.$

$\nabla f = \underline{(A + A^T) x}$

$\| f(x + v) - f(x) - D_x f(v) \|$

$= \| (x + v)^T A (x + v) - x^T A x - D \|$

$= \| (x^T (A + A^T) v + \underline{v^T A v} - D \|$

$D_x f = \underline{\langle (A + A^T) x, v \rangle}$

representation of $D_x f$

$\| v^T A v \| = o(\|v\|)$

① $f(x) = a + g^T x$

$\nabla f = g$

② $f(x) = x^T A x.$

$\nabla f = (A + A^T) x$

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$D_x \nabla f = A + A^T$

$\| f(x+v) - f(x) - D_x f(v) \|$

$= \| (x+v)^T A (x+v) - x^T A x - D \|$

$= \| x^T (A + A^T) v - D \|$

$D_x f = \langle (A + A^T) x, v \rangle$

representation of $D_x f$

Compute gradient of

$$f(X) = \log \det X$$

where

$$f : S^n_{>0} \to \mathbb{R}$$

$$X \mapsto \log \det X.$$

$f(X) = \mathrm{trace}(X^T X)$

$f : \mathbb{R}^{n \times n} \to \mathbb{R}$

$X \to f(X)$

$\| \mathrm{tr}((X+V)^T(X+V))$

$- \mathrm{tr}\, X^T X - D \|$

$= \| \mathrm{tr}(\cancel{X^T X} + X^T V + V^T X$

$+ V^T V$

$- \cancel{X^T X}) - D \|$

$= \| \mathrm{tr}(X^T V + V^T X) - D \|$

$\mathrm{tr}(2X^T V) - D \quad \langle 2X, V \rangle$

**Excercise:** Hint use $\langle X, Y \rangle = \mathrm{trace}(XY)$