Defn: [Convex set]

S is convex

if $\forall\ x_1, x_2 \in S$

$\boxed{\theta x_1 + (1-\theta) x_2} \in S \quad , \quad \theta \in [0, 1]$

eg: $S = \{ x \mid \|x\|_p \leq 1 \}$

$x \in \mathbb{R}^n$

Take any $x_1, x_2 \in S$

then $\|\theta x_1 + (1-\theta) x_2\|_p \leq \theta \|x_1\|_p + (1-\theta)\|x_2\|_p$

$$\leq \theta + (1-\theta)$$

$$\leq 1$$

$\theta x_1 + (1-\theta) x_2 \in S$

$\square$

eg. 2 $S = \{ x \in \mathbb{R}^n \mid x^T Q x \leq 1,\ Q > 0 \}$

Note

$Q = R^T R$

then $x^T R^T R x = \|Rx\|_2^2$

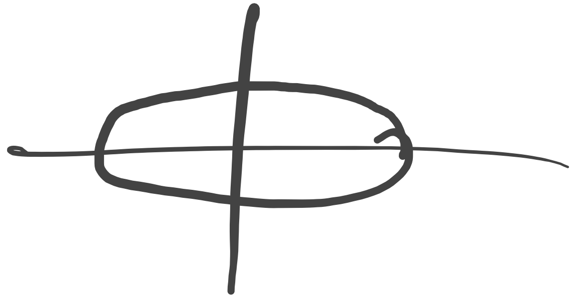
using norm arguments.

S is convex

Ellipsoid.

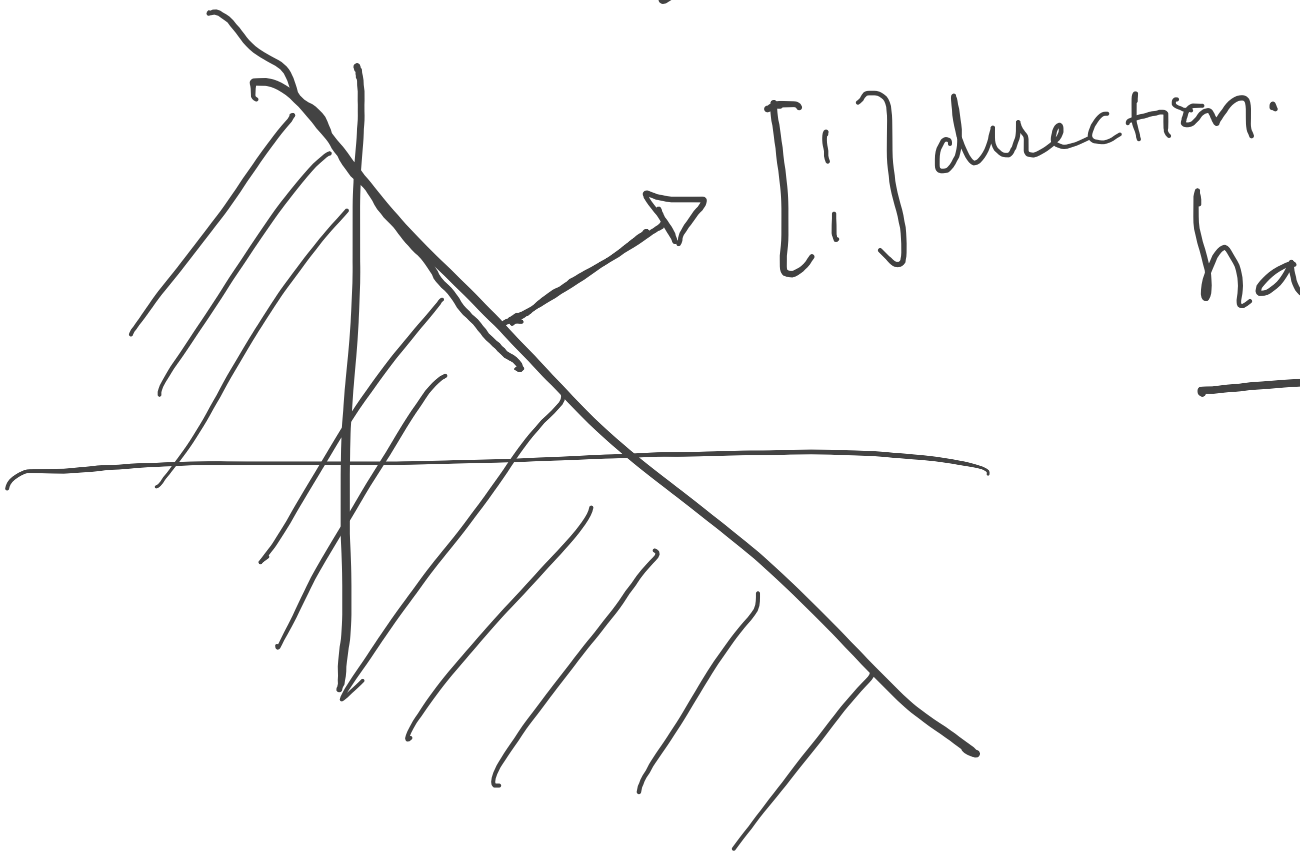
e.g. $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$x_1^2 + 2x_2^2 \leq 1$



eg.3 $S = \{ x \mid a^T x \le b, \ a \in \mathbb{R}^n, \ b \in \mathbb{R} \}$

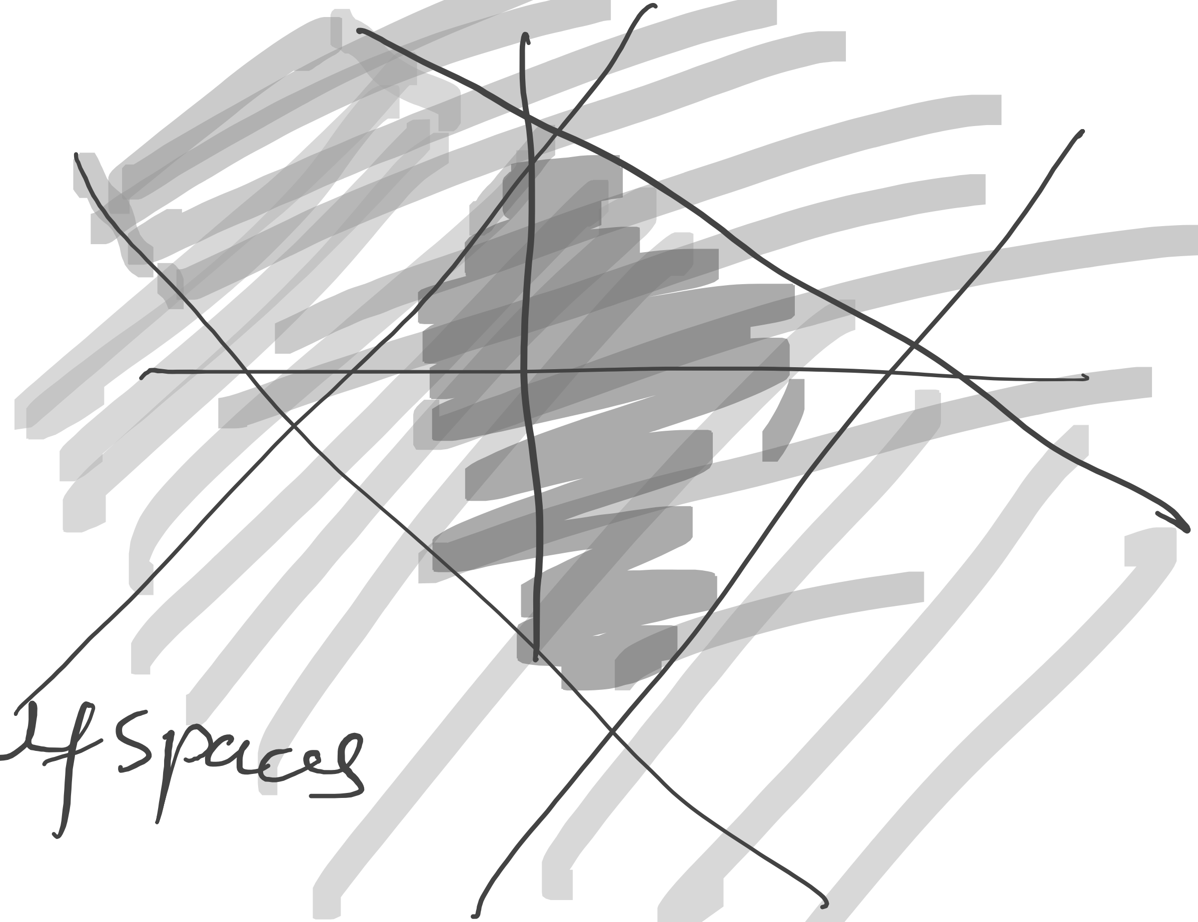
$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ b = 1 \qquad x_1 + x_2 \le 1$



<u>half Spaces.</u>

eg-4

$$\begin{bmatrix} a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ \vdots \\ a_m^T x \le b_2 \end{bmatrix}$$



Intersection of halfspaces is Polyhedral Set.

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A clarification here about mistake made by me in this Class!

① Intersection of half spaces is Called as a Polyhedral Set.

② Polyhedron / polytope is a bounded figure that we are familiar with

③ Polyhedron is a special Polyhedral set i.e. described as an Intersection of half spaces

Def^n [Convex Combination]

given $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$

$$\underline{\underline{x}} = \sum_{i=1}^{m} \boxed{\theta_i x_i} \qquad \theta_i \in [0, 1]$$

$$\sum_{i=1}^{m} \theta_i = 1$$

x is a convex combination of

$x_1, x_2, \cdots, x_m$.

Thm: $\boxed{S \subset \mathbb{R}^n \text{ is convex}}$

if and only if

[all convex combinations of elements of S are in S.]

Proof:

1) ~~S is~~ all convex combinations of S in S [by def^n]

$\Downarrow$

S is convex.

2) S is convex $\Rightarrow$ all convex combinations in S

Base case

$m = 2$, $\theta x + (1-\theta) x_2 \in S \quad \forall x_1, x_2 \in S.$

For

$m = p-1$, $\displaystyle \sum_{i=1}^{p-1} \theta_i x_i \in S \quad \forall x_i \in S$

$\theta_i \in [0, 1]$

$\sum \theta_i = 1$

To show

$m = p$

$\displaystyle \sum_{i=1}^{p} \theta_i x_i \in S$, $\theta_i \in [0, 1]$

$\sum \theta_i = 1$

$$\sum_{i=1}^{p} \theta_i x_i = \theta_1 x_1 + \sum_{i=2}^{p} \theta_i x_i$$

$$= \underbrace{\theta_1 \underbrace{x_1}_{\in S}} + \left(\sum_{j=2}^{p} \theta_j\right) \underbrace{\left[\sum_{i=2}^{p} \left(\frac{\theta_i}{\sum_{j=2}^{p} \theta_j} \right) x_i \right]}_{\in S} \in S$$

$$\sum_{i=2}^{p} \theta_i = (1 - \theta_1)$$

[p-1 pts. Convex Combination]

$\square$

$$\sum_{i=2}^{p} \frac{\theta_i}{\sum_{j=2}^{p} \theta_j} x_i$$

$$\theta_2, \theta_3, \ldots, \theta_p$$

$$\left(\frac{\theta_i}{\theta_2 + \theta_3 + \ldots + \theta_p} \right) \quad i = 2, \ldots, p$$

Thm: $S_1, \; S_2 \subseteq \mathbb{R}^n$ convex

$$\Downarrow$$

$S_1 \cap S_2$ is convex

Proof: Take any $x_1, x_2 \in S_1 \cap S_2$

then $x_1 \in S_1, \quad x_1 \in S_2$
$x_2 \in S_1, \quad x_2 \in S_2$
$\Big] \Rightarrow$

$\theta x_1 + (1-\theta) x_2 \in S_1$

$\theta x_1 + (1-\theta) x_2 \in S_2$

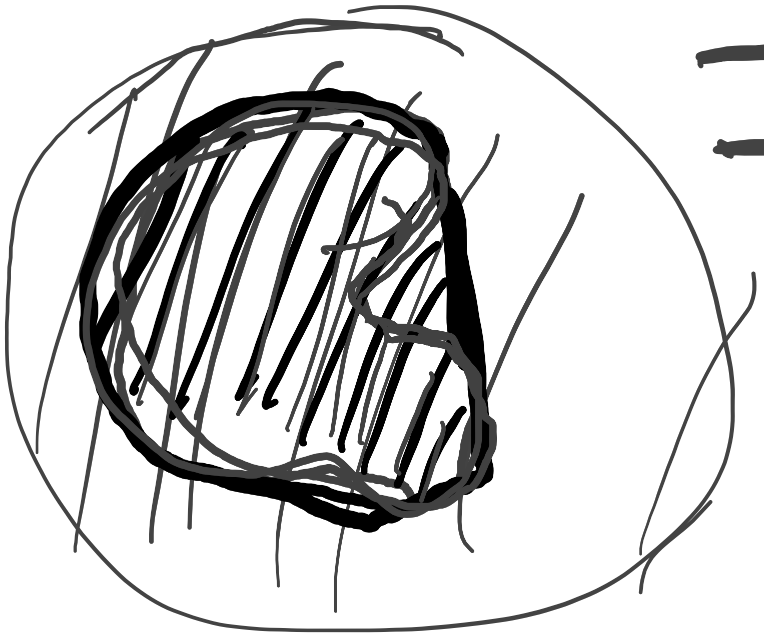
$$\Downarrow$$

$\theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2$

$\square$

Defn: $S \subset \mathbb{R}^n$ be any set.

Convex Hull of S is the smallest convex set that contains S.



- $\supseteq S$
- $M \supseteq S$ convex
- $\mathrm{Conv}(S)$

$\boxed{\mathrm{Conv}(S) \text{ denotes convex hull of } S.}$

Thm: $\boxed{\text{Conv}(S)}$ = {Set of all Convex Combinations of pts in S}

$= \underline{C(S)}$

$$\underline{C(S)} = \left\{ x = \sum_{i=1}^{p} \theta_i x_i \;\middle|\; x_i \in S,\ \theta_i \in [0,1],\ \sum_{i=1}^{p} \theta_i = 1,\ p \in \mathbb{Z}_+ \right\}$$

Proof:- 1) $S \subset C(S)$ obviously true.

To show that it is smallest convex set.

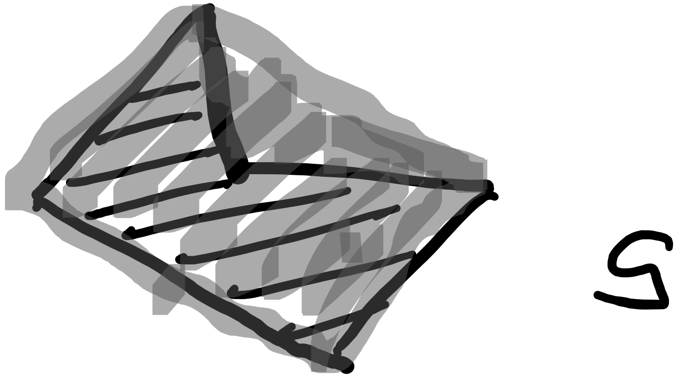
2) If M is a convex set s.t $S \subset M$

and $M \subset C(S)$

$\Downarrow$

$M = C(S)$

Excercise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