Disturbance Decoupling Control for LIGO

1 LIGO Suspension System

Detecting a Gravitational wave requires a very high sensitivity instrument. LIGO detector is an interferometer that is built for this purpose and spans a hugh geographical coverage. High sensitivity requirement also causes disturbances to easily enter the important measurement channels of the interferometer. LIGO interferometer is designed to have suspended mirrors which must be isolated from the ground disturbances for determining the detection of wave with a high degree of confidence. A typical suspended mirror in the LIGO interferometer consists of a quadruple pendulum system as shown in Figure 1. There are four stages, which include two metal masses, one penultimate mass and one test mass. The control actuators are in first stage which can be used to control the motion of all four masses.

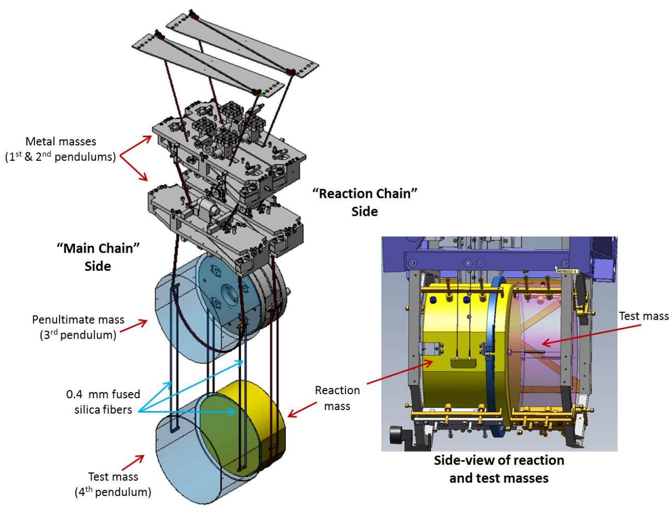

Figure 1: Quadruple Pendulum Based Vibration Isolation. Diagram taken from `https://www.ligo.caltech.edu/page/vibration-isolation`

2 State space model

We assume that measurements from all the masses are available. In the linear regime the overall state space representation is written in the form of following equations which are in standard form from where several existing results pertaining to the disturbance decoupling and rejection theory such as H_∞ or H_2 norm minimization based control could be utilized.

$$\begin{aligned}
&\dot{x} = Ax + Bu + Ed, \text{ governing ODE} \\
&y = Hx, \text{variable of interest} \\
&z = Cx, \text{ measured variable} \\
&A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p} \\
&C \in \mathbb{R}^{n\times m}, H \in \mathbb{R}^{n\times r} \\
&E \in \mathbb{R}^{n\times l}
\end{aligned}$$

3 Disturbance Decoupling Problem

With respect to equations discussed above we list the following problems that are of interest to us –

1. Can the matrix A be modified using u and x to decouple y from d?
2. Can we estimate state variables from measured variables z?

The problems mentioned have been tackled in the following papers. Decoupling using linear geometric control is considered in [1] and [2]. Required computational algorithms appear in [3] and are also discussed in textbook by [4]. Further, H_2 and H_∞ norm optimal control synthesis are developed in [5].

We provide a brief review of disturbance decoupling problem. Furthermore, in Section 4, we note that for the data used in [6] it is possible to decouple the displacement of mass 4 from the disturbances.

3.1 Decoupling Condition

For system given by

$$\begin{aligned}
\dot{x} &= Ax + Ed, \\
y &= Hx,
\end{aligned}$$

we have

$$\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Ed(\tau)d\tau, \\
y(t) &= He^{At}x_0 + \int_0^t He^{A(t-\tau)}Ed(\tau)d\tau,
\end{aligned}$$

Lemma 1. *Decoupling is possible if and only if H, A, E satisfy*

$$He^{At}E = 0 \textit{ for all } t \geq 0.$$

(i.e., transfer function from d to y is zero).

Lemma 2. *$He^{At}E = 0$ for all $t \geq 0$ holds if and only if $HA^iE = 0$ for $i = 1, 2.., n-1$.*

Using the condition in lemma 2, define $\mathcal{V} = \operatorname{Im}[E, AE, A^2E, ..., A^{n-1}E]$ and note $A\mathcal{V} \subset \mathcal{V}$, i.e., $\mathcal{V}$ is an A-invariant subspace of $\mathbb{R}^n$. This gives us

Lemma 3. *Decoupling is possible if and only if there exists an A-invariant subspace $\mathcal{V} \subset \mathbb{R}^n$ such that*

$$\operatorname{Im} E \subset \mathcal{V} \subset \ker H.$$

3.2 Disturbance Decoupling State-Feedback

Now consider,

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed, \\ y &= Hx, \end{aligned}$$

Our goal here is to solve the following problem.

Problem 1. *Design $u = Fx$ such that*

$$\dot{x} = (A + BF)x + Ed, y = Hx$$

is disturbance decoupled.

In other words, compute the matrix F such that there exists an $(A + BF)$-invariant set $\mathcal{V}$ such that

$$\operatorname{Im} E \subset \mathcal{V} \subset \ker H.$$

For computing F, we first need to compute a controlled invariant subspace which is defined next.

Definition 4. *$\mathcal{V}$ is a controlled invariant subspace of $\dot{x} = Ax + Bu$ if for all $x_0 \in \mathcal{V}$, it is possible to find $u(t)$ such $x(t) \in \mathcal{V}$ for all $t \geq 0$.*

Theorem 5. *[1] The following statements are equivalent*

1. *$\mathcal{V}$ is controlled invariant.*
2. *$A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B$.*
3. *There exists F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.*

Given $\mathcal{V}$, one can compute F by solving for condition (2) of Theorem 5. All F's that satisfy condition (3) of Theorem 5 are called as "Friends" of $\mathcal{V}$. The space $\mathcal{V}$ that lies inside $\ker H$ and contains $\operatorname{Im} E$ is sought for decoupling. We next give a procedure to compute $\mathcal{V}$.

3.3 Finding largest $\mathcal{V}$ in $\ker H$ [1]

The following iterative procedure terminates in finitely many steps and gives largest controlled invariant subspace contained in $\ker H$

1. Choose $\mathcal{V}_0 = \ker H$.
2. Iterate with $\mathcal{V}_{i+1} = A^{-1}(\mathcal{V}_i + \operatorname{Im} B) \cap \ker H$.

3. Stop if $\mathcal{V}_{i+1} = \mathcal{V}_i$.

4. Return $\mathcal{V}^* = \mathcal{V}_{i+1}$.

5. $\mathcal{V}^*$ is the largest controlled invariant set in $\ker H$.

Then we immediately have following theorem.

Theorem 6. *[1] Disturbance decoupling possible if and only if* $\operatorname{Im} E \subset \mathcal{V}^*$.

3.4 Computing F

To compute a Friend of $\mathcal{V}^*$ which is a state feedback matrix F we follow the steps listed next [1].

1. Given $\mathcal{V}^* = \operatorname{Im} V$.

2. Solve for Q, M such that $AV = VQ + BM$.

3. Solve for F such that $M = FV$.

However, the F computed as above does not ensure stable $(A + BF)$. For guaranteeing stability, a notion of controllability subspace plays key role.

Definition 7. *A subspace is called controllability subspace if it satisfies –*

1. *there exists control input to reach the origin from any initial condition in it, and*

2. *while doing so the trajectory remains inside the subspace.*

Now let $\langle A, B\rangle := \operatorname{Im}[B, AB, A^2B, ..., A^{n-1}B]$. Next theorem gives a procedure to compute the largest controllability subspace in $\ker H$.

Theorem 8. *[1] Let F be a friend of $\mathcal{V}^*$. The largest controllability subspace in* $\ker H$ *is*

$$\mathcal{R}^* = \langle A + BF, \operatorname{Im} B \cap \mathcal{V}^*\rangle.$$

Lemma 9. *Disturbance decoupling possible with stability guarantee if and only if* $\operatorname{Im} E \subset \mathcal{R}^*$.

If the decoupling condition in above lemma is met then following steps (from [1]) allow us to do pole placement.

1. Split $\mathbb{R}^n = \mathcal{R}^* \oplus \mathcal{W}$.

2. Split $\operatorname{Im} B = (\operatorname{Im} B \cap \mathcal{R}^*) \oplus (\operatorname{Im} B \cap \mathcal{W})$

3. Do similarity transform $q = T^{-1}x$ with $T = [\mathcal{R}^*, \mathcal{W}]$

[1]Functions to do the related subspace computations are available at `https://bitbucket.org/deepakp1988/ddp/src/master/ddp.py`

4. We get

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\overline{A}_{11} & \overline{A}_{12}\\ \overline{A}_{21} & \overline{A}_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix}\overline{B}_1 & 0\\ 0 & \overline{B}_2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

5. Set $\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}\overline{F}_{11} & \overline{F}_{12}\\ \overline{F}_{21} & \overline{F}_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}$ to get

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\overline{A}_{11}+\overline{B}_1\overline{F}_{11} & \overline{A}_{12}+\overline{B}_2\overline{F}_{12}\\ \overline{A}_{21}+\overline{B}_2\overline{F}_{21} & \overline{A}_{22}+\overline{B}_2\overline{F}_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}$$

6. Set $\begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}\overline{F}_{11} & \overline{F}_{12}\\ \overline{F}_{21} & \overline{F}_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}$ to get

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\overline{A}_{11}+\overline{B}_1\overline{F}_{11} & \overline{A}_{12}+\overline{B}_2\overline{F}_{12}\\ \overline{A}_{21}+\overline{B}_2\overline{F}_{21} & \overline{A}_{22}+\overline{B}_2\overline{F}_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}$$

7. Choose $\overline{F}_{21}$ s.t. $\overline{A}_{21}+\overline{B}_2\overline{F}_{21} = 0$.

8. And choose F_{11} and F_{22} such that eigenvalues of $\overline{A}_{11}+\overline{B}_1\overline{F}_{11}$ and $\overline{A}_{22}+\overline{B}_2\overline{F}_{22}$ are in $\mathbb{C}^-$

4 Example

We take data from the paper [6] and consider the following system.

$$\dot{x} = Ax + Bu + Ed, y = Hx.$$

where

$$A = \begin{bmatrix}0_{4\times 4} & I_{4\times 4}\\ A_{22} & 0_{4\times 4}\end{bmatrix}, A_{22} = \begin{bmatrix}-297.30 & 163.50 & 0.00 & 0.00\\ 162.90 & -267.20 & 104.20 & 0.00\\ 0.00 & 57.80 & -74.20 & 16.40\\ 0.00 & 0.00 & 16.40 & -16.40\end{bmatrix}, B = \begin{bmatrix}0_{4\times 4}\\ B_2\end{bmatrix}$$

$$B_2 = \begin{bmatrix}0.05 & 0.00 & 0.00 & 0.00\\ 0.00 & 0.04 & 0.00 & 0.00\\ 0.00 & 0.00 & 0.03 & 0.00\\ 0.00 & 0.00 & 0.00 & 0.25\end{bmatrix},$$

$$E^\top = \begin{bmatrix}0.00 & 0.00 & 0.00 & 0.00 & 131.40 & 0.00 & 0.00 & 0.00\end{bmatrix},$$

$$H = \begin{bmatrix}0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00\end{bmatrix}$$

The codes for the computation done are available at `https://bitbucket.org/deepakp1988/ddp/src/master/`. For placing the closed loop poles at

$$\begin{bmatrix}-1.00 & -1.50 & -2.00 & -2.50 & -3.00 & -3.50 & -4.00 & -4.50\end{bmatrix},$$

we set $u = Fx$ with

$$F = \begin{bmatrix}6413.42 & -3536.27 & 5.07 & 0.00 & -74.02 & 8.01 & 2.44 & -0.00\\ -3650.35 & 5777.25 & -2322.21 & -0.00 & -13.52 & -124.45 & -2.77 & 0.00\\ 8.39 & -2317.81 & 2788.25 & 0.00 & 4.45 & -2.13 & -179.79 & -0.00\\ -0.00 & -0.00 & -65.60 & -6.40 & -0.00 & 0.00 & -0.00 & -34.00\end{bmatrix}$$

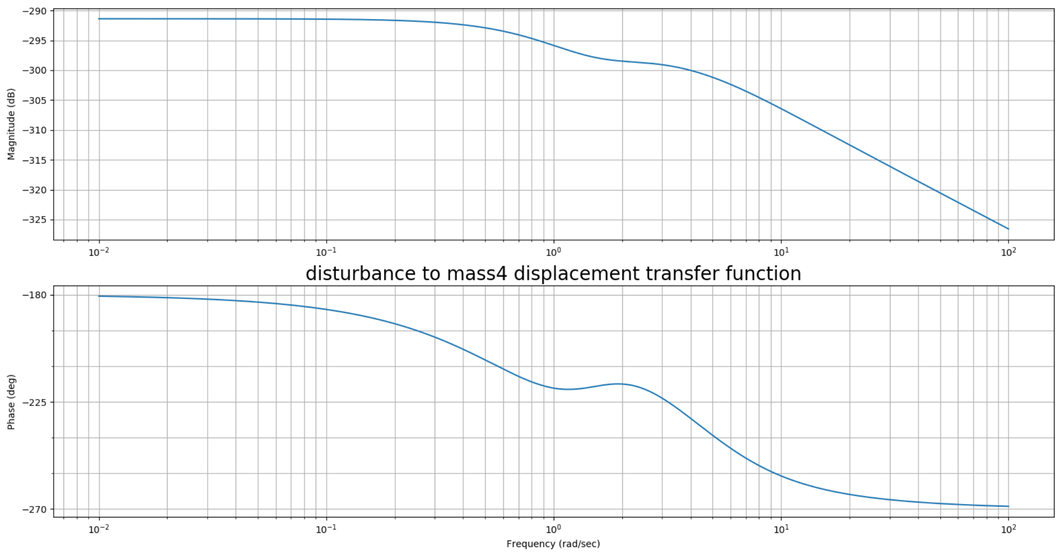

Figure 2: Transfer function from disturbance to displacement at stage 4

Then the transfer function from disturbance to displacement at Stage 4 is shown in Figure 2. The transfer function shows that displacement at Stage 4 is decoupled from the disturbance signal. The maximum amplitude in the Bode plot in Figure 2 is -290 dB.

Further, for simulating the displacement at Stage 4 versus time, we choose a disturbance signal which is a sine wave of 10 Hz and amplitude 0.01 units. The plot of displacement at Stage 4 versus time is seen to be decoupled from the disturbance in Figure 3. Note that displacement at stage 4 is decoupled from the disturbance signal. We also note from Figure 4 that displacements at Stages 1,2, and 3 remain bounded.

Now, though the decoupling is achieved here, it needs to be checked if the gain values in the feedback matrix are of practically appropriate magnitude. Thus, in the future we also would like to put constraints on the gain values while solving the problem.

5 Conclusions

Following are the outcomes:

1. It is seen that for the experiment data from the paper [6], the decoupling condition in Lemma 9 is met and we are able to do a state feedback based decoupling with pole placement. The method is such that all the disturbance effects are absorbed in a subspace which is orthogonal to output matrix.

2. However, the control law obtained from the algorithm for disturbance decoupling in continuous time is usually implemented through a digital state feedback controller, which will cause performance degradation. In future, we propose to develop a self-triggered control strategy for the digital controller so that the norm of the output of the system under the disturbance input remains bounded within a certain tolerance

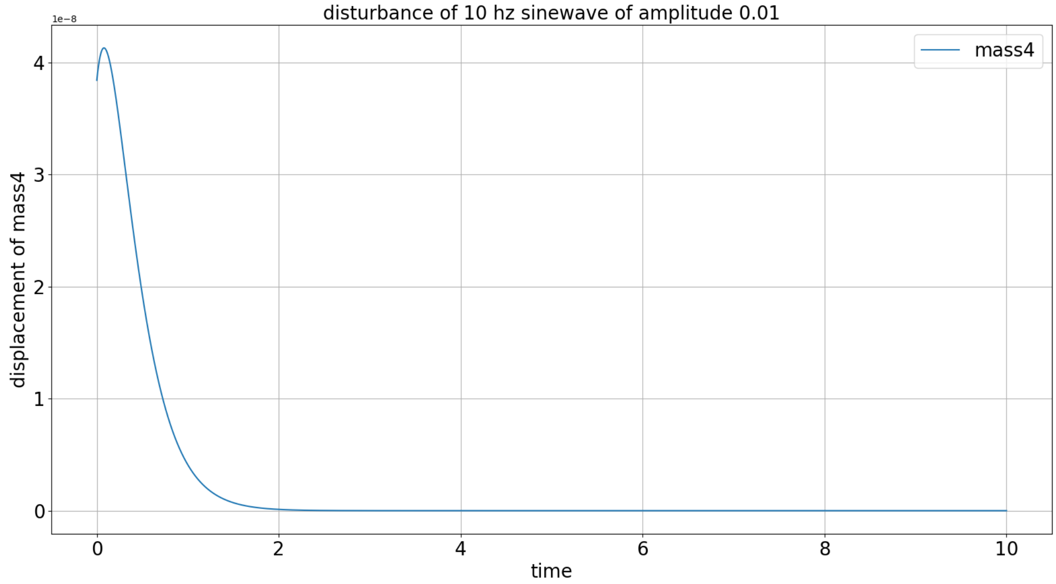

Figure 3: Displacement at stage 4 v/s time

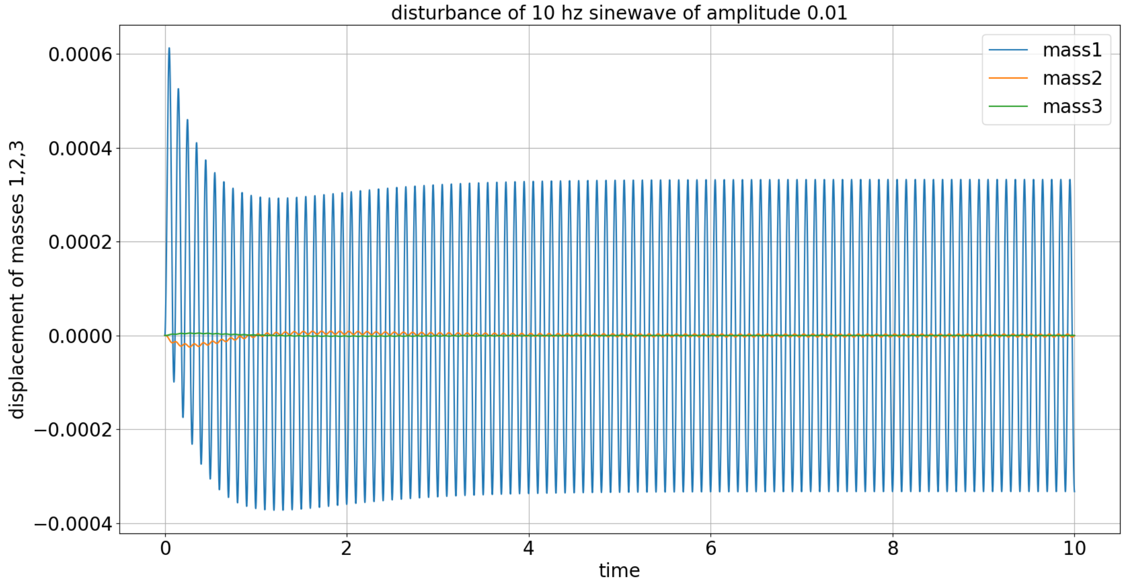

Figure 4: Displacement at stages 1,2,3 v/s time

for all time. We already have partial results for this problem in the form of a heuristic procedure.

3. We have developed a disturbance decoupling library in Python for doing the calculations required in this report. The library is available at the link `https://bitbucket.org/deepakp1988/ddp/src/master/ddp.py`.

A few more related questions that we propose to work on in the future are listed below:

1. Pole placement with actuator and performance constraints.
2. Disturbance Decoupling in Non-linear settings.
3. Robustness against parameter variations and uncertainty.
4. Estimating state variable from a measured variable with unknown disturbance signal.
5. Adaptive decoupling.
6. Numerical issues.

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