Computation of Feedback Control for Time Optimal State Transfer using Groebner Basis

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Abstract

Computation of time optimal feedback control law for a controllable linear time invariant system with bounded inputs is considered. Unlike a recent paper by the authors, the target final state is not limited to the origin of state-space, but is allowed to be in the set of constrained controllable states. Switching surfaces are formulated as semi-algebraic sets using Groebner basis based elimination theory. Using these semi-algebraic sets, a nested switching logic is synthesized to generate the time optimal feedback control. However, the optimal control law enforces an unavoidable limit cycle in the time-optimal trajectory, for most non-origin target points. The time-period of this limit-cycle is dependent on the target position. This dependence is algebraically characterized and a method to compute the time-period of the limit-cycle is provided. As a natural extension, the set of constrained controllable states is also computed.

1. Introduction

Recent advances in computer algebra packages have facilitated application of tools from algebraic geometry (in particular Groebner basis) to solve important problems in control theory (see [1, 2, 3]). In this work, we apply Groebner basis methods to the problem of computation of feedback control (from a constrained set $|u| \leq 1$) for the minimum time state-transfer of a linear time invariant (LTI) system. The Pontryagin's minimum principle (PMP) provides the open loop optimal solution, according to which,

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the input switches between extreme admissible values ± 1 [4]. The feedback solution, however, requires knowledge of the so called *switching surfaces* in the state-space. If the equations for the switching surfaces (in terms of the state variables) can be computed, then it is possible to synthesize the optimal feedback using the sign of the current state with respect to switching surface. A partial solution to this synthesis problem was provided in [5] for a class of systems with positive eigenvalues and later on extended to non-zero distinct rational eigenvalues in [6]. In these papers, a time optimal feedback control law was synthesized (using Groebner basis) which transfers admissible initial states to the *origin* and keeps the state-trajectory arbitrarily close to the origin. In this article, we consider a class of systems with distinct and rational eigenvalues and extend our earlier results from [5, 6] for minimum time state transfer to *non-origin* target points.

Optimal feedback is preferred over open loop solutions for robustness to disturbances as well as reduction in real time computational load. Time optimal feedback control has been employed, among others, in space-craft attitude control [7], robotic manipulators [8], pursuit evasion games [9] and multi-agent consensus tracking [10]. Control to non-origin targets is desirable when linearized equations of the corresponding non-linear system are valid over a wide range of operating conditions and tracking of different set points (not necessarily the equilibrium point) may be needed. Such situations occur regularly in a variety of applications (e.g. see [11] and references therein). The analytical foundations for the switching surfaces in time optimal control was laid out in [12, 13] and later extended for non-origin target points in [14, 15, 16].

In this article, we use algebraic geometry techniques (i.e., Groebner basis based implicitization algorithm) for computing the semi-algebraic representation of switching surfaces (dependent purely on state-variables). We also construct a nested switching logic using the computed expressions for these surfaces. Interestingly, unlike the earlier case (i.e., [5, 6]), the state-trajectory is forced into a limit cycle about the target by the optimal feedback law for most non-origin target points. The period of this limit cycle is dependent on the position of the target point in state-space. We characterize the time period of the limit cycles thus created by the optimal control strategy. A preliminary version of this article was presented in [10].

2. Preliminaries and Analysis

We consider a linear time invariant system with n-dimensional state-space:

$$\dot{x}(t) = Ax(t) + bu(t); \quad x(0) = x_0 \tag{1}$$

where $x : [0, \infty) \to \mathbb{R}^n$ is the state vector and the input $u(t)$ is a measurable function $u : [0, \infty) \to \mathbb{R}$ which belongs to a set of admissible inputs $U = \{u \in L^\infty[0, \infty) : |u(t)| \leq 1 \text{ a.e. } t \in [0, \infty)\}$. We assume that the eigenvalues of matrix A are distinct and rational and the pair (A, b) is controllable. Let us denote by X_p the set of initial conditions from which system (1) can be transferred to $p \in \mathbb{R}^n$ in some time $t > 0$. We assume access to state-variables and compute the time optimal feedback law $h : X_p \to [-1, 1]$ to transfer any initial condition $x_0 \in X_p$ to the target point p. The eigenvalues of A are assumed to be distinct and rational. The *reachable set at time* $t > 0$ (denoted by $R_p(t)$) to point p is the set of all the points $x \in \mathbb{R}^n$ that can be driven to p in time t using the admissible control $u(t) \in U$ [16].

$$R_p(t) = \left\{ e^{-At}p - \int_0^t e^{-A\tau} bu(\tau)d\tau : u(t) \in U \right\} \tag{2}$$

The set of all the states that can be driven to p using $u(t) \in U$ is called the *reachable set* to the point p and is $X_p = \bigcup_{t \in [0,\infty)} R_p(t)$. When p is the origin, we get the set of initial conditions $R_0(t)$ that can be driven to the origin with input $u(t) \in U$ in time t: $R_0(t) = \left\{ \int_0^t e^{-A\tau} bu(\tau)d\tau : u(t) \in U \right\}$. Then, the *set of null-controllable states* [12] is given by $X_0 = \bigcup_{t \in [0,\infty)} R_0(t)$. From point p the *attainable set* at time t is the set of all the states that can be reached from the point p using admissible control $u(t) \in U$ in time $t > 0$ [16].

$$\mathcal{A}_p(t) = \left\{ e^{At}p + \int_0^t e^{A(t-\tau)} bu(\tau)d\tau : u(t) \in U \right\} \tag{3}$$

Taking the union of set (3) for all time $t > 0$ we get the *attainable set* from the point p which is $\mathcal{A}_p = \bigcup_{t \in [0,\infty)} \mathcal{A}_p(t)$ [16].

A point $p \in \mathbb{R}^n$ is said to be *a constrained controllable* for system (1), if it is in the interior of X_p i.e. $x \in \text{int } (X_p)$. In other words all points in the neighborhood of a constrained controllable point p must be transferable to p. We restrict ourselves by considering the target states which are constrained controllable points because of following reason. If a point p is not a constrained controllable point then for some initial conditions in the neighbourhood of p it is impossible to *transfer* the state-trajectory to target point p. Thus, such a target point is not useful for practical purposes. Hence-forth we assume p to be a constrained controllable point.

Next we review the notion of switching surfaces, which will be used subsequently to construct the time optimal feedback law. It is well known that the optimal control for minimum time state-transfer is bang-bang with at most $n-1$ switches [4]. Let $M^{+}_{p,k}$ be the set of initial conditions which can be steered to the target p using bang-bang input with at most $k-1$ switches and initial input $u = 1$. To characterize the set $M^{+}_{p,k}$, we first define the set $V_k := \{(t_1, t_2, ..., t_k) : 0 \leq t_1 \leq t_2 \leq ... \leq t_k < \infty\}$ for $k = 1, ..., n$. Next, we use (2) to define functions $F^{+}_{p,k}$ and $F^{-}_{p,k}$ as follows.

$$F^{\pm}_{p,k} : V_k \to \mathbb{R}^n$$
$$F^{\pm}_{p,k} : (t_1, ..., t_k) \mapsto e^{-At_k} p \pm \left(-\int_0^{t_1} + ... + (-1)^k \int_{t_{k-1}}^{t_k} \right) e^{-A\tau} b d\tau \tag{4}$$

Then, $M^{+}_{p,k} = \{x : x = F^{+}_{p,k}(v), \ \ \forall v \in V_k\}$ and similarly, $M^{-}_{p,k}$ is the set of such conditions with initial input $u = -1$ defined by $M^{-}_{p,k} = \{x : x = F^{-}_{p,k}(v), \ \ \forall v \in V_k\}$. Thus the set of all states that can be driven to p in atmost $k-1$ switches is defined as

$$M_{p,k} = M^{+}_{p,k} \cup M^{-}_{p,k} \tag{5}$$

The sets $M_{p,k}$ for $k = 1, ..., n$ obey the inclusion relation $M_{p,0} \subset M_{p,1} \subset \ldots \subset M_{p,n}$. From the existence and uniqueness of time optimal control to transfer any point in X_p to the target point p (see [4]), we get $M_{p,n} = X_p$. In other words, the reachable set to point p using only bang-bang input (with at most $n-1$ switches) is in fact the entire reachable set to p (i.e. X_p) and hence by (5), $X_p = M^{+}_{p,n} \cup M^{-}_{p,n}$.

3. Using $M_{p,k}$ for Feedback

The nested sequence of sets $M_{p,k}$ $(k = 1, ..., n-1)$ can be used to synthesize time optimal feedback control. Assume $p = 0$, temporarily, to simplify the following explanation. To drive the system from any state in X_0 to $p = 0$ in minimum time, the control value must ideally switch as follows. Consider $x_0 \in M^{+}_{0,n}$. Then initially, input $u = +1$ should be applied, which pushes x from $M^+_{0,n}$ to the manifold $M^-_{0,n-1}$. As soon as $x \in M^-_{0,n-1}$, the input should switch to $u = -1$, which then pushes x to $M^+_{0,n-2}$ and so on. Finally input at $(n-1)^{th}$ switch pushes x from $M^+_{0,1}$ (if n is odd) or $M^-_{0,1}$ (if n is even) to the target $p = 0$. Similar sequence is valid for $x_0 \in M^-_{p,n}$ with opposite signs of input [6]. This switching algorithm works ideally for $p = 0$, because the sets $M^+_{0,n} \setminus M^+_{0,n-1}$ and $M^-_{0,n} \setminus M^-_{0,n-1}$ are disjoint and $M^+_{0,n} \cap M^-_{0,n} = M_{0,n-1}$ is a set of measure zero in M_n (see [12]). Hence the initial input is determined uniquely from whether $x_0 \in M^+_{0,n}$ or $x_0 \in M^-_{0,n}$.

However, for most non-origin target points (i.e., $p \neq 0$), the set $M^+_{p,n} \cap M^-_{p,n}$ is of non-zero measure in $M_{p,n}$. This implies that points in the set $M^+_{p,n} \cap M^-_{p,n}$ can be driven to p by more than one, distinct bang-bang inputs with at most $n-1$ discontinuities. However, uniqueness of time-optimal control guarantees that only one of them is time-optimal. All points in the set X_p other than those which lie in the set $M^+_{p,n} \cap M^-_{p,n}$, belong to exactly one of the sets $M^+_{p,n}$ or $M^-_{p,n}$. All the initial conditions $x_0 \in X_p \setminus (M^+_{p,n} \cap M^-_{p,n})$ can be driven to p in minimum time by using the time optimal switching (exactly as in $p = 0$ case). The situation is demonstrated by an example next.

Example 1. For a second order LTI system (1) with $A = diag(1, 2)$, $b = [\ 1 \quad 1\ ]^T$ and $|u| \leq 1$, the corresponding structure of X_p with $p = \begin{bmatrix} 0.33 & 0 \end{bmatrix}^T$ is shown in Figure 1. This figure illustrates that $M^+_{p,2} \cap M^-_{p,2}$ is non-empty. For an initial condition $x_0 \in M^-_{p,2} \setminus (M^+_{p,2} \cap M^-_{p,2})$ shown in the figure, input $u = -1$ steers the state-trajectory $x(t)$ towards $M^+_{p,1}$. As soon as $x(t) \in M^+_{p,1}$, the input $u = +1$ directs the state-trajectory towards p.

For initial conditions $x_0 \in M^+_{p,n} \cap M^-_{p,n}$, we have two choices of initial control namely $u = +1$ and $u = -1$. Only one of them drives the system to p in minimum time. This initial choice can be computed beforehand by using PMP. However after the initial input has been applied, all the future control inputs are computed as per the algorithm proposed. If choice of initial input goes wrong, although time-optimality will be lost, the target point will be achieved in finite time.

Before giving an algorithm to compute $M_{p,k}$, we note that, for any real similarity

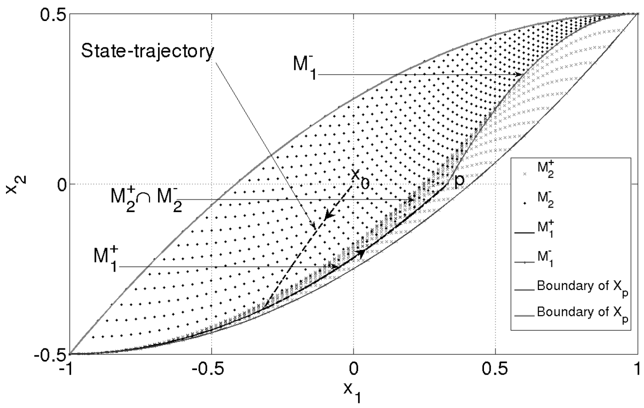

Figure 1: $X_0 = M_2^+ \cup M_2^-$ for case of Example 1

transformation $\widehat{x} = Tx$ on a system, the corresponding set $\widehat{M}_{p,k}^{\pm} = \{\widehat{x} = Tx : \ x \in M_{p,k}^{\pm}\}$[12]. As system matrix A has distinct rational eigenvalues, we assume A to be diagonal without loss of generality.

4. Parametric Representation

All $x \in M_{p,k}$ are characterized by the functions $F_{p,k}^{+}$ or $F_{p,k}^{-}$ (given by (4)) defined over V_k. Note that A is assumed to be diagonal. To use $M_{p,k}$ for state based switching we must eliminate variables $t_1, ..., t_k$ from (4). For that purpose we first convert the expressions defining $M_{p,k}$ to polynomial and rational functions by suitable substitution of variables. First we assume that all the eigenvalues of matrix A are non-zero. However, if any eigenvalue turns out to be zero then a similar process is valid, albeit with a slight modification which is discussed later in this section. Note that since the eigenvalues of A are assumed to be distinct, repeated zero eigenvalues are not allowed.

4.1. All the eigenvalues of A are non-zero:

We rewrite the state variables $x_i, \ \ i = 1, ..., n$ defined by (4) as: $x_i = f_{p,k,i}^{\pm}(e^{-\lambda_i t_1}, ..., e^{-\lambda_i t_k})$ for $i = 1, \ldots, n$ where $\lambda_i \in \lambda(A)$. Here $f_{p,k,i}^{\pm}$ denotes the i^{th} component of function $F_{p,k}^{\pm}$ defined by (4). Also, since A is assumed to be diagonal $f_{p,k,i}^{\pm}$ depends only on the i^{th} eigenvalue of A i.e., λ_i. Let the first q eigenvalues $\lambda_1, ..., \lambda_q$ be positive and the remaining $n - q$ eigenvalues $\lambda_{q+1}, ..., \lambda_n$ be negative. Since $\lambda_i \in \mathbb{Q}$, we write $\lambda_i = \frac{n_i}{d_i}$ for $i = 1, ..., n$, where n_i and d_i are co-prime. Let $l = lcm(d_1, ..., d_n)$ be the least common multiple of $d_1, ..., d_n$. Then by substituting $z_j = e^{-\frac{t_j}{l}}$ $\forall j = 1, ..., k$ we get, $x_i = f^+_{p,k,i}(z_1^{p_i}, \ldots, z_k^{p_i})$, $i = 1, ..., q$ and $x_i = f^+_{p,k,i}(z_1^{-p_i}, \ldots, z_k^{-p_i})$, $i = q+1, ..., n$. Here $p_i = |\lambda_i l|$. Thus x_i's for $i = 1, ..., q$ are now polynomials in $z_1, ..., z_k$ and for $i = q+1, ..., n$, x_i's are rational functions in $z_1, ..., z_k$. Hence, in general, we write $x_i = \frac{N^+_{p,k,i}(z_1,...,z_k)}{D^+_{p,k,i}(z_1,...,z_k)}$, $i = 1, ..., n$ where $\frac{N^+_{p,k,i}}{D^+_{p,k,i}} = f^+_{p,k,i}$ and $D^+_{p,k,i} = 1$ for $i = 1, ..., q$. This gives rational parametric representations of $M^+_{p,k}$ and $M^-_{p,k}$:

$$
\begin{aligned}
M^+_{p,k} =& \{(x_1, \ldots, x_n) : x_i = \frac{N^+_{p,k,i}(z_1, ..., z_k)}{D^+_{p,k,i}(z_1, ..., z_k)} \\
& \forall i = 1, \ldots, n, \;\; 0 < z_k \leq z_{k-1} \leq \ldots \leq z_1 \leq 1\}
\end{aligned} \tag{6}
$$

Note that the eigenvalues of A can be of mixed signs. Specifically if all eigenvalues of A have same signs then by appropriately substituting either $z_j = e^{-\frac{t_j}{l}}$ or $z_j = e^{\frac{t_j}{l}}$, we can obtain a polynomial parametric representation where $D^+_{p,k,i} = 1$ for all $i = 1, .., n$. The variables $z_1, ..., z_k$ must be eliminated from (6) to represent $M^+_{p,k}$ and $M^-_{p,k}$ purely in terms of the state variables $x_1, ..., x_n$. Such a representation can then be used directly for a state-feedback based switching of the input values between ± 1. For that we use the Groebner basis based elimination method which is described in Section 5.1.

4.2. One eigenvalue of A is zero:

Without loss of generality we assume that zero eigenvalue corresponds to the state x_n. The first $n-1$ state variables are rewritten identically as described above. But, the remaining state variable x_n is a linear function in variables $t_1, ..., t_k$ given by $x_n = \pm(2t_1 - 2t_2 + ... + (-1)^{k-1} 2t_{k-1} + (-1)^k t_k)$. For the sake of simplicity we assume k to be even. Then, $x_n = +(2t_1 - 2t_2 + ... + (-1)^{k-1} 2t_{k-1} + t_k$. At this stage, since x_n is linear in its arguments, we eliminate one variable (say t_1) from variables $t_1, ..., t_k$ as follows:

1. Substitute $t_1 = \frac{1}{2}(x_n - \sum_{j=2}^{k-1} 2(-1)^{j-1} t_j + (-1)^{k-1} t_k)$ in the functions describing x_i for all $i = 1, ..., n-1$.

2. Then x_i for $i = 1, ..., n-1$ becomes

$$
x_i = f^+_{p,k,i}\left(\frac{e^{-\frac{1}{2}\lambda_i x_n} \prod_{q=1}^{\frac{k-2}{2}} e^{-\lambda_i t_{2q+1}}}{e^{-\frac{1}{2}\lambda_i t_k} \prod_{q=1}^{\frac{k-2}{2}} e^{-\lambda_i t_{2q}}}, e^{-\lambda_i t_2}, ..., e^{-\lambda_i t_k} \right) \tag{7}
$$

Recall that, $\lambda_i \in \mathbb{Q}$. Hence we write $\lambda_i = \frac{n_i}{d_i}$ for $i = 1, ..., n-1$. Let $l = lcm(d_1, ..., d_{n-1})$ be the least common multiple of $d_1, ..., d_{n-1}$. We convert the above function (defined by (7)) into polynomials or rational function by substituting $z_j = e^{-\frac{t_j}{l}}$ $\forall j = 2, ..., k-1$ and $w = e^{-\frac{x_n}{l}}$. After substitution, $x_i = f^+_{p,k,i}\left(\frac{w^{-\frac{1}{2}p_i}\prod_{q=1}^{\frac{k-2}{2}} z_{2q+1}^{-p_i}}{z_k^{-\frac{1}{2}p_i}\prod_{q=1}^{\frac{k-2}{2}} z_{2q}^{-p_i}}, z_2^{p_i}, \ldots, z_k^{p_i}\right)$, $i = 1, ..., q$ and $x_i = f^+_{p,k,i}\left(\frac{w^{\frac{1}{2}p_i} z_k^{-\frac{1}{2}p_i}\prod_{q=1}^{\frac{k-2}{2}} z_{2q}^{-p_i}}{\prod_{q=1}^{\frac{k-2}{2}} z_{2q+1}^{-p_i}}, z_2^{-p_i}, \ldots, z_k^{-p_i}\right)$ for $i = q+1, ..., n-1$. Here $p_i = |\lambda_i l|$ is a positive integer for $i = 1, ..., n-1$. Thus, we write,

$$x_i = \frac{N^+_{p,k,i}(z_2, ..., z_k, w)}{D^+_{p,k,i}(z_2, ..., z_k, w)}, \qquad i = 1, ..., n-1 \tag{8}$$

and,

$$x_n = (-2\log z_1 + ... + (-1)^{k-2} 2\log z_{k-1} + 2\log z_k) \tag{9}$$

where $w = e^{-\frac{x_n}{l}}$. Similar, expression for x_i easily follows for an odd numbered k. Note that (9) is not in polynomial or rational form. However the state variables x_i for $i = 1, ..., n-1$ are in rational parametric form in variables $z_2, z_3, ..., z_k, w$ (see (8)). This gives the following parametric representations of $M^+_{p,k}$ and $M^-_{p,k}$:

$$\begin{aligned} M^+_{p,k} = &\{(x_1, \ldots, x_n) : \text{ (8) and (9) hold} \\ &\text{and} \quad 0 < z_k \leq \ldots \leq z_2 \leq 1\} \end{aligned} \tag{10}$$

Since we have obtained (8) by already eliminating t_1 (i.e. z_1) we now proceed to eliminate variable $z_2, z_3, .., z_k$ from (8) to form an implicit representation involving only state variables $x_1, ..., x_{n-1}$ and $w = e^{-\frac{x_n}{l}}$.

Remark 2. Note that the set $M^+_{p,k}$ is not algebraic, since the terms in (9) are logarithmic. Now (8) is obtained by eliminating the variable t_1, and we only consider (8) for future computations. If we treat w as a separate variable, (8) can be considered as a rational function in variables $z_2, z_3, ..., z_k, w$. Hence, the Groebner basis computation required for implicitly describing switching surfaces remains identical to the previous case.

5. Switching Surface and Switching algorithm

The computation of switching surfaces by elimination (using Groebner basis) of variable $z_1, ..., z_{n-1}$ from the parametric description of the set $M^{\pm}_{p,n-1}$ was presented in [5] and [6]. Here, we recount the elimination procedure for the sake of completeness.

5.1. Implicitization

We have following parametric representation for switching surface $M^{+}_{p,n-1}$:

$$x_i = \frac{N^{+}_{p,n-1,i}(z_1, ..., z_{n-1})}{D^{+}_{p,n-1,i}(z_1, ..., z_{n-1})}, \qquad i = 1, ..., n \tag{11}$$

with $z_1, ..., z_{n-1}$ satisfying the inequality

$$0 < z_{n-1} \leq z_{n-2} \leq \ldots \leq z_1 \leq 1 \tag{12}$$

First, we follow standard implicitization steps to eliminate the variables z_1,...,z_{n-1} from (11) (see [17]). For simplicity of notation, we drop the subscript p from $N^{+}_{p,n-1}$ and $D^{+}_{p,n-1}$. The procedure is as follows.

1) Using (11) form an ideal $J^{+}_{p,n-1} = \langle D^{+}_{n-1,1}x_1 - N^{+}_{n-1,1}, \ldots, D_{n-1,n}x_n - N^{+}_{n-1,n}, 1 - D^{+}_{n-1,1}D^{+}_{n-1,2}...D^{+}_{n-1,n}y\rangle$. The additional element $1 - D^{+}_{n-1,1}D^{+}_{n-1,2}...D^{+}_{n-1,n}y$ is added in order to prevent the denominators $D^{+}_{n-1,1}$,..., $D^{+}_{n-1,n}$ from becoming zero. Note that $J^{+}_{p,n-1} \subset \mathbb{Q}[y, z_1, ..., z_{n-1}, x_1, ..., x_n]$.

2) Compute Groebner basis $G^{+}_{p,n-1}$ of $J^{+}_{p,n-1}$ w.r.t. lexicographic ordering as $y \succ z_1 \succ z_2 \succ ... \succ z_{n-1} \succ x_1 \succ \ldots \succ x_n$.

3) The element $g^{+}_{p,n-1} \in G^{+}_{p,n-1} \cap \mathbb{Q}[x_1, \ldots, x_n]$ (i.e., the element in $G^{+}_{p,n-1}$ not involving variables $y, z_1, ..., z_{n-1}$) defines the smallest variety containing the parametric representation (11).

Remark 3. If one eigenvalue of A is zero, we follow exactly the same steps while treating w as a separate variable. We eliminate variables $z_2, ..., z_{n-1}$ from $n-1$ equations and then substitute $w = e^{-\frac{x_n}{l}}$ in the Groebner basis $G^{+}_{p,n-1}$.

Next, we express inequality (12) purely in terms of state-variables x_1,...,x_n by solving for the variables $z_1, ..., z_{n-1}$. The procedure is as follows. In the description of set $M^{+}_{p,n-1}$ let us denote the respective z_i by z^{+}_i $(i = 1, ..., n-1)$. To solve for a parameter z^{+}_k:

1) Compute the Groebner basis (say G^{+}_{p,z_k}) using the lexicographic ordering .. $\succ$ .. $\succ$ $\ldots \succ z_k^+ \succ x_1 \succ \ldots \succ x_n$. Note that in this ordering all parameters other than z_k^+i.e., z_i^+ ($i = 1, ..., n-1$ and $i \neq k$) are ordered higher (and z_k^+ itself is ordered higher than all state variables).

2) Consider the set $G^{+}_{p,z_k} \cap \mathbb{Q}[z_k^+, x_1 \ldots, x_n]$ and let there be m elements denoted by $g^+_{p,j} = 0 \quad \forall j = 1, \ldots, m$ in this set. These elements are polynomials that have pure terms in variables $(z_k^+, x_1, \ldots, x_n)$.

3) Now, if among these m polynomials i.e., $g^+_{p,j} = 0 \quad \forall j = 1, \ldots, m$, at least one polynomial has all the monomials involving z_k^+ with same power then, we express z_k^+ as a closed form expression in terms of $x_1, ..., x_n$.

4) Otherwise, to check whether current state $(x_1, ..., x_n)$ satisfies (12) we solve single variable polynomial equation with z_k^+ as the only variable.

5) Repeat steps 1 to 4 for all parameters $z_1^+, ..., z_{n-1}^+$ and check whether any set of the solutions $z_i^+, i = 1, ..., n-1$ satisfies (12).

Remark 4. Note that all the intensive computations in the procedure (i.e elimination of variables using Groebner basis) are being done off-line. The only on line computation required is that of evaluating multivariate polynomials and solving single variable polynomials (if required). The case of solving single variable polynomials arise only if z_k^+ for any $k = 1, ..., n-1$ does not have a closed form expression in terms of $x_1, ..., x_n$.

5.2. Switching algorithm

Now using the expressions obtained above, we can check if a point x in the state-space belongs to the switching surface $M_{p,n-1}$. We first note that since $M_{p,n-1}$ is a set of measure zero in $M_{p,n}$, generically $x_0 \in M_{p,n} \backslash M_{p,n-1}$. As discussed in section 2 we must first compute the initial input $u(0)$ (which can be computed by checking $x_0 \in M^{\pm}_{p,n} \backslash M_{p,n-1}$). As per the switching logic described in Section 2, ideally, the sign of the input should change as soon the state trajectory has reached $M_{p,n-1}$. Since we now have an expression for $M_{p,n-1}$ in terms of state variables, whether $x \in M_{p,n-1}$ can be checked continuously. Now, the state trajectory will travel along $M_{p,n-1}$ until it hits $M_{p,n-2}$. This will carry on for all $M_{p,k}$ with $k < n-2$. Hence, it appears that for realizing the optimal switching ideally, we need to compute $M_{p,k}$ for all $k = 0, ..., n-1$

and check whether $x(t) \in M_{p,k}$ for $k < n$ instantaneously. This optimal switching law can be stated by the following function

$$h(x) = \begin{cases} +1 & \text{if } x \in M^{+}_{p,k} \backslash M_{p,k-1} \text{ for } k = 1, ..., n-1 \\ -1 & \text{if } x \in M^{-}_{p,k} \backslash M_{p,k-1} \text{ for } k = 1, ..., n-1 \end{cases}$$

However, in simulation or practice, unavoidable delays in computation (say δ) would make the state-trajectory to overshoot $M_{p,n-1}$ and enter $M_{p,n} \backslash M_{p,n-1}$ slightly as soon as the first switch occurs. Now, as the state trajectory is in $M_{p,n} \backslash M_{p,n-1}$, we can reuse $M_{p,n-1}$ for the second switch. This situation repeats for all the $n-1$ switches and the knowledge of $M_{p,n-1}$ is enough to construct the entire switching law. Hence, we use the implicit description of $M_{p,n-1}$ to implement the switching logic using algorithm 1.

Algorithm 1 Switching logic

Require: current $x(t)$ and past $(u(t))$
  if $x(t) \in M_{p,n-1}$ **then**
    $u(t+\delta) = -u(t)$
  else
    $u(t+\delta) = u(t)$
  end if

5.3. Examples

Example 5. Consider system of the form (1) with $A = diag(1, 2, 3)$ and $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. Let the target state be $p = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}^T$. We compute the parametric expression for states using (4). By substituting $z_i = e^{-t_i}$ we convert this parametric expression F_2^+ into a polynomial form. The resulting expression along the inequalities $0 \le z_2 \le z_1 \le 1$ describe the surface M_2^+. Then to eliminate z_1 and z_2 we first form an ideal $J^+ = \langle x_1 - 2z_1 + \frac{9}{10}z_2 + 1, x_2 - z_1^2 + \frac{2}{5}z_2^2 + \frac{1}{2}, x_3 - \frac{2}{3}z_1^3 + \frac{7}{30}z_2^3 + \frac{1}{3} \rangle$ from the polynomials of F_2^+. The Groebner basis G^+ with ordering $z_2 \succ z_1 \succ x_1 \succ x_2 \succ x_3$ for the ideal J^+ is calculated [1]. Due to lack of space we omit the Groebner basis. We select an element from Groebner basis which is only in terms of state variables x_1, x_2 and x_3. Let us denote it by

[1] Computed using a computer algebra package Singular[18]

$g_{p,2}^{+}(x_1, x_2, x_3)$. Now, $g_2^{+}(x_1, x_2, x_3) = 0$ and condition $0 < z_2 \leq z_1 \leq 1$ together describe M_2^{+}. Further, we also select an element from Groebner basis which is linear in z_1. Solving this equation gives $z_1 = -\frac{-\frac{2675}{8014}x_1^3 - \frac{8025}{8014}x_1^2 + \frac{22}{4007}x_1 x_2 - \frac{8003}{8014}x_1 + \frac{22}{4007}x_2 + \frac{168507}{160280}x_3 + \frac{3109}{160280}}{x_1^2 + 2x_1 - \frac{163609}{80140}x_2 - \frac{3329}{160280}}$. Similar expression for z_2 can be found from the Groebner basis of ideal J with ordering $z_1 \succ z_2 \succ x_1 \succ x_2 \succ x_3$. Thus, we get $M_{p,2}^{+} = \{(x_1, x_2, x_3) : g_{p,2}^{+}(x_1, x_2, x_3) = 0,\ 0 < z_2 \leq z_1 \leq 1\}$. Similarly, $M_{p,2}^{-}$ can be calculated. The feedback logic can be implemented as per algorithm 1. We compare the switching instants generated using algorithm 1 with those generated by optimal open loop input computed using PMP for the sample initial condition, $x_0 = \begin{bmatrix} -0.2 & -0.14 & -0.1 \end{bmatrix}^T$. The initial value of the optimal input is $u(0) = +1$ and the optimal open-loop switching instants are $t_1 = 0.0704$ and $t_2 = 0.1953$. At $t_3 = 0.6419$ the system reaches p. We infer that the respective open loop switching instants are matching with the closed loop switching instants.

6. Limit Cycles

For the origin target point, it was shown in [5, 6] that the trajectory can not only be transferred to the origin but also kept within any arbitrarily small neighborhood of the origin by accurate switching. However, for almost all the target points, the trajectory, after passing through the target goes into a limit cycle (about the target). This limit cycle appears due to the difference in the relative directions of the vector fields associated with the two inputs (± 1) at the target point. In this section, we analyze the conditions for occurrence of such a limit cycle and propose an algorithm to characterize the period of the limit cycle.

6.1. Existence of a Limit Cycle

The switching algorithm proposed in Section 5 repeatedly forces the state trajectory back to the target p every time it overshoots p. This gives rise to a limit cycle about the target point p. We illustrate the situation by following example.

Example 6. Consider Example 1 from Section 2 with the target state $p = \begin{bmatrix} 0.1 & 0 \end{bmatrix}^T$. Figure 2 shows the phase diagram corresponding to the state-trajectory starting with the initial condition $x_0 = \begin{bmatrix} 0.2 & 0 \end{bmatrix}^T$.

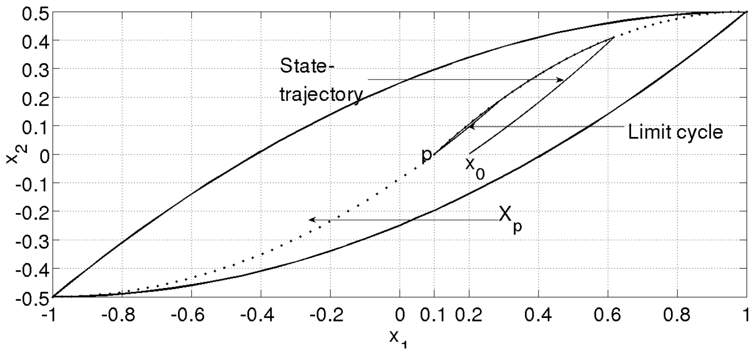

Figure 2: The limit cycle near $p = [0.1\ 0]^T$

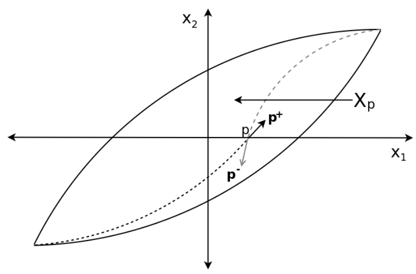

Figure 3: The vector fields corresponding to the two inputs ± 1

From Figure 2, we observe that the state-trajectory after reaching the point p is forced into a limit cycle (with a period of 0.410 seconds). This happens because the switching curve is not smooth at the target p. We illustrate this by Figure 3 where the velocity vectors $\mathbf{p}^+$ and $\mathbf{p}^-$ corresponding to the two inputs ± 1 are not exactly opposed to one another and hence the switching curve is non-smooth at point p. Therefore, after reaching p and overshooting slightly into $M^+_{p,2}$, the state trajectory is forced (by algorithm 1) in the direction of $\mathbf{p}^+$ until it reaches the switching surface again. The next input switch at this instant brings it back to p along the switching curve. This repetition results in a limit cycle.

6.2. Non occurrence of limit cycle for some target points

If the switching curve is smooth at the target point p, then the state-trajectory does not form a limit cycle. We analyze this situation using the concept of average velocity (Chapter 3, Section 1.1 of [19]). Let the input to the system be $u = 1$ for time Δt_1 followed by $u = -1$ for time Δt_2. Now, the change in the state value Δx

in time $\Delta t = \Delta t_1 + \Delta t_2$ is given by $\Delta x = \dot{x}_+ \Delta t_1 + \dot{x}_- \Delta t_2$ (where $\dot{x}_+ = Ax + b$ and $\dot{x}_- = Ax - b$). Assuming $\Delta t_1 = \alpha \Delta t$ for some $\alpha \in (0, 1)$ we get, $\frac{\Delta x}{\Delta t} = \alpha \dot{x}_+ + (1 - \alpha)\dot{x}_-$. We define the *average velocity* as $\dot{x}_{avg} = \alpha \dot{x}_+ + (1 - \alpha)\dot{x}_-$ by taking limit as $\Delta t \to 0$. The state trajectory can be kept in an arbitrarily small neighbourhood of the target point, if and only if there exists $\alpha \in (0, 1)$ such that $\dot{x}_{avg} = 0$ at the target p (Chapter 3, Section 1.1 of [19]). Now, $\dot{x}_{avg} = \alpha \dot{x}_+ + (1 - \alpha)\dot{x}_- = 0$ i.e. $\dot{x}_+$ and $\dot{x}_-$ are linearly dependent and are pointing in opposite direction. Hence, $Ax + b = -\frac{1-\alpha}{\alpha}(Ax - b)$. Therefore all the target points which do not impose limit cycle are characterized by the set $L_0 := \{x | Ax = (1 - 2\alpha)b, \text{ for some } 0 < \alpha < 1\}$. For points not lying in the set L_0, there does not exist any $\alpha \in (0, 1)$ to create $\dot{x}_{avg} = 0$. Therefore, for such points the limit cycle is unavoidable.

6.3. Time period of the Limit Cycle

We now compute the time period of limit cycle for points $x \in X_p \backslash L_0$. By substituting the initial condition $x_0 = p$ in (4), we get $p = e^{-At_n} p \pm (-\int_0^{t_1} + \int_{t_1}^{t_2} - ... + (-1)^n \int_{t_{n-1}}^{t_n} e^{-A\tau} b d\tau)$ and $(I - e^{-At_n})p = F^{\pm}_{0,n}(t_1, ..., t_n)$.

Definition 7. The set of all the target points which gives rise to a limit cycle of time period $t_n \leq T$ is given by, $L_T = \{p | (I - e^{-At_n})p = F^{\pm}_{0,n}(t_1, ..., t_n) : 0 < t_1 \leq ... \leq t_n \leq T\}$.

The boundary of this set L_T is $\partial L_T = \{p | (I - e^{-AT})p = F^{\pm}_{0,n}(t_1, ..., t_{n-1}, T); 0 < t_1 \leq ... \leq t_{n-1} \leq T\}$.

Lemma 8. *For $T_1 > T_2$, $L_{T_2} \subset L_{T_1}$.*

Theorem 9. *L_T is a convex set.*

Proof. We consider two cases. 1) *Matrix A has non-zero eigenvalues:* $R_0(T)$ is a convex set [12]. We observe from the definition of ∂L_T that $(I - e^{-AT})^{-1} \partial R_0(T) = \partial L_T$ and $(I - e^{-AT})^{-1} R_0(T) = L_T$. Also, for all $T_1 < T_2 < T$, $R_0(T_1) \subset R_0(T_2) \subset R_0(T)$ [12]. Now, by convexity of $R_0(T)$, we have

$$\bigcup_{\lambda \in [0,1]} (\lambda \partial R_0(T_1) + (1 - \lambda)\partial R_0(T_2)) = R_0(T_2)$$

Thus,

$$\bigcup_{\lambda \in [0,1]} ((I - e^{-AT_2})^{-1}(\lambda \partial R_0(T_1) + (1 - \lambda)\partial R_0(T_2))) = L_{T_2}$$

and hence,

$$\bigcup_{\lambda\in[0,1]} \lambda(I - e^{-AT_2})^{-1}\partial R_0(T_1) + (1-\lambda)\partial L_{T_2} = L_{T_2} \subset L_T. \quad (13)$$

But, $\partial R_0(T_1) \subset R_0(T_2)$. Thus, $(I - e^{-AT_2})^{-1}\partial R_0(T_1) \subset L_{T_2} \subset L_T$. Now, choose any $p_1, p_2 \in L_T$. For these points p_1 and p_2 there exist $T_1 < T$ and $T_2 < T$ respectively such that $p_1 \in [I - e^{-AT_2}]^{-1}\partial R_0(T_1) \subset L_T$ and $p_2 \in \partial L_{T_2} \subset L_T$. By (13), it follows that $\lambda p_1 + (1-\lambda)p_2 \in L_T$ for all $\lambda \in [0, 1]$.

2) *Matrix A has one zero eigenvalue ($\lambda_n(A) = 0$):* Without loss of generality, we consider $A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix}$ where $\tilde{A} \in \mathbb{R}^{n-1\times n-1}$ is a diagonal matrix. Let

$$F^{\pm}_{0,n}(t_1, ..., t_n) = \begin{bmatrix} \widetilde{F}^{\pm}_{0,n}(t_1, ..., t_n) \\ \pm\sum_{i=1}^{n-1} 2(-1)^i t_i + (-1)^{n-1}t_n \end{bmatrix}$$

Then, $\tilde{R}_0(T) = \{x \in \mathbb{R}^{n-1} : x = \widetilde{F}^{\pm}_{0,n}(t_1, ..., t_n), 0 < t_1 \leq ... \leq t_n \leq T\}$ i.e. $\tilde{R}_0(T)$ is a projection of $R_0(T)$ on hyperplane $x_n = 0$. Therefore, $\tilde{R}_0(T)$ is also a convex set. Let $\widetilde{L}_T = \{\widetilde{p} \in \mathbb{R}^{n-1} | \widetilde{p} = (\widetilde{I} - e^{-\widetilde{A}t_n})^{-1}\widetilde{F}^{\pm}_{0,n}(t_1, ..., t_n), 0 < t_1 \leq ... \leq t_n \leq T\}$. Now, $L_T = \widetilde{L}_T \times \langle e_n \rangle$ (where $\langle e_n \rangle$ is span of $e_n = \begin{bmatrix} 0 & 0 & ... & 1 \end{bmatrix}^T$ i.e. the n^{th} standard unit vector and symbol $\times$ denotes Cartesian product of two sets). Then since $\tilde{R}_0(T)$ is a convex set $\widetilde{L}_T$ is also a convex set (using same arguments as in case 1). Hence, L_T is a convex set. □

We convert the representation of $L^{\pm}_T$ to rational or polynomial parametric representation (depending on eigenvalues of A) using the procedures described in Section 4. Compute Groebner basis G^+_n and G^-_n for L^+_T and L^-_T respectively using standard implicitization steps described in Section 5. Using the computed Groebner basis G^+_n and G^-_n, we compute the time period of limit cycle for the given target p. We also provide an easy to check sufficient condition to test whether a target point $p \in L_T$ for a given time period T.

6.3.1. *Computation of time period*

The time period of the limit cycle for a given target point p is computed as follows. First select $g_n^+ \in G_n^+[z_n, p_1, ..., p_n]$ and solve $g_n^+ = 0$ for z_n. Then select $g_{n-1}^+ \in G_n^+[z_n, z_{n-1}, p_1,..., p_n]$ and using the values of z_n obtained in the previous step, solve $g_{n-1}^+ = 0$ for z_{n-1} and so on. Lastly, select $g_1^+ \in G_n^+[z_n, z_{n-1}, ..., z_1, p_1, ..., p_n]$ and using values of $z_n, z_{n-1}, ..., z_2$ obtained in previous step, solve $g_1^+ = 0$ for z_1. Repeat the above procedure for G_n^-. We obtain several solutions for $z_1, z_2, ..., z_n$. Now, among these solutions select the one that satisfies the inequality $0 < z_n \leq z_{n-1} \leq \cdots \leq z_1 \leq 1$. Then, $t_i = -l \log(z_i)$ for $i = 1, ..., n-1$ are the switching instants of limit cycle about p. The *time period of limit cycle is* $T = t_n = -l \log(z_n)$ (refer to Section 4).

For the case of one zero eigenvalue, first compute the values of $z_n, z_{n-1}, ..., z_2$ using first $n-1$ steps described above. Finally to compute t_1 we use $x_n = \pm(2t_1 - 2t_2 + ... + (-1)^{k-1}2t_{n-1} + (-1)^k t_n)$ where $t_i = -l \log z_i$. Among several solutions thus obtained, we choose, in particular, the solution $t_1, \ t_2, \ \cdots, \ t_n$ satisfying $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$. Then the *time period of limit cycle is* $T = t_n$.

6.3.2. *Characterizing* L_T

Next we characterize the set of points p that gives rise to a limit cycle of period T i.e. ∂L_T . Substitute $z_n = e^{-\frac{T}{l}}$. Then the equation $g_n^+(z_n, p_1, ..., p_n) = 0$ describes the smallest variety containing the parametric representation of ∂L_T^+ for a given period T. Similarly, the equation $g_n^-(z_n, p_1, ..., p_n) = 0$ describes the smallest variety containing the parametric representation of ∂L_T^- . The following sufficient condition provides an easy test to check whether $x \in L_T$:

Theorem 10. *If for all* $\mu \in [0, 1]$, $g_n^+(\mu p) \neq 0$ *and* $g_n^-(\mu p) \neq 0$ *then* $p \in L_T$.

Remark 11. From the sequence of switching events described at the end of Section 2, we conclude that for odd ordered systems (i.e n is odd) limit cycle trajectory follows the sequence $p \to M_n^\pm \to M_{n-1}^\mp \to ... \to M_1^\pm \to p \to M_n^\pm$. Similarly for even ordered system (i.e. n is even) limit cycle trajectory follows the sequence $p \to M_n^\pm \to M_{n-1}^\mp \to ... \to M_1^\mp \to p \to M_n^\pm$. We observe that for odd ordered system control input value does not change its sign at the target point p whereas for even ordered system change of sign in control input value occurs at the target point p.

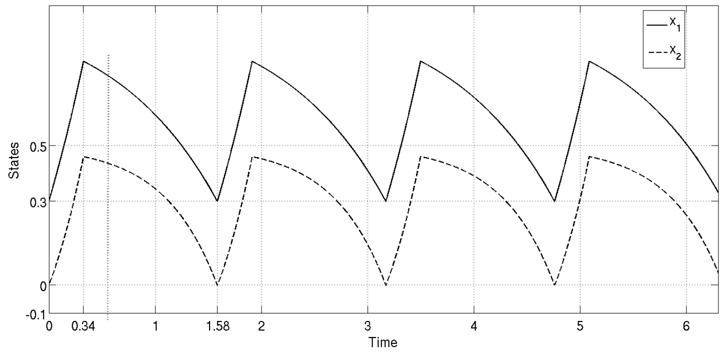

Figure 4: Limit cycle with $p = [0.3\ \ 0]^T$and period $t = 1.5833$

Example 12. Consider the system from Example 1, Section 2. For the target point $p = [p_1\, p_2]^T$, we substitute $z_i = e^{-t_i}$ to get polynomial equations in p_1 and p_2. Moreover the inequality $0 < t_1 \leq t_2 < \infty$ is converted to $0 < z_2 \leq z_1 < 1$. Using the procedure described above we eliminate z_1 and obtain G_2^+ as $s_1 = z_2^2 p_1^2 - 2z_2^2 p_1 + 4z_2^2 p_2 - z_2^2 - 2z_2 p_1^2 + 2z_2 + p_1^2 + 2p_1 - 4p_2 - 1$ and $s_2 = 2z_1 + z_2 p_1 - z_2 - p_1 - 1$. Here $g_2^+ = s_1$. Similarly, we can find g_2^- using $u = -1$ for $0 \leq t \leq t_1$ and then switching accordingly. For $p = [0.3\ \ 0]^T$, solution of $g_2^+ = 0$ gives $(1.51z_2 - 0.31)(z_2 - 1) = 0$. The solution $z_1 = 1$ is not a meaningful solution since it gives a limit cycle of period $t_2 = 0$. The solution $z_2 = \frac{31}{151}$, gives $z_1 = \frac{218}{302}$ and the inequality $0 < z_2 \leq z_1 < 1$ is satisfied by this solution. Hence at $p = [0.3\ \ 0]^T$, we get a limit cycle with period $t_2 = 1.5833$. This can be verified from Figure 4.

Also for a given period say $t_2 = T = 1$ the set of all points p around which the state trajectory goes into a limit cycle of period $T = 1$ with starting input $u = 1$ can be computed from g_2^+. For $t_2 = 1$, we have $z_2 = e^{(-1)} = 0.3679$. With this z_2, $s_1 = 0$ gives $p_1^2 + 4.3276p_1 - 8.6554p_2 - 1 = 0$. Similarly, for starting input $u = -1$, the set of such points can be calculated from g_2^-. This set is shown in Figure 5.

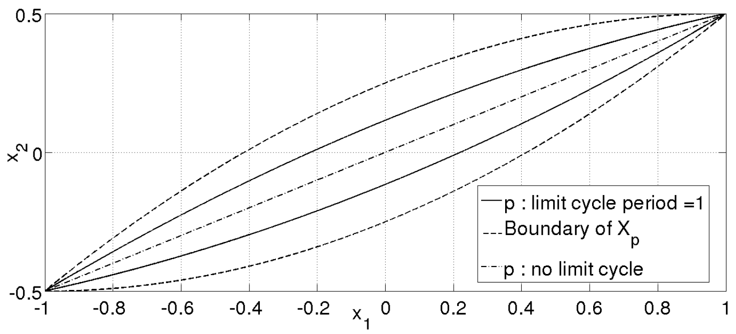

Figure 5: Set of points p around which limit cycle has period 1

7. Constrained Controllable and Reachable Sets

Let $CCP = \{p \in \mathbb{R}^n : p \in \text{int } X_p\}$ be the set of all constrained controllable points for (1). In this section we describe an algebraic method to compute CCP. Recall the definition of the null controllable set X_0 and the attainable set $\mathcal{A}_0$ from Section 2. We call a system *semi-stable* if all eigenvalues of A lie in closed left half of the complex plane (i.e., $\lambda(A) \leq 0$). Otherwise (i.e., if $\lambda(A) > 0$) we call the system *anti-stable*. Under the assumption that system (1) is controllable, the set CCP can be specified for semi-stable and anti-stable systems by $CCP = X_0 \cap \mathcal{A}_0$ [16]. Using (2) we have, $X_0 = \bigcup_{t\geq 0}\{x : x = -\int_0^t e^{-A\tau} bu(\tau)d\tau,\ u(\tau) \in U\}$ and from (3), $\mathcal{A}_0 = \bigcup_{t\geq 0}\{x : x = \int_0^t e^{A\tau} bu(\tau)d\tau,\ u(\tau) \in U\}$.

Theorem 13. *If for system (1), $A = diag(A_1, A_2)$, with $A_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ anti-stable ($\lambda(A_1) > 0$) and $A_2 \in \mathbb{R}^{m\times m}$ semi-stable ($\lambda(A_2) \leq 0$), b is partitioned accordingly as $b = \begin{bmatrix} b_1^T & b_2^T \end{bmatrix}^T$ and origin lies in CCP, then $CCP = X_0^1 \times \mathcal{A}_0^2$ where X_0^1 is the null controllable set for anti-stable system $\dot{x}_1 = A_1x_1 + b_1u$ and $\mathcal{A}_0^2$ is attainable set for the stable system $\dot{x}_2 = A_2x_2 + b_2u$.*

Proof. We have $X_0 = X_0^1 \times \mathbb{R}^m$ and similarly $\mathcal{A}_0 = \mathbb{R}^{(n-m)} \times \mathcal{A}_0^2$ [20]. Therefore, $CCP = X_0 \cap \mathcal{A}_0 = (X_0^1 \times \mathbb{R}^m) \cap (\mathbb{R}^{(n-m)} \times \mathcal{A}_0^2) = (X_0^1 \cap \mathbb{R}^{(n-m)}) \times (\mathbb{R}^m \cap \mathcal{A}_0^2) = X_0^1 \times \mathcal{A}_0^2$. □

Thus, to check whether given $x = [\ x_1 \quad x_2\ ]^T \in CCP$, we independently check whether $x_1 \in X_0^1$ and $x_2 \in \mathcal{A}_0^2$. But, the attainable set $\mathcal{A}_0^2$ of $\dot{x}_2 = A_2x_2 + b_2u$ is same as the reachability set of $\dot{x}_2 = -A_2x_2 + b_2u$. Hence, it is enough to compute the null-controllable region X_0 just for anti-stable systems to characterize CCP for any given system. Next we characterize the reachable set X_p for a point $p \in CCP$. The following lemma from [16] can be used directly.

Lemma 14. *$X_p = X_0$ if and only if $p \in \mathcal{A}_0 \bigcap X_0 = CCP$.*

From this lemma, it is clear that the semi-algebraic characterization of X_p for any point $p \in CCP$ is same as that of null-controllability region X_0 of the given system. As mentioned above, X_0 can be computed using the method proposed in [5, 6].

8. Conclusion

In this article, we extend previous synthesis results to the case of non-zero target states and relax the assumption on eigenvalues to include zero eigenvalue. In practice, the switching feedback law drives the system to a some neighborhood of p, in near optimal time. The presence of limit cycles, even for time optimal transfer, is an interesting feature of this development. A characterization of the target points for which limit cycle exists is provided. Properties of the limit cycle for different target points are studied. The proposed method is currently limited to controllable diagonalizable systems with rational eigenvalues. Relaxing some of these assumptions is the subject of current and future research.

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