MA6L001 Mathematical Methods (Probability & Statistics)
Tutorial Sheet No. 1

The following fifteen problems in bold face are considered for Assignment No. 1

1. **Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition noted. This is continued until two consecutive defectives are produced or four items have been checked, which ever occurs first. Describe the sample space for this experiment.**

2. Consider $\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Define $P(R) =$ area of $R = (b-a)(d-c)$ where R is the rectangular region that is a subset of Ω of the form $R = \{(u, v) : a \le u < b, c \le v < d\}$. Let T be the triangular region $T = \{(x, y) : x \ge 0, y \ge 0, x + y < 1\}$. Show that T is an event, and find $P(T)$.

3. **State True or False with valid reasons for the following statements.**

   (a) **The probability that exactly one of the events A or B occurs is equal to $P(A) + P(B) - 2P(A \cap B)$.**

   (b) **Let A and B two events with $P(A) = \frac{1}{2}$ and $P(B^c) = \frac{1}{4}$. Then, A and B can be mutually exclusive events.**

   (c) **If A and B are two independent events, then A^c and B^c are independent events.**

   (d) **A box contains a double-headed coin, a double-tailed coin and an unbiased coin. A coin is picked at random and flipped. It shows a head. The conditional probability that it is the double-headed coin is 0.5 .**

   (e) **Consider a parallel system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.99, then the minimum number of components in this system is 3.**

4. Consider the flights starting from Delhi to Bombay. In these flights, 90% leave on time and arrive on time, 6% leave on time and arrive late, 1% leave late and arrive on time and 3% leave late and arrive late. What is the probability that, given a flight late, it will arrive on time?

5. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ both fail}) = 0.15$, $P(B \text{ fails alone}) = 0.15$. Evaluate the following probabilities (a) $P(A \text{ fails}/B \text{ has failed})$ (b) $P(A \text{ fails alone } /A \text{ or } B \text{ fail})$.

6. **An aircraft has four engines in which two engines in each wing. The aircraft can land using atleast two engines. Assume that the reliability of each engine is $R = 0.93$ to complete a mission, and that engine failures are independent.**
   a) Obtain the mission reliability of the aircraft.
   b) If at least one functioning engine must be on each wing, what is the mission reliability?

7. A batch of N transistors is dispatched from a factory. To control the quality of the batch the following checking procedure is used; a transistor is chosen at random from the batch, tested and placed on one side. This procedure is repeated until either a pre-set number $n(n < N)$ of transistors have passed the test (in which case the batch is accepted) or one transistor fails (in this case the batch is rejected). Suppose that the batch actually contains exactly D faulty transistors. Find the probability that the batch will be accepted.

8. **Consider the random variable X that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of a particular spot in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of X is as follows:**

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$F(x)$	**0.250**	**0.546**	**0.898**	**0.932**	**0.955**	**0.972**	**0.981**	**0.989**	**0.995**	**0.998**	**1.000**

Find (a) $P(X = 10)$ **(b)** $P(X \leq 5/X > 2)$ **.**

9. For what values of α, p does the following function represent a probability mass function $p_X(x) = \alpha p^x, x = 0, 1, 2, \ldots$. Prove that the random variable having such a probability mass function satisfies the following memoryless property $P(X > a + s/X > a) = P(X \geq s)$.

10. Let X be a random variable such that $P(X = 2) = \frac{1}{4}$ and its distribution function is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 4 \\ \beta x^2, & 4 \leq x < 8/\sqrt{3} \\ 1, & x \geq 8/\sqrt{3} \end{cases} .$$

   (a) Find α, β if 2 is the only jump discontinuity of F.

   (b) Compute $P(X < 3/X \geq 2)$.

11. **A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM.**

   (a) What is the probability that the student must wait more than five minutes?

   (b) If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes?

12. In a torture test a light switch is turned on and off until it fails. If the probability that the switch will fail any time it is turned 'on' or 'off' is 0.001, what is probability that the switch will fail after it has been turned on or off 1200 times?.

13. The life time (in hours) of a certain piece of equipment is a continuous random variable X, having pdf

$$f_X(x) = \begin{cases} \frac{xe^{-x/100}}{10^4}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} .$$

   If four pieces of this equipment are selected independently of each other from a lot, what is the probability that atleast two of them have life length more than 20 hours?.

14. The probability of hitting an aircraft is 0.001 for each shot. Assume that the number of hits when n shots are fired is a random variable having a binomial distribution. How many shots should be fired so that the probability of hitting with two or more shots is above 0.95?

15. **An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?**

16. Suppose the duration (measured in minutes) of a telephone conversation between two persons is a random variable X with cumulative distribution function

$$P(X \leq t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-0.04t}, & 0 \leq t < \infty \end{cases}$$

   Given that the conversation has been going on for 20 minutes, compute the probability that it continues for at least another 10 minutes.

17. **Suppose that the life length of two electronic devices say D_1 and D_2 have normal distributions $N(40, 36)$ and $N(45, 9)$ respectively. (a) If a device is to be used for 45 hours, which device would be preferred? (b) If it is to be used for 42 hours which one should be preferred?**

18. Let X be uniformly distributed random variable on the interval $(0, 1)$. Define $Y = a + (b - a)X, \quad a < b$. Find the distribution of Y.

19. Consider a nonlinear amplifier whose input X and output Y are related by its transfer characteristic

$$Y = \begin{cases} X^{\frac{1}{2}}, & X > 0 \\ -|X|^{\frac{1}{2}}, & X < 0 \end{cases}$$

Find pdf of Y if X has $N(0, 1)$ distribution.

20. Let the phase X of a sine wave be uniformly distributed in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Define $Y = \sin X$. Find the distribution of Y.

21. Find the probability distribution of a binomial random variable X with parameter n, p, truncated to the right at $X = r, r > 0$.

22. Find pdf of a doubly truncated normal $N(\mu, \sigma^2)$ random variable, truncated to the left at $X = \alpha$ and to the right at $X = \beta$.

23. Suppose that two teams are plying a series of games, each of which is independently won by team A with probability 0.5 and by team B with probability 0.5. The winner of the series is the first team to win four games. Find the expected number of games that are played.

24. Let X be a uniformly distributed random variable on the interval $[a, b]$ where $-\infty < a < b < \infty$. Find the distribution of the random variable $Y = \frac{X-\mu}{\sigma}$ where $\mu = E(X)$ and $\sigma^2 = Var(X)$. Also, find $P(-2 < Y < 2)$.

25. **Suppose Shimla's temperature is modeled as a random variable which follows normal distribution with mean 10 Celcius degrees and standard deviation 3 Celcius degrees. Find the mean if the temperature of Shimla were expressed in Fahreneit degrees.**

26. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$.

    (a) Find $E[(X - E(X))^4]$ if it exists.

    (b) Find $P(-1/2 < X \leq 3)$ and $P(X = 0)$.

27. The mgf of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$.
(a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

28. **Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.**

29. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

30. **The moment generating function of a discrete random variable X is given by $M_X(t) = \frac{1}{6} + \frac{1}{2}e^{-t} + \frac{1}{3}e^t$. If μ is the mean and σ^2 is the variance of this random variable, find $P(\mu - \sigma < X < \mu + \sigma)$.**

31. Let X and Y be independent random variables. The range of X is $\{1, 3, 4\}$ and the range of Y is $\{1, 2\}$. Partial information on the probability mass function is as follows:

$$p_X(3) = 0.50; \quad p_Y(2) = 0.60; \quad p_{X,Y}(4, 2) = 0.18 \ .$$

(a) Determine p_X, p_Y and $p_{X,Y}$ completely.

(b) Determine $P(|X - Y| \geq 2)$.

32. Find k, if the joint probability density of (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} ke^{-3x_1-4x_2}, & x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

Also find the probability that the value of X_1 falls between 0 and 1 while X_2 falls between 0 and 2.

33. **Consider a transmitter sends out either a 0 with probability p, or a 1 with probability $(1-p)$, independently of earlier transmissions. Assume that the number of transmissions within a given time interval is Poisson distributed with parameter λ. Find the distribution of number of 1's transmitted in that same time interval?**

34. The random variable X represents the amplitude of cosine wave; Y represents the amplitude of a sine wave. Both are independent and uniformly distributed over the interval (0,1). Let R represent the amplitude of their resultant, i.e., $R^2 = X^2 + Y^2$ and θ represent the phase angle of the resultant, i.e., $\theta = \tan^{-1}(\mathrm{Y/X})$. Find the joint and marginal pdfs of θ and R.

35. **Consider the metro train arrives at the station near your home every quarter hour starting at 5:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable, that is $\mathcal{U}([7:10, 7:30])$.**

    (a) Find the distribution of time you have to wait for the first train to arrive?

    (b) Also, find its mean waiting time?

36. Aditya and Aayush work independently on a problem in Tutorial Sheet 4 of Probability and Stochastic Processes course. The time for Aditya to complete the problem is exponential distributed with mean 5 minutes. The time for Aayush to complete the problem is exponential distributed with mean 3 minutes.

    (a) What is the probability that Aditya finishes the problem before Aayush?

    (b) Given that Aditya requires more than 1 minutes, what is the probability that he finishes the problem before Aayush?

    (c) What is the probability that one of them finishes the problem a minute or more before the other one?

37. **Let X and Y be two random variables such that $\rho(X,Y) = \frac{1}{2}$, $Var(X) = 1$ and $Var(Y) = 4$. Compute $Var(X - 3Y)$.**

38. **Suppose that 30 electronic devices say $D_1, D_2, \ldots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes operative etc. Assume that the time to failure of D_i is an exponentially distributed random variable with parameter $= 0.1(hour)^{-1}$. Let T be the total time of operation of the 30 devices. What is the probability that T exceeds 350 hours?**

39. **Suppose that X_i , $i = 1, 2, \ldots, 30$ are independent random variables each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \ldots + X_{30}$.**
    (a) Using central limit theorem evaluate $P(S \geq 3)$.
    (b) Compare the answer in (a) with exact value of this probability.