

Department of Mathematics
MTL 725 (Stochastic Processes and Applications)
Tutorial Sheet 3 Answers

1 e^{-5t}

3

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 & 0 \\ 0 & 2\mu & -(\lambda + 2\mu) & \lambda & 0 \\ 0 & 0 & 3\mu & -(\lambda + 3\mu) & \lambda \\ 0 & 0 & 0 & 4\mu & -4\mu \end{pmatrix}.$$

4 **Kolmogorov Forward Equations** $\frac{d}{dt}P(t) = P(t)Q$.

$$\frac{d}{dt}P_{i0}(t) = -\lambda P_{i0}(t) + \mu P_{i1}(t),$$

$$\frac{d}{dt}P_{i1}(t) = \lambda P_{i0}(t) - (\lambda + \mu)P_{i1}(t) + 2\mu P_{i2}(t),$$

$$\frac{d}{dt}P_{i2}(t) = \lambda P_{i1}(t) - 2\mu P_{i2}(t).$$

Kolmogorov Backward Equations $\frac{d}{dt}P(t) = Q P(t)$.

$$\frac{d}{dt}P_{0j}(t) = -\lambda P_{0j}(t) + \lambda P_{1j}(t),$$

$$\frac{d}{dt}P_{1j}(t) = \mu P_{0j}(t) - (\lambda + \mu)P_{1j}(t) + \lambda P_{2j}(t),$$

$$\frac{d}{dt}P_{2j}(t) = 2\mu P_{1j}(t) - 2\mu P_{2j}(t).$$

5 (a) **Forward Kolmogorov Equations**

$$\frac{dp_0(t)}{dt} = -\alpha p_0(t) + \beta p_1(t),$$

$$\frac{dp_1(t)}{dt} = \alpha p_0(t) - \beta p_1(t).$$

(b)

$$p_1(t) = \frac{\alpha}{\alpha + \beta} \left(1 - e^{-(\alpha + \beta)t} \right),$$

$$p_0(t) = 1 - p_1(t) = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} e^{-(\alpha + \beta)t}.$$

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$$P(t) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{e^{-3t/2}}{3} \begin{pmatrix} 2 \cos\left(\frac{\sqrt{3}}{2}t\right) & -\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) & -\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ -\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) & 2 \cos\left(\frac{\sqrt{3}}{2}t\right) & -\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ -\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) & -\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) & 2 \cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}.$$

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$$\mathbb{E}[T_{0 \rightarrow 1}] = \frac{1}{\lambda_0} = \frac{1}{\lambda}.$$

14

$$\Pr\{X(t) = 0 \mid X(0) = 0\} = e^{-\lambda t},$$

$$\Pr\{X(t) = 1 \mid X(0) = 0\} = 1 - e^{-\lambda t},$$

$$\Pr\{X(t) = 2 \mid X(0) = 0\} = 0.$$

16 0.8

21 (a) $F_{T_j}(t) = \Pr(T_j \leq t) = 1 - e^{-\lambda t}$, $t \geq 0$. (b) $f(t_1, t_2, t_3) = \lambda^3 \exp[-\lambda(t_1 + t_2 + t_3)]$, $t_1, t_2, t_3 \geq 0$.

22 (a) $e^{-5/4}$ (b) $5e^{-2}$

23 (a) $512/6e^{-24}$ (b) 4

24 yes

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