

Department of Mathematics
MTL 725 (Stochastic Processes and Applications)
Tutorial Sheet No. 5

1. Find the renewal function corresponding to the lifetime density $f(x) = \lambda^2 x e^{-\lambda x}$, $x \geq 0$.
2. Let $X_1, X_2, \dots$ be the inter-occurrence times in a renewal process. Suppose $P\{X_i = 1\} = 1/3$, $P\{X_i = 2\} = 1/3$, $P\{X_i = 3\} = 1/3$, $i = 1, 2, \dots$. Let N_n be the number of renewals up to discrete-time n . Compute $P\{N(2) = k\}$.
3. Consider a renewal process $\{N(t), t \geq 0\}$. Let $H(t) = E[N(t) - N(t-a)1_{(a \leq t)}]$. Find a renewal equation that H satisfies.
4. Let $N_1(t)$ and $N_2(t)$ be independent renewal processes with the same inter-renewal distribution, and consider the sum $N(t) = N_1(t) + N_2(t)$ (called superposition). Assuming that $N(t)$ is a renewal process. Prove that it is a Poisson process if and only if $N_1(t)$ and $N_2(t)$ are Poisson processes.
5. Prove $E[T_{N(t)}] = \mu E[N(t) + 1] - E[X_{N(t)}]$. If $N(t)$ is a Poisson process, show that $E[T_{N(t)}] < \mu E[N(t)]$.
6. Consider a renewal process $\{N(t), t \geq 0\}$ with for $n = 1, 2, \dots$, $P\{X_n = k\} = p(1-p)^{k-1}$, $k = 1, 2, \dots$. Prove that, $P\{N(t) = n\} = \binom{[t]}{n} p^n (1-p)^{[t]-n}$, $n = 0, 1, \dots$.
7. Consider a renewal process $\{N(t), t \geq 0\}$ with X_i follows uniform distribution between 0 and 1. Prove that, $M(t) = e^t - 1$, $0 \leq t \leq 1$.
8. Consider a sequence of independent Bernoulli trials in which each trial results in a success or failure with respective probabilities p and $q = 1 - p$. Let $N(t)$ denote the number of successes in n trials, where n is an integer. Show that N_n is a discrete time renewal process, called a Bernoulli Process. (The parameter n may denote discrete time or any integer referring to sequential information.) Justify that the inter-renewal times have the geometric distribution $P(X_1 = n) = pq^{n-1}$, $n \geq 1$. Find the distribution and mean of N_n , and do the same for the renewal time T_n . Show that the moment-generating function of T_n is

$$E[e^{\alpha T_n}] = \left(\frac{pe^{\alpha}}{1 - qe^{\alpha}} \right)^n, 0 < \alpha < -\log q.$$

9. A system consists of two components with independent lifetimes X_1 and X_2 , where X_1 is exponential distributed with rate λ , and X_2 has a uniform distribution on $[0, 1]$. The components operate in parallel, and the system lifetime is $\max\{X_1, X_2\}$ (the system is operational if and only if at least one component is working). When the system fails, it is replaced by another system with an identical and independent lifetime, and this is repeated indefinitely. The number of system renewals over time forms a renewal process $\{N(t), t \geq 0\}$. Find the distribution and mean of the system lifetime. Find the distribution and mean of $N(t)$. Determine the portion of time that (a) two components are working, (b) only type 1 component is working, and (c) only type 2 component is working.
10. An infinite number of jobs are to be processed one at a time by a single server. There are m types of jobs, and the probability that any job is of type i is p_i , where $p_1 + \dots + p_m = 1$. The service time of a type i job has a distribution F_i with mean μ_i . The service times and types of the jobs are independent. Let $N(t)$ denote the number of jobs completed by time t . Show that $\{N(t), t \geq 0\}$ is a renewal process and specify its inter-renewal distribution and mean. Let $N_i(t)$ denote the number of type i jobs processed up to time t . Show that $\{N_i(t), t \geq 0\}$ is a delayed renewal process and specify $\lim_{t \rightarrow \infty} \frac{N_i(t)}{t}$.
11. The status of a system is represented by an alternating renewal process $\{X(t), t \geq 0\}$ with renewal times T_n , where the mean sojourn time in a working state 1 is λ and the mean sojourn time in a non-working state 0 is μ . The system availability is measured by the portion of time it is working, which is $\lim_{t \rightarrow \infty} \frac{\int_0^t X(s) ds}{t}$. Determine this quantity and show that it is equal to the cycle-availability measured by $\lim_{n \rightarrow \infty} \frac{\int_0^{T_n} X(s) ds}{T_n}$.

12. Consider time $T_n = \sum_{k=1}^n X_k$, where the X_k are IID with distribution F and $F(0) = P\{X_k = 0\} > 0$. The associated point process $\{N(t), t \geq 0\}$ is a renewal process with instantaneous renewals (or batch renewals). Let $\tilde{T}_n$ are the times at which batches of events arrive, and at time $\tilde{T}_n$, a batch of a_n events arrive. Hence, $N(t) = \sum_{n=1}^{\infty} a_n I_{\tilde{T}_n \leq t}$, where a_n is the number of renewals exactly at time $\tilde{T}_n$. Specify the distribution of a_n . Are the a_n IID? Are they independent of $\tilde{T}_n$? Specify the distribution of $\tilde{T}_1$ in terms of F .
13. Consider an non-homogeneous renewal process $\{N(t), t \geq 0\}$ whose inter-point times $X_n = T_n - T_{n-1}$ are independent with distributions F_n . Assuming it is finite, prove that $E[N(t)] = \sum_{n=1}^{\infty} F_1(t)F_2(t) \cdots F_n(t)$.
14. Consider a production-inventory system that produces a product at a constant rate of c units per unit time and the items are put in inventory to satisfy demands. The products may be discrete or continuous (e.g., oil, chemicals). Demands occur according to a Poisson process $\{N(t), t \geq 0\}$ with rate λ , and the demand quantities $X_1, X_2, \dots$ are independent, identically distributed positive RVs with mean μ , and are independent of the arrival times. Then, the inventory level at time t would be

$$Z_x(t) = x + ct - \sum_{n=1}^{N(t)} X_n, t \geq 0,$$

where x is the initial inventory level. Consider the probability $R(x) = P\{Z_x(t) \geq 0\}$ of never running out of inventory. Show that if $c < \lambda\mu$, then $R(x) = 0$ no matter how high the initial inventory level x is. Hint: a SLLN to show that $Z_x(t) \rightarrow \infty$ as $t \rightarrow \infty$ if $c < \lambda\mu$, where x is fixed. Find the limit of $Z_x(t)$ as $t \rightarrow \infty$ if $c > \lambda\mu$.

15. Consider a system in which customers arrive at a processing station according to a renewal process with inter-arrival distribution F and are processed by a single server. The processing or service times are independent of the common distribution G and are independent of the arrival process. Also, customer arrivals during service time are blocked from being served — they either go elsewhere or go without service. Prove that,

$$P\{\tilde{T}_1 \leq t\} = \int_0^{\infty} P\{T_{N(x)+1} \leq t\} dG(x).$$

Deduce the same result, when inter-arrival times are independent exponential distributions with parameter λ and processing times are independent exponential distributions with parameter μ .