

Department of Mathematics
MTL 725 (Stochastic Processes and Applications)
Tutorial Sheet No. 6

1. Consider $M/G/1/\infty$ system with LIFO preemptive policy. Assume that $\rho = \lambda E(S) < 1$. Prove that the stationary distribution π is

$$\pi_n = (1 - \rho)\rho^n, \quad n = 0, 1, \dots$$

2. Consider a single-server queue with arrival following the Poisson process with a mean rate of 10 per hour. Currently, the server has an average service time of 5 minutes, and service times follow an exponential distribution. A training program is expected to reduce the variability in service times, although the average service time will increase slightly. After the training, the average service time is expected to be 5.5 minutes, and the standard deviation will decrease from 5 minutes (the exponential case) to 4 minutes. Management wants to know if it is worth giving the server additional training.
3. Let $X(t)$ be the number of customers in the system at time t in an $M/G/1$ queueing model. Assume that, the arrival times $0 < \tau_1 < \tau_2 < \dots$, with the initial condition $X(0) = 0$. Further, assume that $\rho < 1$. Let $T = \inf\{t > \tau_1 : X(t) = 0\}$, denoting the time when the system first reaches an empty. Now, $T = \tau_1 + Y$, where Y represents the duration for which the server is busy. Find $E[T]$. Also, show that $E[Y] = \frac{\rho}{\lambda(1-\rho)}$.
4. Compute the transition probabilities of an embedded Markov chain for an $M/G/1$ queue where service is uniformly distributed on (a, b) , $0 < a < b < \infty$.
5. Compute the mean number of customers in the system and mean waiting time for $M/G/1$ queue where service time is beta distributed with parameters α and β .
6. A certain assembly-line operation is assumed to be a $M/G/1$ queueing system, with input rate $5/h$ and service times with mean 9 minutes and variance 90 minutes. Find $E(X)$ and $E(R)$. Is the operation improved or degraded if the service times are forced to be exponential distribution with the same mean?
7. Customers arrive at an ATM machine according to a Poisson process with rate $\lambda = 60/h$. The following transaction times are observed (in seconds):

28, 71, 0, 70, 51, 62, 36, 25, 35, 87, 69, 27, 56, 25, 36.

- (a) Would an $M/M/1$ queueing system be an appropriate model for this system?
- (b) Estimate the mean number of people waiting in line at ATM.
- (c) The bank wishes to keep the number in queue less than or equal to one. What is the mean transaction time needed to achieve this goal?
8. Consider a $GI/M/1$ queueing system, where the inter-arrival times are uniformly distributed from 0 to 6 minutes and the expected service time is 2 minutes. Find, the following measures in steady-state.
- (a) Mean queue length.
- (b) Mean time spent in queue for each customer.
9. Find the steady-state probabilities of $M/G/1/2$. Assume that G follows Erlang distribution with parameters 2 and $1/3$.
10. Derive the stationary system-size probabilities for the $M/M/2/2$ loss system using only Little's formula and fundamental steady-state identities.
11. Consider a system where trucks provide coils to a steelworks warehouse. They arrive with random inter-arrival time, distributed according an exponential distribution with a mean of 20 minutes. The time needed by a team of workers to unload a truck is a RV and it has an deterministic distribution with a mean of half an hour. Evaluate how many teams must work in parallel to reduce the mean time of each truck in the system to less than one hour and the mean number of trucks at the steelworks to less than three.

12. Consider an $GI/M/c$ queueing system with service in order of arrival. Prove that, the equilibrium conditional probability that a blocked customer will still be waiting in the queue when the next customer arrives is equal to the generalized occupancy i.e., the state of having all servers busy and a queue of length n .
13. Suppose that, in a production plant a certain number of identical machines will be installed. These machines require maintenance after 5 days, i.e., 120 hours, in average. This working time is a RV with an exponential distribution. Some operators will be in charge for the maintenance. The time needed by each of them for each machine is a RV with deterministic distribution and a mean of 2 hours. Owing to a suitable rotation of work-shifts a team of operators will be available 24h a day. According to the design the number of machines should be 20 and the number of operators in the maintenance team should be 20. Consider this scenario as $M/G/c/c$ loss system and find out what is the mean fraction of time during which each operator would be busy? How long will a machine stop last, in average?
14. Determine the lowest value of the parameter k of Erlang distribution which will allow us to approximate the mean line delay of a $D/M/c/c$ loss system to no worse than 2.0% by an $E_k/M/c/c$ loss system when $\lambda = 4$, $\mu = 1.5$, and $c = 3$.
15. Consider a $GI/G/\infty$ self-service system with stable condition. Prove that
 - (a) Mean number of customers in the system is

$$\frac{E(S)}{E(A)} = E(N) = \lambda \int_0^{\infty} (1 - F_s(t)) dt$$

where $\lambda = \frac{1}{E(A)}$.

- (b) Mean number of busy servers greater than or equal to r is given by

$$\lambda \int_r^{\infty} (1 - F_s(t)) dt.$$