

Department of Mathematics
MTL 725 (Stochastic Processes and Applications)
Tutorial Sheet No. 7

1. Let $\{X_t, t \geq 0\}$ be a martingale with respect to a given filtration. Show that

$$E[X_t] = E[X_0] \quad \forall \quad t \geq 0.$$

Is the converse true?

2. If X and Y are two martingales with respect to a given filtration $\mathcal{F}$, then can you say that the sum $X + Y$ is also a martingale with respect to $\mathcal{F}$.
3. Let S_n be a symmetric random walk such that $S_n = Y_1 + Y_2 + \dots$ where $\{Y_i, i = 1, 2, \dots\}$ are IID RVs with

$$P(Y_i = 1) = 1 - P(Y_i = -1) = \frac{1}{2}.$$

Let $\mathcal{F}_n = \sigma(Y_1, Y_2, \dots, Y_n)$. Show that

$$X_n = (-1)^n \cos(\pi S_n), \quad n = 1, 2, \dots$$

is a martingale with respect to $\{\mathcal{F}_n\}_{n \geq 1}$.

4. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate 1. Which of the following are martingales.
- (a) $\{N(t) - t, t \geq 0\}$.
- (b) $\{(N(t))^2 - t, t \geq 0\}$.
5. Let $\{M(t), t \geq 0\}$, where $M(t) = N(t) - \lambda t$ be the compensated Poisson process. Show that $\{M(t), t \geq 0\}$ is a martingale.
6. A gambler wins or loses one rupee in each round of betting, with equal chances and independently of the past events. She starts betting with the determination that she will stop when either she wins m rupees or she loses n rupees.
- (a) What is the probability that she will be winning when she stops playing further?
- (b) What is the expected number of rounds of betting before she stops playing?
7. Let $\{X_n, n = 1, 2, \dots\}$ be IID RVs with $E[X_n] = 0$ and $E[X_n^2] = \sigma^2$ for $n \geq 1$. Define

$$Y_n = \left(\sum_{k=1}^n X_k \right)^2 - n\sigma^2.$$

Then, show that $\{Y_n, n = 1, 2, \dots\}$ is a martingale.

8. Let $X_1, X_2, \dots$ be independent RVs with

$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n}. \end{cases}$$

Let $Y_1 = X_1$, and for $n \geq 2$,

$$Y_n = \begin{cases} X_n & Y_{n-1} = 0 \\ nY_{n-1} | X_n| & Y_{n-1} \neq 0 \end{cases}.$$

- (a) Show that $\{Y_n, n = 1, 2, \dots\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \dots, Y_n)$.

(b) Show that $\lim_{n \rightarrow \infty} P(Y_n = 0) = 1$.

9. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0}, P)$ be a filtered probability space and $\{Y_n, n = 0, 1, \dots\}$ be a sequence of absolutely integrable RVs adapted to the filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Assume that there exist real numbers $u_n, v_n, n = 0, 1, \dots$, such that

$$E[Y_{n+1} | \mathcal{F}_n] = u_n Y_n + v_n.$$

Find two real sequences a_n and $b_n, n = 0, 1, \dots$, such that the sequence $M_n = a_n Y_n + b_n, n = 0, 1, \dots$, is martingale w.r.t. the same filtration.

10. Let $\{X_n, n = 0, 1, \dots\}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0}, P)$ such that $E[|X_n|] < \infty$ for all $n = 0, 1, \dots$. Also, X_n is $\mathcal{F}_n$ -measurable and for all bounded stopping times τ , $E[X_\tau] = E[X_0]$. Prove that $\{X_n, n = 0, 1, \dots\}$ is a martingale.
11. (**Jensen's Inequality**) Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let X be an integrable RV defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\varphi(x)$ is also integrable. Prove that $\varphi(E[X|\mathcal{G}]) \leq E[\varphi(X)|\mathcal{G}]$ a.s. for any σ -field $\mathcal{G}$ contained in $\mathcal{F}$.
12. Suppose $X, X_1, X_2, \dots$ are square integrable RVs such that $X_n \rightarrow X$ in L^2 as $n \rightarrow \infty$. Prove that $E[X_n|\mathcal{G}] \rightarrow E[X|\mathcal{G}]$ in L^2 as $n \rightarrow \infty$ for any sub σ -field $\mathcal{G}$ on Ω contained in $\mathcal{F}$. (Hint: Use Jensen inequalities, Problem 11, with $\varphi(x) = x^2$)
13. Use Jensen's inequality to show that for any convex, measurable function φ

$$E\left(\varphi(\sigma t + W(t)e^{-\sigma W(t)})\right) \geq \varphi(t)e^{\frac{\sigma^2}{2}t},$$

where $\{W(t), t \geq 0\}$ is a Brownian motion.