

Department of Mathematics
MTL 725 (Stochastic Processes and Applications)
Tutorial Sheet No. 8

1. Let $\{W(t), t \geq 0\}$ be a Brownian motion. Calculate $P(W(6) - W(2) < 0, W(1) > 0)$.
2. Show that $f(W(t))$ is continuous in mean square where f is a Lipschitz continuous function and $\{W(t), t \geq 0\}$ is a Brownian motion.
3. Find $E[W(1) + 2W(2) + 3W(3)]^2$ where $\{W(t), t \geq 0\}$ is a Wiener process.
4. Let $\{W(t), t \geq 0\}$ be a Brownian motion and $\{\mathcal{F}_t\}_{t \geq 0}$ be its natural filtration. Show that the process $\{X(t), t \geq 0\}$ defined by

$$X(t) = 2e^{3W(t)+4t}, \quad t \geq 0,$$

is a Markov process.

5. For $r \in \mathbb{R}^+$, consider the stochastic process $\{X(t), t \geq 0\}$

$$X(t) = 3 \exp\{(r-2)t + 2W(t)\}, \quad t \geq 0,$$

where $\{W(t), t \geq 0\}$ is the Brownian motion. Show that for any $t > 0$,

$$E[e^{-rt}(X(t) - 3)^+] = 3\Phi(m_+) - 3e^{-rt}\Phi(m_-)$$

where $\Phi(\cdot)$ is the CDF of standard normal distribution and $m_{\pm} = \frac{\sqrt{t}}{2}(r \pm 2)$.

6. Define $X(t) = e^{-t}W(e^{2t})$, where $\{W(t), t \geq 0\}$ is a Brownian motion. Show that

$$E[X(t)X(s)] = e^{-|t-s|}, \quad \forall -\infty < s, t < \infty.$$

7. Show that for any $T > 0$, $V(t) = W(t+T) - W(T)$ is a Wiener process if $\{W(t), t \geq 0\}$ is a Wiener process.
8. Show that $-W(t)$ is a Wiener process if $\{W(t), t \geq 0\}$ is a Wiener process.
9. Prove that, for a Brownian motion $\{W(t), t \geq 0\}$, $E(|W(t) - W(s)|^2) = |t - s|$.
10. Let $\{W(t), t \geq 0\}$ be a Brownian motion and $\{\mathcal{F}_t\}_{t \geq 0}$ be its natural filtration. Consider, the Brownian motion $\{X(t), t \geq 0\}$ with drift μ then there holds that

$$X(t) = \mu t + W(t)$$

is a Markov process.

11. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the natural filtration of $\{W(t), t \geq 0\}$. Let b be a positive real number. Show that the process Y defined by $Y(t) = W^2(bt) - bt$ is a martingale.
12. Let $\{W(t), t \geq 0\}$ be a Brownian motion. Let $T = \inf\{t > 0 : W(t) \notin [-a, b]\}$. Let $a, b > 0$. Prove that
 (i) $P(W(T) = -a) = \frac{b}{a+b}$ (ii) $P(W(T) = b) = \frac{a}{a+b}$ (iii) $E[T] = ab$.
Hint: $W(t \wedge T)$ is a martingale and $P(T < \infty) = 1$.
13. Let $\{W(t), t \geq 0\}$ be a Brownian motion. For $b > 0$, let $T = \inf\{t > 0 : W(t) = b\}$. Show that $T < \infty$ a.s.. Further, show that $E[T] = \infty$.
Hint: Take a limit in Problem 12.
14. Let $\{W(t), t \geq 0\}$ be a Brownian motion. Prove that, for $\lambda > 0$

$$(a) \quad P(\sup_{s \leq t} W(s) \geq \lambda) \leq e^{-\lambda^2/2t},$$

$$(b) \quad P(\sup_{s \leq t} W(s) \geq \lambda) \leq 2e^{-\lambda^2/2t}.$$

Hint: $E[e^{cW(t)}|\mathcal{F}_s] \geq e^{cW(t)}$. Use Doob's inequality.

15. Consider a two-dimensional Wiener process $\{W(t), t \geq 0\}$, where $W(t) = (W_1(t), W_2(t))$. Find the probability that $|W(t)| < R$ where $R > 0$. Here, $|(x_1, x_2)|^2 = (x_1)^2 + (x_2)^2$.
16. Suppose that $(W^1(t), W^2(t), \dots, W^d(t))$ is a d -dimensional Wiener process. Show that if $i \neq j$, then $W^i(t)W^j(t)$ is a martingale.
17. Let $\{W(t), t \geq 0\}$ be the Brownian motion. Prove that the following processes are martingales.
 - (a) $\{(W(t))^3 - 3tW(t), t \geq 0\}$.
 - (b) $\{(W(t))^4 - 6t(W(t))^2 + 3t^2, t \geq 0\}$.
18. Show that Wiener process is 0.5-self-similar.
19. Consider a stochastic process $\{X(t), t \geq 0\}$,

$$X(t) = 2t + W(t), \quad t \geq 0,$$

where $\{W(t), t \geq 0\}$ is a Brownian motion. For $m > 0$, define the first passage time of $X(t)$ in level m as

$$\tau_m = \min\{t \geq 0 : X(t) = m\}.$$

- (a) Show that the process $\{Z(t), t \geq 0\}$,

$$Z(t) = \exp \left\{ \sigma X(t) - \left(2\sigma + \frac{1}{2}\sigma^2 \right) t \right\}, \quad t \geq 0,$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(W(s), 0 \leq s \leq t)$.

- (b) Show that

$$E \left[\exp \left\{ \sigma m - \left(2\sigma + \frac{1}{2}\sigma^2 \right) \tau_m \right\} 1_{\{\tau_m < +\infty\}} \right] = 1.$$

- (c) Show that

$$E[e^{-\alpha\tau_m}] = \exp\{2m - m\sqrt{2\alpha + 4}\}, \quad \forall \alpha > 0.$$

Moreover, prove that $E[\tau_m] < +\infty$.

20. Use the reflection principle to prove that if $\{W(t), t \geq 0\}$ is a Brownian motion and $T = \inf\{t > 0 : W_t \in (0, \infty)\}$, then $P(T = 0) = 1$. In other words, Brownian motion enters the interval $(0, \infty)$ immediately. By symmetry it enters the interval $(-\infty, 0)$ immediately. Conclude that Brownian motion hits 0 infinitely often in every time interval $[0, t]$.
21. Let $\{X(t), t \geq 0\}$ be a compound Poisson process. Let $0 = t_0 < t_1 < \dots < t_n$. Prove that the increments $X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ are stationary and independent. Also, show that the distribution of $X(t_j) - X(t_{j-1})$ is same as the distribution of $X(t_j - t_{j-1})$.
22. Prove that, any Lévy process is a Markov process.
23. Consider, a Hawkes process $\{N(t), t \geq 0\}$ with exponential exciting function where the intensity function is given by

$$\lambda(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN(s), \quad \alpha, \beta > 0.$$

Under, the assumptions that $\frac{\alpha}{\beta} < 1$, prove

- (a) Law of large numbers for $\{N(t), t \geq 0\}$.
- (b) Central limit theorem for $\{N(t), t \geq 0\}$.