Department of Mathematics
MTL 390 (Sampling Distribution)
Tutorial Sheet No. 4
(Answers to Selected Problems)

1. $E(\bar{X}) = \theta$
   $CRLB = \frac{\sigma^2}{n}$
   $Var(\bar{X}) = \frac{\sigma^2}{n} = CRLB$

2. $E(S^2) = \sigma^2$
   $Var(S^2) = \frac{\sigma^2}{n-1} \longrightarrow 0$ as $n \longrightarrow \infty$

3. $Y = (\prod_{i=1}^n X_i)^{1/n}$
   $E(Y) = \frac{\theta}{(1+\frac{1}{n})^n} \longrightarrow \frac{\theta}{e}$ as $n \longrightarrow \infty$
   $E(Y^2) = \frac{\theta^2}{(1+\frac{2}{n})^n} \longrightarrow \frac{\theta^2}{e^2}$ as $n \longrightarrow \infty$
   $Var(Y) \longrightarrow 0$ as $n \longrightarrow \infty$

5. $E(\alpha\bar{X} + (1-\alpha)S^2) = \lambda$
   Hence $\alpha\bar{X} + (1-\alpha)S^2$ is an unbiased estimator of λ.
   Define a function of X_1 as
   $$h(X_1) = \begin{cases} 1 & X_1 = 0 \\ 0 & X_1 \neq 0 \end{cases}$$
   $E[h(X_1)] = e^{-\lambda}$
   $h(X_1)$ is an unbiased estimator of $e^{-\lambda}$.

6. $\bar{X} - S^2$

7. Unbiased estimator of mean is $\bar{X}$ and variance is S^2.
   $\bar{X} = 1.5$
   $Var(X) = 16.6$

8. (a) $\hat{\mu} = 85.75$
   $\hat{\sigma^2} = 34$
   (b) $\hat{\mu} = 85.75$
   $\hat{\sigma^2} = 34$

9. $\hat{\theta} = \begin{cases} X_{(\frac{n+1}{2})} & \text{n is odd} \\ \frac{X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)}}{2} & \text{n is even} \end{cases}$

10. (a) $\hat{\mu} = \bar{X}$
    $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$
    (b) Let the sample size be m.
    $\hat{p} = 1 - \frac{\sum_{i=1}^m {X_i}^2}{m\bar{X}}$
    $\hat{n} = \frac{\sum_{i=1}^m X_i}{\hat{p}}$.

11. $\hat{\lambda} = \frac{r}{\bar{X}}$

12. $\hat{\lambda_2}$ is more efficient then $\hat{\lambda_1}$.

14. $\hat{r} = \bar{X}$
    $E(\bar{X}) = r$
    $Var(\bar{X}) = \frac{r}{n} \longrightarrow 0$ as $n \longrightarrow \infty$

15. (a) $\sum_{i=1}^{n} {X_i}^2$ is a sufficient statistics.

(b) $\hat{\theta} = \frac{\sqrt{1+\frac{4\sum_{i=1}^{n} {X_i}^2}{n}}-1}{2}$

(c) $CRLB = \frac{2\theta^2}{n(2\theta+1)}$

17. $c = \frac{1}{n}\sqrt{\frac{\pi}{2}}$
Y is not an efficient estimator as

$$Var(Y) = \frac{(\pi-2)\theta}{2n} \neq CRLB = \frac{2\theta^2}{n}$$

18. $CRLB = \frac{1}{n}$
UMVUE for θ^2 is $\bar{X}^2 - 1$.
$E(\bar{X}^2 - 1) = \theta$
$Var(\bar{X}^2 - 1) = \frac{4\theta^2}{n} + \frac{2}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$

19. $\bar{X}^2 - \frac{1}{n}\bar{X}$ is UMVUE for λ^2.

20. (a) $\hat{\theta} = \frac{\bar{X}}{1+\bar{X}}$
(b) $\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i+n-1}$

24. Critical region is

$$\sum_{k=1}^{50} X_i > 2664.4$$

power of the test for true mean μ_1 is

$$\begin{cases} .8869, & \mu_1 = 55 \\ 1 & \mu_1 \geq 58 \end{cases}$$

28.

$$k(\theta) = \begin{cases} .9958, & \theta = .01 \\ .9138 & \theta = .05 \\ .7361 & \theta = .1 \\ .3758 & \theta = .20 \end{cases}$$