Department of Mathematics
MTL 390 (Estimation)
Tutorial Sheet No. 4

1. Prove that $\overline{X}$, the mean of a random sample of size n from a distribution that is $N(\theta, \sigma^2)$, $-\infty < \theta < \infty$ is an efficient estimator of θ for every known $\sigma^2 > 0$.

2. Assuming population to be $\mathcal{N}\ (\mu, \sigma^2)$, show that sample variance is a consistent estimator for population variance σ^2.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from uniform distribution on an interval $(0, \theta)$. Show that $(\prod_{i=1}^{n} X_i)^{1/n}$ is consistent estimator of θe^{-1}.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution with pmf $P(x, p) = (1-p)^{x-1}p$, $x = 1, 2, \ldots$. Prove that maximum likelihood estimator of p is $\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson distribution $P(\lambda)$. Show that $\alpha\overline{X} + (1-\alpha)s^2, 0 \leq \alpha \leq 1$, is a class of unbiased estimators for λ. Also find an unbiased estimator for $e^{-\lambda}$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from binomial distribution $B(1, p)$. Find an unbiased estimators for p^2 if it exists.

7. Suppose that 200 independent observations $X_1, X_2, \ldots, X_{200}$ are obtained from random variable X. We are told that $\sum_{i=1}^{200} X_i = 300$ and that $\sum_{i=1}^{200} X_i^2 = 3754$. Using these values obtain unbiased estimates for $E(X)$ and $Var(X)$.

8. Consider the number of students attended Probability and Statistics lecture classes for 42 lectures. Let $X_1, X_2, \ldots, X_{32}$ denote the number students attended in randomly chosen 32 lecture classes. Suppose the observed data is as follows

100	90	85	95	88	82	92	84	88	87	82	88	82	91	92	91
82	90	82	87	92	70	84	79	88	81	82	78	81	82	90	79

   (a) Using method of moments, find the estimators for the population mean and population variance.

   (b) Assume that, X_i's are i.i.d random variables each having normal distribution with mean μ and variance σ^2 find ML estimators for the population mean and population variance.

9. Find the maximum likelihood estimator of θ based on a sample of size n from the two sided exponential family with pdf given as follows:

$$f(x) = \frac{1}{2} e^{-|x-\theta|}, \qquad -\infty < x < \infty$$

   Is the estimator unbiased?

10. Using method of moments, find the estimators of the parameters for the following population distributions (a) $\mathcal{N}(\mu, \sigma^2)$ (b) $B(n, p)$.

11. Consider a queueing system in which the arrival of customers follow Poisson process. Let X be the distribution of service time, which has gamma distribution. That is the pdf of X is given by

$$f_X(x) = \frac{\lambda(\lambda x)^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad x > 0\ .$$

   Suppose that r is known. Let $X_1, X_2, \ldots, X_n$ be a random sample on X. Obtain the ML estimate of λ based on this sample.

12. Let X_1, X_2 and X_3 be three independent random variables having the Poisson distribution with the parameter λ. Show that

$$\hat{\lambda_1} = \frac{X_1 + 2X_2 + 3X_3}{6}$$

is an unbiased estimator of λ. Also compare the efficiency of $\hat{\lambda_1}$ with that of the alternate estimator.

$$\hat{\lambda_2} = \frac{X_1 + X_2 + X_3}{3} \; .$$

13. Let X be random variable with Cauchy distribution

$$f(x) = \frac{1}{\Pi} \frac{1}{1 + (x - \theta)^2}, \quad -\infty < x < \infty.$$

Find the Cramer-Rao lower bound for the estimation of the location parameter θ.

14. Prove that method of moment estimator is consistent for the estimation of $r > 0$ in the Gamma family

$$f(x) = \frac{e^{-x} x^{r-1}}{\Gamma(r)}, \quad 0 < x < \infty.$$

15. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with both mean and variance equal to an unknown parameter θ.
    (a) Is there a sufficient statistics?
    (b) What is the MLE?
    (c) What is the Cramer-Rao lower bound?

16. Prove that for the family of uniform distribution on $[0, \theta]$, $max(x_1, x_2, \ldots, x_n)$ is the MLE for θ.

17. Consider the normal distribution $N(0, \theta)$. With a random sample $X_1, X_2, \ldots, X_n$ we want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c \sum_{i=1}^{n} |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

18. If $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, 1)$. Find a lower bound for the variance of an estimator of θ^2. Determine the minimum variance unbiased estimator of θ^2 and then compute its efficiency.

19. Suppose that the random sample arises from a distribution with pdf

$$f(x; \theta) = \begin{cases} \theta x^{\theta - 1}, & 0 < x < 1, \quad \theta \in \Omega = \{\theta; \; 0 < \theta < \infty\} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\hat{\theta} = -\frac{n}{\ln \Pi_{i=1}^{n} X_i}$ is the maximum likelihood estimator of θ. Further prove that in a limiting sense, $\hat{\theta}$ is the minimum variance unbiased estimator of θ and thus θ is asymptotically efficient.

20. Let $X_1, X_2, \ldots, X_n$ be random sample from a Poisson distribution with mean θ. Find the minimum variance unbiased estimator of θ^2.

21. Let $X_1, X_2, \ldots, X_n$ be random sample from a distribution with pdf

$$f(x; \theta) = \begin{cases} \theta^x (1 - \theta), & x = 0, 1, 2, \ldots; \quad 0 \le \theta \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Show that $\sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.

(c) Determine the minimum variance unbiased estimator of θ.

22. Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size 10 from a distribution with the following pdf

$$f(x;\theta) = \frac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty$$

for all real θ. Find the likelihood ratio test λ for testing $H_0: \ \theta = \theta_0$ against the alternative $H_1: \ \theta \neq \theta_0$.

23. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent random samples from the two normal distributions $N(0, \theta_1)$ and $N(0, \theta_2)$.

(a) Find the likelihood ratio test λ for testing the composite hypothesis $H_0: \ \theta_1 = \theta_2$ against the composite alternative hypothesis $H_1: \ \theta_1 \neq \theta_2$.

(b) The test statistic λ is a function of which F statistic that would actually be used in this test.

24. Let $X_1, X_2, \ldots, X_{50}$ denote a random sample of size 50 from a normal distribution $N(\theta, 100)$. Find a uniformly most powerful critical region of size $\alpha = 0.10$ for testing $H_0: \ \theta = 50$ against $H_1: \ \theta > 50$.

25. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the following pdf

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$. Find a sufficient statistics for θ and show that a uniformly most powerful test of $H_0: \ \theta = 6$ against $H_1: \ \theta < 6$ is based on this statistics.

26. If $X_1, X_2, \ldots, X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find a best critical region for testing $H_0: \ \theta = 1$ against $H_1: \ \theta = 2$.

27. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size 20 from a Poisson distribution with mean θ. Show that the critical region C is defined by $\sum_{i=1}^{20} x_i \geq 4$.

28. Let X have following pmf

$$p(x;\theta) = \begin{cases} \theta^x (1-\theta)^{1-x}, & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

We test the simple hypothesis $H_0: \ \theta = \frac{1}{4}$ against the alternative composite hypothesis $H_1: \ \theta < \frac{1}{4}$ by taking a random sample of size 10 and rejecting H_0 if and only if the observed values $x_1, x_2, \ldots, x_n$ of the sample observations are such that $\sum_{i=1}^{n} x_i < 1$. Find the power function $k(\theta)$, $\ 0 < \theta \leq \frac{1}{4}$, of this test.