

Department of Mathematics
MTL 5005 (Introduction to Probability)
Tutorial Sheet No. 3

1. Let us write

$$b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

Show that as k goes from 0 to n , $b(k; n, p)$ first increases monotonically and then decreases monotonically. The greatest value is assumed when $k = m$, where m is an integer such that $(n+1)p - 1 < m \leq (n+1)p$ except that $b(m-1; n, p) = b(m; n, p)$ when $m = (n+1)p$.

2. A shipment of 120 burglar alarms contains 5 that are defective. If 3 of these burglar alarms are randomly selected and shipped to a customer,
 (a) find the exact probability that he will get exactly one bad unit.
 (b) approximate probability that he will get exactly one bad unit by means of the binomial distribution.
3. Show that as k goes from 0 to ∞ the ratio $a_k = b(k; n, p)/P(k, \lambda)$ first increase, then decrease, reaching their minimum when k is the largest integer not exceeding $\lambda + 1$.
4. Let X be a Poisson distributed random variable with parameter λ . Prove that

$$\sum_{i=0}^n P(X = i) = \frac{1}{n!} \int_{\lambda}^{\infty} e^{-x} x^n dx.$$

5. Let X be a binomial r.v. with parameter n and p and Y be a Pascal r.v. with parameters r and p . Show that
 (i) $P(Y \leq n) = P(X \geq r)$ (ii) $P(Y > n) = P(X \leq r)$.
6. Let $X \sim \text{EXP}(\lambda)$. Prove that the random variable X has “no memory”, that is,

$$P(X > r + s | X > r) = P(X > s)$$

for any two positive real numbers r and s .

7. Find the probability density function of an exponential random variable truncated to the left at point $a > 0$.
8. Let $X \sim U[0, 1]$. (i) Prove that $Y = -\frac{1}{\lambda} \log(1 - X) \sim \text{EXP}(\lambda)$.
 (ii) Find the pdf of $U = X/(1 + X)$.
9. Let $X \sim N(\mu, \sigma^2)$. Prove that $Z = \frac{(X-\mu)}{\sigma} \sim N(0, 1)$ and $Z^2 \sim \chi^2$ with 1 degrees of freedom.
10. Verify the following as a consequence of definition of expectation:
 (a) If X is a bounded random variable, then $E(X)$ exists.
 (b) If $X \in (a, b)$ with probability 1, then $a < E(X) < b$.
 (c) If X is symmetrical abt a point μ , then $E(X) = \mu$.
 (d) If $X \in \{1, 2, \dots, n, \dots\}$ with probability 1, then $E(X) = \sum_{n=1}^{\infty} P(X \geq n)$.
11. Let A, B, C be events in a sample space with $P(A) = 0.1$, $P(B) = 0.4$, $P(C) = 0.3$, $P(A \cap B) = 0.1$, $A \cap C = B \cap C = \phi$. Find mean and variance of a random variable $X = I_A + I_B - I_C$, where $I_{[\cdot]}$ is the indicator function of the set $[\cdot]$.

12. Show that $E(|X|) = \int_{-\infty}^0 F_X(x) dx + \int_0^{\infty} (1 - F_X(x)) dx$.

13. Show that the matrix $\sum = (\mu_{i+j})_{k \times k}$ is non negative definite where μ_p is the central moment of order p of the random variable X .
14. The r^{th} factorial moment of a random variable X defined by $\mu'_{(r)} = E[X(X-1)\dots(X_r+1)]$. If X is a discrete random variable with p.m.f. $p_X(i) > 0$, $i = 0, 1, \dots, r$, and 0 elsewhere, show that $\mu'_{(k)} = 0$, $k = r+1, r+2, \dots$.
15. From a point on the circumference of the circle of radius r , a chord is drawn in a random direction. Show that the expected value of length of the chord is $\frac{4r}{\pi}$ and its variance is $2r^2(1 - \frac{8}{\pi^2})$.
16. For the random variable X with p.m.f. $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \dots$, show that $\mu_2 = \mu_3 = \lambda$, $\mu_4 = \lambda + 3\lambda^2$.
17. Show that for a random variable with p.d.f. $f_X(x) = \frac{k\alpha^k}{(x+\alpha)^{k+1}}$, $x > 0$, the α^{th} absolute moment exists for X , for $\alpha < k$.
18. For a random variable with p.d.f. $f_X(x) = \frac{1}{\alpha!}e^{-x}x^\alpha$, $x > 0$, where α is a positive integer, show that $P(0 < X < 2(\alpha+1)) > \frac{\alpha}{\alpha+1}$ by using Chebyshev inequality.
19. For the random variable with p.m.f. $p_X(x) = \binom{r+x-1}{x} p^r q^x$, $x = 0, 1, 2, \dots$, where $q = 1 - p$, find the m.g.f. and hence the mean and variance of X .
20. Find the mean, variance of the random variable X having the distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ p + (1-p)(1 - e^{-\lambda x}), & 0 \leq x < T \\ 1, & x \geq T \end{cases}$$

21. For the Laplace distribution, i.e.e the random variable having p.d.f. given by: $f_X(x) = \frac{1}{2\lambda} \exp\left(-\frac{|x-\mu|}{\lambda}\right)$, $x \in R$, $\lambda > 0$, $0 < \mu < \infty$, find m.g.f. and its mean and variance.
22. Let X be a random variable having m.g.f $M(t)$, $t > 0$. Then show that for $t > 0$, $P(tX > s^2 + \log(M(t))) < e^{-s^2}$.
23. (Jensen's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) < E(g(X))$. Hence show that $E(X) \leq (E(|X|))^{1/r}$.
24. Let $g(X) \geq 0, \forall x \in [0, \infty)$ be a non-decreasing even function. Show that for any random variable X such that $E(g(X))$ exists $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.
25. X has the power series distribution if it has a p.m.f. $P(X = j) = \frac{a_j \theta^j}{f(\theta)}$, $x = 0, 1, 2, \dots$, $\theta > 0$, $a_j \geq 0$ and where $f(\theta) = \sum_{j=0}^{\infty} a_j \theta^j$. Find the m.g.f. of X . Use it to find the factorial moments of X .
26. Show that the sequence of moments determine the probability distribution of the random variable uniquely if it is a bounded random variable.
27. Show that absolute moment of no order exists for the random variable having p.d.f $f_X(x) = \frac{1}{2|x|(\ln|x|)^2}$ for $|x| > e$.
28. Let X be a discrete r.v. with MGF

$$M_X(t) = \alpha + \beta e^{2t} + \gamma e^{4t}, \quad E(X) = 3, \quad Var(X) = 2.$$

- (i) Find α, β and γ ? (ii) Find the PMF of X ? (iii) Find $E(2^X)$?

29. The MGF of a r.v. X is given by $M_X(t) = \exp(\mu(e^t - 1))$. (a) What is the distribution of X ? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.
30. Let X be a normal distributed random variable with mean μ and variance σ^2 . Find the moment generating function (MGF) of the random variable X . Using MGF, find $E(X^4)$.
31. Let X be a r.v. with characteristic function given by $E[e^{iXt}] = \frac{1}{7} (2 + e^{-it} + e^{it} + 3e^{2it})$. Determine (i) $P(-1 \leq X \leq \frac{1}{2})$ (ii) $E(X)$.
32. Let Φ be the characteristic function of a random variable X . Prove that $1 - |\Phi(2u)|^2 \leq 4(1 - |\Phi(u)|^2)$.

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