

Department of Mathematics
MTL 5005 (Introduction to Probability)
Tutorial Sheet No. 4

1. Let

$$F(x, y) = \begin{cases} 1, & x + 2y \geq 1 \\ 0, & x + 2y \leq 1 \end{cases}$$

Does F define a distribution function in 2-dimensional plane ?

2. Let

$$F(x, y) = \begin{cases} 0, & x < 0, y < 0 \text{ or } x + y < 1 \\ 1, & \text{otherwise} \end{cases}$$

Show that F is not a distribution function in R^2 .

3. Let $S = (0, 1] \times (0, 1]$. A point (u, v) is chosen at random in S . Define $X(u, v) = u + v$, $Y(u, v) = \frac{u}{v}$. Show that (X, Y) is a 2-dimensional random variable on the probability space $(S, \beta_2 \cap S, P)$ where P is appropriately chosen. Find the joint and marginal distribution functions of (X, Y) . Using distribution functions compute $P(1 < X < 2.5, 1 < Y < 3)$, $P(X \leq 1 | Y \leq 2)$.

4. Six dice are rolled. Let:

$$\begin{aligned} X &= \text{number of dice with score a multiple of 2} \\ Y &= \text{number of dice with score a multiple of 3} \\ Z &= \text{number of dice with score a multiple of 4} \end{aligned}$$

Find the joint p.m.f. of (X, Y, Z) . Also find the marginal p.m.f. of X , (X, Z) , (Y, Z) and the conditional p.m.f. of X given (Y, Z) .

5. A point (u, v) is chosen as follows. First a u is chosen at random in the interval $(0, 1)$, then a point v is chosen at random on the interval $(0, u)$. Find the joint p.d.f. of (X, Y) , defined by $X(u, v) = u$, $Y(u, v) = v$. Also find the conditional p.d.f. of X , given $Y = y$ and that of Y , given $X = x$.
6. Find the possible marginal and conditional distributions if (X, Y) have the following distributions:
- (a) $P(X = j, Y = k) = q^{k-2}p^2, k = j + 1, j + 2, \dots; j = 0, 1, 2, \dots$
- (b) $P(X = j, Y = k) = \frac{15!}{j!k!(15-j-k)!} \left(\frac{1}{2}\right)^j \left(\frac{1}{3}\right)^k \left(\frac{1}{6}\right)^{15-j-k}$, for all admissible values of j and k .
7. Suppose 2 cards are drawn from a deck of 52 cards. Let X = number of aces obtained and Y = number of queens obtained. Discuss whether or not random variable X, Y are independent.
8. Suppose a two dimensional random variable has a joint p.d.f.:

$$f_{X,Y}(x, y) = \begin{cases} kx(x - y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

- (a) Evaluate the constant k
- (b) Compute $P(X + Y < 1)$, $P(XY < 1)$

9. Let (X, Y, Z, U) be a 4-dimensional random variable with joint p.d.f.

$$f_{X,Y,Z,U}(x, y, z, u) = e^{-u}, \quad 0 < x < y < z < u < \infty$$

- (a) Find the marginal p.d.f. of X .

10. The life length of an electronic device has exponential distribution with mean 1000 hours. Let X_1, X_2, X_3 be the life length of 3 such devices operating independently.

- (a) Find the probability that $X_1 < X_2 < X_3$. Does the answer change if the life length distribution is not exponential.
- (b) Compute $P(X_1 + X_2 < 4000, X_1 > 2000)$
11. Let X_1, X_2 be i.i.d. random variables each having p.m.f. $P(X = \pm 1) = \frac{1}{2}$. Define $X_3 = X_1 X_2$. Show that random variables X_1, X_2, X_3 are pairwise independent but not mutually independent.
12. Let (X, Y) be uniformly distributed on the region $\{(x, y) : 0 < x < y < 1\}$.
- (a) Find $V(2X + 3Y - 4)$, $Cov(X + Y, X - Y)$, $E(X^3 + XY^2 - X^2Y)$.
- (b) Find MGF of (X, Y) and use this to compute correlation coefficient between X, Y .
13. Let $X_1, X_2, \dots, X_{m+n}$ be i.i.d. random variables each having a finite second order moment. Find the correlation coefficient between S_n and $S_{m+n} - S_n$, $n > m$, where $S_k = \sum_{i=1}^k X_i$, $k = 1, 2, \dots, m + n$.
14. The joint pdf of (X, Y) is given by:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}(1 + xy(x^2 - y^2)), & |x| \leq 1, |y| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Compute

- (a) $MGF(X, Y)$
- (b) $MGF(X + Y)$.
15. Let X, Y be discrete random variables with respective pmfs given by

$$p_X(x_1) = p_1, p_X(x_2) = 1 - p_1, p_Y(y_1) = p_2, p_Y(y_2) = 1 - p_2$$

Show that X, Y are independent iff X, Y are uncorrelated.