

Department of Mathematics
MTL 5005 (Introduction to Probability)
Tutorial Sheet No. 5

1. Let $\{X_n\}$ be a sequence of random variables defined on the probability space $([0, \infty), \beta_1, P)$, $P(\{0\}) = 0$.

(a) Define $X_n(s) = \frac{1}{s} \left(1 - \frac{1}{n}\right)$, $s \in [0, \infty)$. Then show that $X_n \xrightarrow{a.s} X$, where $X(s) = \frac{1}{s}$, $s \in [0, \infty)$.

(b) Define $X_n(s) = \frac{1}{ns}$, $s \in [0, \infty)$. Then show that $X_n \xrightarrow{a.s} 0$.

(c) Let $P(I) = \int_I e^{-x} dx$. Define

$$X_n(s) = \begin{cases} 0, & \text{if } s \text{ is rational} \\ (-1)^n, & \text{if } s \text{ is irrational} \end{cases}$$

Then show that $\{X_n\}$ diverges almost surely.

2. Let $S = [0, 1]$ and P be uniform distribution.

(a) Define

$$X_n(s) = e^{-n^2(ns-1)}, \quad 0 \leq s \leq 1; \quad n = 1, 2, \dots$$

Show that $X_n \xrightarrow{a.s} 0$, but $X_n \not\xrightarrow{m.s} 0$.

(b) Define

$$X_n(s) = \begin{cases} 3^n, & \text{if } 0 \leq s \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Show that $X_n \xrightarrow{p} 0$, but $X_n \not\xrightarrow{m.s} 0$.

3. Consider a sequence of random variables $\{X_n\}$ with $P(X_n = 0) = 1 - \frac{1}{n^\alpha}$, $P(X_n = \pm n) = \frac{1}{2n^\alpha}$. Determine the value of α for which the sequence obeys WLLN.

4. Consider a sequence of random variables $\{X_n\}$ with $E(X_n) = m$ and

$$\text{Cov}(X_i, X_j) = \begin{cases} \sigma^2, & i \neq j \\ a\sigma^2, & i = j \pm 1, \text{ where } |a| < 1, \sigma^2 > 0 \text{ are given constants} \\ 0, & \text{otherwise} \end{cases}$$

Show that WLLN holds for $\{X_n\}$.

5. Use CLT to show that $\lim_{n \rightarrow \infty} e^{-n} \sum_{i=0}^n \frac{n^i}{i!} = 0.5$

6. Consider a sequence of independent random variables $\{X_n\}$ such that

$$P(X_n = 0) = 1 - \frac{2}{n^3}, \quad P(X_n = \pm n) = \frac{1}{n^3}, \quad n > 1$$

Does the sequence $\{X_n\}$ obey CLT ?

7. If $X_n \xrightarrow{p} 0$, then find the median of $X_n \rightarrow 0$ as $n \rightarrow \infty$.

8. Consider a sequence $\{X_n\}$ of identically distributed random variables with the property that $nP(|X_i| > n) \rightarrow 0$ as $n \rightarrow \infty$. Show that $\frac{1}{n} \max_{1 \leq i \leq n} X_i \xrightarrow{p} 0$

9. Suppose $|X_n - X| \leq Y_n$, almost surely for some random variable X , then show that if $E(Y_n) \rightarrow 0$, then $E(X_n) \rightarrow E(X)$ and $X_n \xrightarrow{p} X$.

10. Show that $X_n \xrightarrow{2} X \Rightarrow E(X_n) \rightarrow E(X)$, $E(X_n^2) \rightarrow E(X^2)$ as $n \rightarrow \infty$.

11. Let $\{X_i\}$ be a sequence of independent random variables, such that each X_i has mean 0 and variance 1. Show that

$$\sqrt{n} \frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} \xrightarrow{L} Z \sim N(0, 1)$$

12. Does WLLN hold for the following sequences

(a) $P(X_k = \pm 2^k) = \frac{1}{2}$

(b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$

13. For what values of α , does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k = \pm k^\alpha) = \frac{1}{2}$, $k = 1, 2, \dots$.

14. Let $\{X_n\}$ be a sequence of independent random variables with the following probability distribution. In each case, does the Lindeberg Condition for CLT hold ?

(a) $P(X_k = \pm \frac{1}{2^n}) = \frac{1}{2}$

(b) $P(X_k = \pm \frac{1}{2^{n+1}}) = \frac{1}{2^{n+3}}$, $P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$

15. From an urn containing 10 identical balls numbered 0 through 9, n balls are drawn with replacement,

- (a) What does the law of large number tell you about the appearance of 0's in n drawings.
 (b) How many drawings must be made in order that with probability atleast 0.95, the relative frequency of occurrence of 0's will be between 0.09 and 0.11 ?
 (c) Use CLT to find the probability that among n numbers thus chosen, the number 5 will appear between $\frac{n-35n}{10}$ and $\frac{n+35n}{10}$ times, if (i) $n = 25$ (ii) $n = 100$.